Wavelets, Sampling and Synchronization

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Two big examples

Classical (Shannon) sampling:

$$PW = \{f \in L^2; \hat{f} = 0 \text{ off } [-1/2, 1/2]\}$$

$f \in PW$ then

$$f(t) = \sum_k f(k)\mathrm{sinc}\,(t-k) = \sum_k f(k+\alpha)\mathrm{sinc}\,(t-k-\alpha)$$

$$\sum_k f(k+\alpha)\mathrm{sinc}\,(t-k) = f(t-\alpha)$$

Two big examples

Haar sampling:

$$V_H = \{f \in L^2; f\big|_{[k,k+1)} = \text{ constant }\}; \quad \varphi_H = \chi_{[0,1)}$$

$f \in V_H$ then

$$f(t) = \sum_k f(k+\alpha)\varphi_H(t-k) \quad (0 < \alpha < 1)$$

$$\sum_k f(k+\alpha)\varphi_H(t-k) = f(t+[\alpha]) \quad (\alpha \in \mathbb{R})$$

PSI spaces and scaling functions

PW, V_H are PSI spaces:

- $V = V(\varphi) \subset L^2$ is closed
- $\{\varphi(t-k)\}$ an orthonormal basis for V

$$f \in V(\varphi) \Leftrightarrow f = \sum_k c_k \varphi(t-k) \text{ with } \{c_k\} \in \ell^2(\mathbb{Z})$$

If φ generates a PSI space and

$$\frac{1}{2}\varphi\left(\frac{t}{2}\right) = \sum_k h_k \varphi(t-k) \qquad (1)$$

we say φ is a scaling function.

$$(1) \iff \hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$$

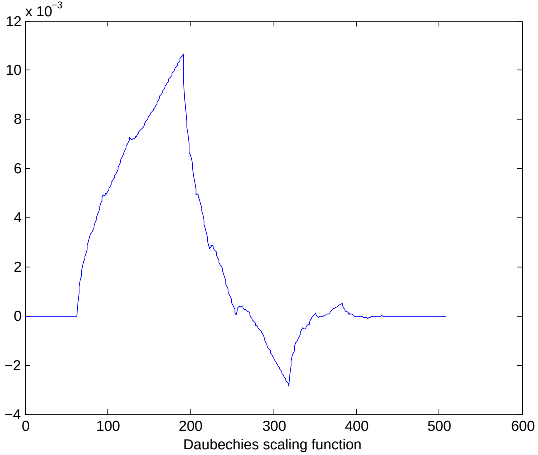
$$\{\varphi(\cdot - k)\} \text{ orthonormal } \implies |m_0(\xi)|^2 + |m_0(\xi + 1/2)|^2 \equiv 1 \text{ (QMF)}$$

Daubechies' ${}_2\varphi$ scaling function

$$\frac{1}{2}\varphi\left(\frac{t}{2}\right) = \left(\frac{1+\sqrt{3}}{8}\right)\varphi(t) + \left(\frac{3+\sqrt{3}}{8}\right)\varphi(t-1) + \left(\frac{3-\sqrt{3}}{8}\right)\varphi(t-2) + \left(\frac{1-\sqrt{3}}{8}\right)\varphi(t-3)$$

- Supported on $[0,3]$
- $|\varphi(x)-\varphi(y)| \leq C|x-y|^{0.550}$

Daubechies' ${}_2\varphi$ scaling function



τ-cycle condition

$$\tau : \mathbb{T} \to \mathbb{T} \quad \tau(z) = z^2$$

$\{z_1, z_2, \ldots, z_n\} \subset \mathbb{T}$ is a τ-cycle if $\tau z_j = z_{j+1}$ and $\tau z_n = z_1$.

QMF m_0 satisfies the τ-cycle condition if there exists no τ-cycle $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{T}$ with $|m_0(z_j)| = 1$ for all j.

Theorem

If m_0 is a QMF with $m_0(1) = 1$ and φ is the associated scaling function, then $\{\varphi(\cdot - k)\}$ is an orthonormal basis for $V(\varphi) \Leftrightarrow m_0$ satisfies the τ-cycle condition.

Example

$m_0(z) = (1 + z^{-6})/2 \longrightarrow \varphi(t) = \chi_{[0,3]}(t)$,
$\omega = e^{2\pi i/3}$, $|m_0(\omega)| = |m_0(\omega^2)| = 1$

Uniform critical sampling

Zak transform:

$$Z_{\mathbb{C}}f(t,z) = \sum_k f(t+k)z^k \quad (t \in \mathbb{R},\ z \in \mathbb{C})$$

$$f(t) = \sum_k c_k \varphi(t-k) \Rightarrow Zf(t,z) = C(z)Z\varphi(t,z)$$

(V,φ) a PSI, $f \in V(\varphi)$ with samples $\{f(\alpha+k)\}$.

Theorem (Janssen 1993)

If $\inf_\xi |Z\varphi(\alpha,\xi)| > 0$ *then*

$$f(t) = \sum_k f(\alpha+k)S_\alpha(t-k); \quad S_\alpha(t) = \sum_\ell \left(\frac{1}{Z\varphi(\alpha,\cdot)}\right)^{\wedge}(\ell)\varphi(t-\ell)$$

Periodic nonuniform sampling

Theorem (Djokovic, Vaidyanathan 1997)

Suppose φ *is continuous, supported in* $[0, M]$,
$0 \leq t_0 < t_1 < \cdots < t_{L-1} < 1$ *and* $\{Z\varphi(t_j, z)\}_{j=0}^{L-1}$ *co-prime*

Euclid's algorithm $\longrightarrow$ *polynomials* $P_0, \ldots, P_{L-1}$, $\deg(P_j) \leq M-2$
with $\sum_{j=0}^{L-1} P_j(z) Z\varphi(t_j, z) = 1$.

$f \in V(\varphi)$ *then*

$$f(t) = \sum_k \sum_{j=0}^{L-1} f(t_j + k) S_j(t-k); \quad S_j(t) = \sum_{m=0}^{M-2} p_{jm}\varphi(t-m)$$

Periodic nonuniform sampling – validity

- $S_j \in V(\varphi)$ supported in $[0, 2M-2]$
- Sampling rate: L samples per unit time

Theorem (H, Lakey 2005)

Suppose φ is a continuous scaling function supported on $[0, M]$. Then the 2^{M-2} polynomials $\left\{ Z\varphi\left(\frac{\ell}{2^{M-2}}, z\right)\right\}_{\ell=0}^{2^{M-2}-1}$ are co-prime.

Proof: Contradicts the τ-cycle condition.

Discrepancy

$V \subset \mathbb{R}$ is translation-invariant if $\tau_a : f(t) \to f(t-a)$ preserves V.

If V is a closed translation-invariant subspace of L^2 then

$$V = V_E = \{f \in L^2; \hat{f} = 0 \text{ off } E\}$$

$$d_\varphi(a) = \sup_{f \in V(\varphi),\ \|f\|_2 = 1} \|\tau_a f - P_\varphi(\tau_a f)\|_2^2$$

Discrepancy: $d_\varphi = \sup_{0 \le a < 1} d_\varphi(a)$

Discrepancy

Theorem (H, Lakey 2009)

Suppose φ is a continuous, compactly supported orthonormal generator for a PSI space $V = V(\varphi)$ and $0 < a < 1$. Then $\tau_a V \not\subset V$, i.e., $d_\varphi(a) > 0$.

Theorem (H, Lakey 2009)

Let $\varphi \in W(L^\infty, \ell^1)$ be an orthogonal generator for the PSI space $V(\varphi)$. Then there exists $a \in (0,1)$ with $d_\varphi(a) = 1$, i.e., $d_\varphi = 1$. If φ is a scaling function then $d_\varphi(1/2) = 1$.

"There exists $f \in V(\varphi)$ such that $\tau_{1/2} f \perp V(\varphi)$"

Note: $\mathrm{sinc} \notin W(L^\infty, \ell^1)$

Synchronization - uniqueness

Incoming data $\{f(\alpha + k)\}$, $f \in V(\varphi)$, α unknown.

Sampling functions S_α depend on knowledge of α:

Janssen: $S_\alpha(t) = \sum_\ell \left(\frac{1}{Z\varphi(\alpha, \cdot)}\right)\hat{}(\ell)\varphi(t - \ell)$

Need to determine translation offset α before reconstruction.

Question 1: Does sampled data with unknown offset determine a unique $f \in V(\varphi)$?

Synchronization - uniqueness

Theorem (H, Lakey 2009)

Let φ be an orthonormal scaling function supported in $[0, M]$. Suppose that for some integer $J \geq 1$, and each $0 \leq \alpha < 2^{-J}$, the polys $\{Z\varphi(\alpha + \ell 2^{-J}, z)\}_{\ell=0}^{2^J-1}$ are co-prime. Let $f, g \in V(\varphi)$ and suppose there exists $\alpha, \beta \in \mathbb{R}$ such that

$$f(\alpha + k2^{-J}) = g(\beta + k2^{-J}) \text{ for all } k \in \mathbb{Z}.$$

Then $f = g(\cdot - m)$ for some integer m.

Conclusion: without knowing the offset α, oversampled data determines a single $f \in V(\varphi)$.

Question 2: Does oversampled data determine the offset?

Synchronization – offset determination

$$
\begin{aligned}
f(\alpha + k) = f(\beta + k) \text{ all } k \quad &\Leftrightarrow \quad Zf(\alpha, z) = Zf(\beta, z) \text{ all } z \\
&\Leftrightarrow \quad C(z)Z\varphi(\alpha, z) = C(z)Z\varphi(\beta, z) \text{ all } z \\
&\Leftrightarrow \quad Z\varphi(\alpha, z) = Z\varphi(\beta, \xi) \text{ all } z \\
&\Leftrightarrow \quad \varphi(\alpha + k) = \varphi(\beta + k) \text{ all } k
\end{aligned}
$$

Theorem (H, Lakey 2009)

Let φ be a continuous compactly supported scaling function for an MRA and $\alpha, \beta \in [0, 1)$ with $\varphi(\alpha + k) = \varphi(\beta + k)$ for all k. Then $\alpha = \beta$.

Proof: Uses ergodicity of the τ operator $\tau : z \mapsto z^2$.

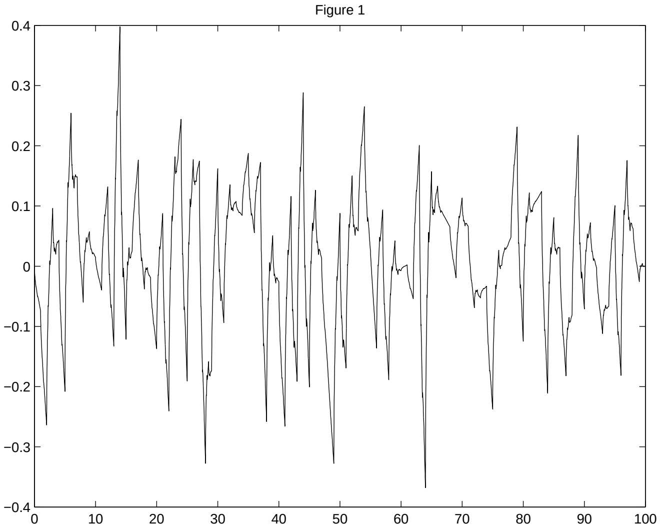
Synchronization – offset determination

Twice oversampled data $\{a_\ell = f(\alpha + \ell/2)\}$, $F = F(\beta)$ defined by

$$\sum_{m,n=0}^{1} \sum_{p} \left| \sum_{\ell} \left[a_{n+2\ell} \varphi\left(\beta + \frac{m}{2} + \ell - p\right) - a_{m+2\ell} \varphi\left(\beta + \frac{n}{2} + \ell - p\right) \right] \right|^2 .$$

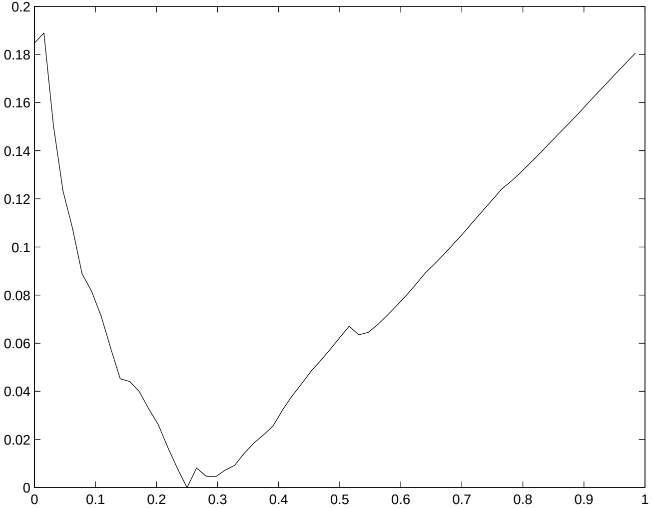
Then α is the unique zero of F.

Synchronization – offset determination



Synchronization – offset determination

Figure 2



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