A Branch-Price-and-Cut Algorithm for a Maritime Inventory Routing Problem

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CARMA Workshop
Newcastle, November 1st, 2009
• IRP in maritime transportation
  • Problem description

• A Time-indexed Column Generation Formulation

• The Pricing Problem
  • problem characteristics

• Cut generation
  • extended VRP cuts
  • new mixed 0-1 cuts

• Branching

• Computational results
Problem Description

Supply Point

Demand Point
Q: Given a production profile for a port, when and how much inventory should be picked up by a vessel?

Supply port constraints:

- Port storage capacity and safety stock
- Min and Max load limits per day
Q: Given a demand profile for a supply port, when and how much inventory should be dropped-off by a vessel?

Demand port constraints:

- Port storage capacity and safety stock
- Min and Max discharge limits per day
Problem Description (Vessel)

Q: Given a **time window** of operation, how to **route** a vessel so that it is available at a port to load/discharge when required?

Vessel constraints:
- Vessel storage capacity
- Draft limits
- No inventory left at the end of voyage
A Time-indexed Column Generation Formulation

The Master Problem

\[
\begin{align*}
\text{min} & \quad \sum_{v \in V} \sum_{r \in R_v} c^r \lambda^r \\
\text{s.t.} & \quad l^r_{j,t} = l^r_{j,t-1} + b^r_{j,t} - \sum_{v \in V} \sum_{r \in R_v} f^r_{j,t} \lambda^r \\
& \quad l^r_{j,t} = l^r_{j,t-1} - b^r_{j,t} - \sum_{v \in V} \sum_{r \in R_v} f^r_{j,t} \lambda^r, \quad j \in J_S, \ t = 1, \ldots, T, \\
& \quad j \in J_D, \ t = 1, \ldots, T, \\
& \quad 0 \leq l^r_{j,t} \leq Q^r_{j,t}, \quad j \in J_S \cup J_D, \ t = 1, \ldots, T, \\
& \quad \sum_{r \in R_v} \lambda^r = 1, \quad v \in V, \\
& \quad \lambda^r \geq 0, \quad v \in V, r \in R_v \\
& \quad \sum_{r \in R_v} z^r_{j,t} \lambda^r \in \{0,1\}, \quad v \in V, j \in J_S \cup J_D, \ t = 1, \ldots, T.
\end{align*}
\]

- cost of voyage \( r \)
- amount loaded/discharged at port \( j \) and time \( t \)
- inventory at port \( j \) and time \( t \)
- production/demand at port \( j \) and time \( t \)
- 0-1 indicator of voyage \( r \)
- loading/discharging at port \( j \) and time \( t \)
The Pricing Problem

Find min cost route and determine quantity loaded/discharged at each port that is visited so that vessel capacity, draft limits, and min/max load/discharge quantities are not exceeded.
The Pricing Problem: Characteristics

For a given path $P$, $A(P) = \{e_1, \ldots, e_K\}$ and $N(P) = \{n_0, n_1, \ldots, n_K\}$, an optimal allocation of load quantities can be obtained by solving the linear relaxation of the \textit{Multi-period Knapsack Problem} (LP-MKP):

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{K} c_{e_i} f_{e_i} \\
\text{s.t.} & \quad \sum_{i=1}^{j} f_{e_i} \leq U_{n_j} \text{ for all } j = 1, \ldots, K, \text{ and} \\
& \quad l_{e_i} \leq f_{e_i} \leq u_{e_i} \text{ for all } i = 1, \ldots, K.
\end{align*}
\]
The Pricing Problem: Characteristics

For a given path $P$, $A(P) = \{e_1, \ldots, e_K\}$ and $N(P) = \{n_0, n_1, \ldots, n_K\}$, an optimal allocation of load quantities can be obtained by solving the linear relaxation of the *Multi-period Knapsack Problem* (LP-MKP):

$$\begin{align*}
\min & \quad \sum_{i=1}^{K} c'_{e_i} f_{e_i} \\
\text{s.t.} & \quad \sum_{i=1}^{j} f_{e_i} \leq U_{n_j} \quad \text{for all } j = 1, \ldots, K, \text{ and} \\
& \quad l_{e_i} \leq f_{e_i} \leq u_{e_i} \quad \text{for all } i = 1, \ldots, K.
\end{align*}$$

**Proposition**

LP-MKP can be solved by:

1. Initializing $f_{e_i}$ to $l_{e_i}$ for all $i = 1, \ldots, K$, and
2. Increase load quantity on arcs *greedily* (i.e. non-decreasing order of $c'_{e_i}$) until we either
   - reach the upper limit $u_{e_i}$, or
   - reach some limit $U_{n_i}$ on the total amount of inventory allowed before entering node $n_i$.

**Corollary**

There exists an optimal allocation $f_{e_i}^*$, $i = 1, \ldots, K$ such that for each $i = 1, \ldots, K$ either:

1. $f_{e_i}^* \in \{l_{e_i}, u_{e_i}\}$, or
2. $f_{e_i}^* = U_{n_k} - \sum_{j \in \{1, \ldots, k\}\setminus\{i\}} f_{e_j}^*$ for some $k \geq i$ and $f_{e_j}^* \in \{l_{e_j}, u_{e_j}\}$ for all $j = i + 1, \ldots, k$. 

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Engineer, F. G.  
Branch-Price-and-Cut for a Maritime IRP  
10/22
Given port $j$ and time interval $[t_1, t_2]$, compute min number of loads/discharges based on excess/deficit inventory and max load/discharge per day.

\[ P_1 \times P_2 = \frac{1}{2} \]

3/2 loads at $j$ during $[t_1, t_2]$. 

\[ \lambda^{p_1} = \frac{1}{2} \]

\[ \lambda^{p_2} = \frac{1}{2} \]
Cuts for IRP: Port capacity cuts

Given port \( j \) and time interval \([t_1, t_2]\), compute min number of loads/discharges based on excess/deficit inventory and max load/discharge per day.

Inventory before \( t_1 \) = \( I_{j,t_1-1} \) = 50

Production during \([t_1, t_2]\) = \( \sum_{t=t_1, \ldots, t_2} b_{j,t} \) = 25 \times 6 = 150

Capacity at \( j \) at \( t_2 \) = \( Q_{j,t_2} \) = 75

Excess inventory = 50 + 150 - 75 = 125

Max load per day = \( F_j^{\text{max}} \) = 75

Min no. of loads at \( j \) during \([t_1, t_2]\) = \( \left\lceil \frac{125}{75} \right\rceil \) = 2
Given port $j$ and time interval $[t_1, t_2]$, compute min number of loads/discharges based on excess/deficit inventory and max load/discharge per day.

$$z_j(t_1, t_2) = \sum_{v \in V} \sum_{r \in R_v} \sum_{t=t_1}^{t_2} z_{j,t}^r \geq \left[ \frac{l_{j,t_{1}-1} + \sum_{t=t_1}^{t_2} b_{j,t} - Q_{j,t_2}}{F_{j}^{\max}} \right]$$
Given port $j$ and time interval $[t_1, t_2]$, compute min number of loads/discharges based on excess/deficit inventory and max load/discharge per day.

$$z_j(t_1, t_2) = \sum_{v \in V} \sum_{r \in R_v} \sum_{t=t_1}^{t_2} z^r_{j,t} \lambda^r \geq \frac{[I_{j,t_1-1} + \sum_{t=t_1}^{t_2} b_{j,t} - Q_{j,t_2}]}{F_{j}}$$

$$0 \leq I_{j,t_1-1} \leq Q_{j,t_1-1}$$
Cuts for IRP: Timing cuts

Given port $j$ and time interval $[t_1, t_2]$, compute timing of departures based on excess/deficit inventory, production/demand rate, and vessel capacity.

- At least 2 loads at $j$ during $[t_1, t_2]$
- At least 2 visits at $j$ during $[t_1, t_2]$
- $0 + (1+2+3)/3 = 2$ days “wait” since $t_1$ to pickup 150 units of inventory during $[t_1, t_2]$. 

![Diagram of timing cuts and loads at port $j$](image-url)
Cuts for IRP: Timing cuts

Given port $j$ and time interval $[t_1, t_2]$, compute timing of departures based on excess/deficit inventory, production/demand rate, and vessel capacity.

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- At least 2 visits at $j$ during $[t_1, t_2]$
- $0 + (1 + 2 + 3)/3 = 2$ days “wait” since $t_1$ to pickup 150 units of inventory during $[t_1, t_2]$.

Inventory loaded during $[t_1, t_2] = f_j(t_1, t_2) = \sum_{v \in V} \sum_{r \in R_v} \sum_{t=t_1,...,t_2} f_{j,t} = 150$

Inventory before $t_1 = I_{j,t_1-1} = 50$

Max load per day = $F_j = 75$

Vessel capacity = $Q = 100$

Production rate during $[t_1, t_2] = b_{j,t} = 25$

$\Rightarrow$ At least 2 visits required and at least one of these must load on or after $t_1 + 3$

$\Rightarrow$ To load 150 units of inventory sum of last load time over all vessels $\geq t_1 + 3$
Given port $j$ and time interval $[t_1, t_2]$, compute timing of departures based on excess/deficit inventory, production/demand rate, and vessel capacity.

If $(k - 1)Q \leq t_j(t_1, t_2) - l_{j,t_1-1} \leq kQ$ then:

1. at least $k$ visits are required, and
2. at least $i$ visits must load on or after

$$t_1 + \left[ \frac{f_j(t_1, t_2) - l_{j,t_1-1} - (i - 1)Q}{b_j^{\text{max}}(t_1, t_2)} \right] - 1$$

for all $i = 1, \ldots, k$.

Amount of inventory loaded at $j$ during $[t_1, t_2]$ in excess of what is available at $t_1-1$.
Given port $j$ and time interval $[t_1, t_2]$, compute timing of departures based on excess/deficit inventory, production/demand rate, and vessel capacity.

If $(k - 1)Q < f_j(t_1, t_2) - l_{j,t_{1-1}} < kQ$ then:

1. at least $k$ visits are required, and
2. at least $i$ visits must load on or after

$$t_1 + \left\lfloor \frac{f_j(t_1, t_2) - l_{j,t_{1-1}} - (i - 1)Q}{b_j^{\max}(t_1, t_2)} \right\rfloor - 1$$

for all $i = 1, \ldots, k$.

Amount of inventory loaded at $j$ during $[t_1, t_2]$ in excess of what is available at $t_{1-1}$

Constant overestimation of production rate at $j$ during $[t_1, t_2]$
Branching for IRP

1. Partition follow-on ports

2. Partition location/timing of first/last load/discharge

3. Partition arc cut-set within network

4. Partition decision to load/discharge at port and time

\[ \sum_{r \in R_v} z_{j,t}^r \lambda_r^r \in \{0,1\}, \]
## Computational Experiments: LP relaxation Results

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<th>Inst. Class</th>
<th>No. inst.</th>
<th>Avg. LP gap (%)</th>
<th>Avg. % gap closed after +PP</th>
<th>Avg. % gap closed after +BC</th>
<th>Avg. % gap closed after +Cuts</th>
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PP – Preprocessing, BC – Boundary constraints, PCC – Port capacity cuts, VCC – Vessel capacity cuts, TC – Timing cuts
## Computational Experiments: IP Results

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<tr>
<th>Inst. Class</th>
<th>No. inst.</th>
<th>No. solved inst.</th>
<th>Avg. gap (%)</th>
<th>Avg. time (s)</th>
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B&C – Default CPLEX11.1
B&C+ - CPLEX11.1 + branching and cut enhancements
BP&C – Our branch-price-and-cut algorithm

Note: 10 hour time limit used
Summary

• A time-indexed column generation formulation
  • Demurrage time and costs (i.e. vessel idle and holding costs)
  • Capacities and production/consumption rates fluctuate over time
  • Enforce draught limits and require no inventory on the vessel at the end of its voyage

• A unique mixed 0-1 pricing problem
  • Extract properties amiable for solving exactly and efficiently through DP

• Cuts
  • Extend VRP capacity cuts to mixed 0-1 case
  • Developed new mixed 0-1 cuts specifically for IRP

• Computational results compare very favorably in terms of producing strong lower bounds as compared to an alternative arc-flow formulation and branch-and-cut approach.
Questions?