**David and Me: a Chronology**

With apologies to
Michael Moore (*Roger*) & Paul Erdos
(*Ramanujan*)

Jonathan M. Borwein, FRSC
Canada Research Chair in IT
Dalhousie University
Halifax, Nova Scotia, Canada

**Special Session in Analysis**

In Honour of
**David Borwein’s 80th Birthday**

CMS, Halifax, June 14, 2004

**Topics**: Analysis, Continuation, Renormalization, Summation, Summability, $\ell^p$, $L^p$

**Qualities**: Elegance, Power, Tenacity (*EPT*)

**Birthday**: 24-03-1924  **Talk Revised**: 09-06-2004
I see some parallels between the shifts of fashion in mathematics and in music. In music, the popular new styles of jazz and rock became fashionable a little earlier than the new mathematical styles of chaos and complexity theory. Jazz and rock were long despised by classical musicians, but have emerged as art-forms more accessible than classical music to a wide section of the public. Jazz and rock are no longer to be despised as passing fads. Neither are chaos and complexity theory.

But still, classical music and classical mathematics are not dead. Mozart lives, and so does Euler. When the wheel of fashion turns once more, quantum mechanics and hard analysis will once again be in style.
1924  DB born ’29 SA ’48 UK  
1950  

1951  JB arrives  
1953  PB arrives  

1955  Bruce Shawyer arrives  

1957  \[ DB + JB + 2 \times 2 \rightarrow \text{Cheese} + £5.00 \]  

1958  DB secretary ICM GA  
1963  

3
'Twas curvig and the graphley trace
   Did max and minim in the plane;
All normless was the function space,
   And the neighbourhoods insane.

"Beware the Mathawock, my son!
   The rules that fright, the proofs that wrack!
Beware the ab ab surd, and shun
   The booleous Algebrack!"

He took his abstract sword in hand:
   Long time the symbful foe he sought -
So rested he by the Geoma tree,
   And stood awhile on nought.

And, as on voidful nought he stood,
   The Mathawock, with pi's aflame,
Transcended o'er the ringley wood,
   And factored as it came!

P,Q!  P,Q! and lambda mu
The abstract blade did lemmas hack!
He left it dead, and with its zed
   He went permuting back.

"And hast thou slain the Mathawock?
   Come to my arms, vectorious boy!
O baseless day! Arroo! - Array!"
   He tupled in his joy.

'Twas curvig, and the graphley trace
   Did max and minim in the plane;
All normless was the function space,
   And the neighbourhoods insane.

D. Borwein

Written at least 30 years ago
1968 First D&JB MAA Monthly solutions

- JB has ‘only’ class from DB (for BS)
- LB ‘makes’ JB a functional analyst

1971 DB and JB’s first joint AMS meeting. DB ‘discovers’ $\log_2$—on a Wang

1975 Victor Klee gets confused at UNB
1977 DB and JB's first joint AMS meeting as father and son professors

1983

E.J.Barbeau, Problem Editor
Canadian Math.Bulletin
Dept.Math.,Univ.of Toronto
Toronto, Ont. M5S 1A1.


For \( p > 1 \), define \( \pi_p = \frac{2}{p} \int_0^1 [t^{1-p} + (1-t)^{1-p}]^{1/p} dt \).

Compute \( \pi_2 \). If \( p^{-1} + q^{-1} = 1 \), show that \( \pi_p = \pi_q \).

Solution by D.Borwein (Univ.of Western Ontario) and D.Russell (York Univ.).

Let \( I_p := \frac{1}{p} \int_0^{\frac{1}{2}} [t^{1-p} + (1-t)^{1-p}]^{1/p} dt \) (* \( \pi_p \), \( p > 1 \), \( p^{-1} + q^{-1} = 1 \).

The value \( \pi_2 = \pi \) is trivial (substitute \( t = \sin^2 \theta \)).

Integration by parts, rearrangement, gives

\[
I_p = 1 - \frac{1}{q} \int_0^{\frac{1}{2}} (1-t)^{-p} [t^{1-p} + (1-t)^{1-p}]^{-1/q} dt .
\]

Substitute \( t = (1 + u^q)^{-1} \) in this last integral to get

\[
I_p = 1 - \int_1^{\infty} u^{-2} (1 + u^p)^{-1/q} (1 + u^q)^{-1/p} du
\]

from which the symmetry is evident.

1983

Mv ~/DB

An elegant arclength duality
Then we started writing papers together … just before DB retired

- He has not changed much—before or since!

\[ M_2(s) := \sum_{m,n \in \mathbb{Z}, (m,n) \neq 0} \frac{(-1)^{m+n}}{(m^2 + n^2)^{s/2}}, \]

\[ M_3(s) := \sum_{(m,n,p) \in \mathbb{Z}, (m,n,p) \neq 0} \frac{(-1)^{m+n+p}}{(m^2 + n^2 + p^2)^{s/2}} \]

**Theorem 1.** Converges in 2D, 3D over increasing cubes \((\text{Re } s > 0)\) \((\ell^\infty)\)

For \(s = 1\), one may sum over circles in 2D but not in 3D \((\ell^2)\)

For \(s = 1\), one may not sum over diamonds in 2D \((\ell^1)\)

• \(\otimes\) Electrochemical stability of NaCl. It upset many chemists that \(M_3(1) = 3\pi \times \sum_n o_3(n) \sech^2(\pi \sqrt{n}/2) \neq \sum_n (-1)^n r_3(n)/\sqrt{n}\) which diverges. Indeed, \(r_3(n)\) is not \(O\left(n^{1/2}\right)\)
We are pleased to announce

The David Borwein Distinguished Career Award(s)

The awards intend to recognize exceptional, broad and continued contributions to Canadian Mathematics. One or two to be awarded every even year at the summer Meeting of the Society with recipients chosen by the Advancement of Math Committee (thus, President, President-elect and Treasurer are central to the decision). A presentation about the prize and winners(s) (perhaps a short montage, audio and video clips) will occur at the banquet.

The award will be a sculpture by Helaman Ferguson entitled something like Mathematics of Salt.

Based on Benson’s formula for Madelung’s constant for NaCl, the sculpture aims to reflect David's love and appreciation of mathematics in general and classical analysis in particular. It will be polished bronze and resemble the Clay Institute Prize (1999), the SIGGRAPH Prize (2003) and ICIAM03 Memorial each designed by Ferguson.
• Hardy—agreeing with Lorenz 1879—proved

\[ M_2(s) = -4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)^s} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s} \]


• We needed general alternating series results (à la Riemann and Hardy) to explain why the physical universe is so perverse

- Looked at **Wigner sums** such as

\[
\mathcal{W}_3(s) := \sum_{(m,n,p) \in \mathbb{Z}}^{1/(m^2 + n^2 + p^2)^{s/2}} - \int \int \int_{R^3} \frac{dV}{(x^2 + y^2 + z^2)^{s/2}}
\]

- Klaus Fuchs—the mathematical physicist and East German atom spy—worked with the wrong sums (even though he attended an Edinburgh Colloquium with DB)

- Analytic continuation “rocks and rules”. But on the boundary of convergence of \( \mathcal{W}_3 \)

\[
\mathcal{W}_3(1) \neq \lim_{s \downarrow 1} \mathcal{W}_3(s)
\]

- More of the pure math driven by the previous physical and chemical lattice sums


**Theorem 2.** (Mann 1953, 1971) Let $I := [0,1]$. Let $f$ be a be continuous self-map of $I$. For every $z_0 \in I$,

$$z_{n+1} \leftarrow f \left( \frac{1}{n} \sum_{k=1}^{n} z_k \right)$$

converges to a fixed point of $f$. 

10
• Compare Banach-Picard and Sarkovsky’s theorem “period 3 implies chaos”

• We made a delicate analysis of what happens for regular summability methods. Also for Lipschitz functions

• The result holds for Hölder means $H_2$

□ It fails for some methods. What happens for $C_2$?


• JB’s unhealthy addiction—to lattice sums, analytic continuation and conditional convergence —continues and reinfects DB

- Ignorance was bliss as we met *Euler sums*:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s, t) = \sum_{n>0} \frac{H_n^t}{n^s}
\]

where

\[
H_n^t := 1 + \frac{1}{2^t} + \cdots + \frac{1}{(n - 1)^t},
\]

and we write $H_n = H_n^1$

- Euler studied these because Goldbach made a transcription error:

*Si non errasset, fecerat ille minnus*

**Theorem 3.**

\[
\sum_{n>0} \left( \frac{H_n}{n} \right)^2 = \frac{\zeta(4)}{4}
\]
• Since, via Parseval

\[ \frac{1}{2\pi} \int_{0}^{\pi} (\pi - t)^2 \log^2(2 \sin \frac{t}{2}) \, dt \]

\[ = \sum_{n=1}^{\infty} \frac{(H_n)^2}{n^2}. \]


David is a Fellow of the EMS and the Editor was the son of an old colleague

– Oh, and the analysis is pretty subtle

✓ The only DJP-Borwein paper

- **Giuga’s 1951 conjecture.** $N > 1$ is prime if (and only if)

$$\sigma_N = \sum_{k=1}^{N-1} k^{N-1} \equiv N - 1 \pmod{N}$$

- We broke the Tokyo supercomputer


- Culmination of work on lattice sums started in 1985, it relies on subtle estimates of average orders and discrepancy
1999 UWO Family Alumni Award

• We analyzed the distribution of events such that surprise—appropriately defined—is maximized. This is the Paradox of the Unintended Hanging.


• \( p \mapsto \text{Vol}_N(B_p) = (2\Gamma)^N(1+1/p)/\Gamma(1+N/p) \) used, echoing '83 CMB solution, to show

\[
\sqrt{\text{Vol}_N(B_p) \text{Vol}_N(B_q)} \geq \text{Vol}_N(B_2)
\]
Example. For the \textit{sinc} function

\[
sinc(x) := \frac{\sin(x)}{x},
\]

consider, the seven —hard to compute numerically accurately—highly oscillatory integrals

\[
I_1 := \int_0^\infty sinc(x) \, dx = \frac{\pi}{2},
\]

\[
I_2 := \int_0^\infty sinc(x)sinc\left(\frac{x}{3}\right) \, dx = \frac{\pi}{2},
\]

\[
I_3 := \int_0^\infty sinc(x)sinc\left(\frac{x}{3}\right)sinc\left(\frac{x}{5}\right) \, dx = \frac{\pi}{2},
\]

\[
\ldots
\]

\[
I_7 := \int_0^\infty sinc(x)sinc\left(\frac{x}{3}\right) \cdots sinc\left(\frac{x}{13}\right) \, dx = \frac{\pi}{2}.
\]
However, 

\[ I_8 := \int_0^\infty \text{sinc}(x) \text{sinc} \left( \frac{x}{3} \right) \cdots \text{sinc} \left( \frac{x}{15} \right) \, dx \]

\[ = \frac{46780792471344073864469}{935615849440640907310521750000} \pi \]

\[ \approx 0.499999999992646 \pi. \]

► When a researcher, using a well-known computer algebra package, checked this he—and the makers—diagnosed a “bug” in the software. Not so!

• Our analysis, via Parseval, links the integral 

\[ I_N := \int_0^\infty \text{sinc}(a_1 x) \text{sinc}(a_2 x) \cdots \text{sinc}(a_N x) \, dx \]

with the volume of the polyhedron 

\[ P_N := \{ x : \left| \sum_{k=2}^N a_k x_k \right| \leq a_1, |x_k| \leq 1, 2 \leq k \leq N \}. \]

where \( x := (x_2, x_3, \cdots, x_N) \)
If we let
\[ C_N := \{ (x_2, x_3, \cdots, x_N) : -1 \leq x_k \leq 1, 2 \leq k \leq N \}, \]
then
\[ I_N = \frac{\pi}{2a_1} \frac{\text{Vol}(P_N)}{\text{Vol}(C_N)}. \]

The value drops precisely when the constraint \( \sum_{k=2}^{N} a_k x_k \leq a_1 \) becomes active and bites the hypercube \( C_N \), as occurs when \( \sum_{k=2}^{N} a_k > a_1 \). Above, \( \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1 \), but on adding \( \frac{1}{15} \), the sum exceeds 1, the volume drops, and \( I_N = \frac{\pi}{2} \) no longer holds.

- A warning about inferring patterns from seemingly compelling computation.
• DB’s analytic power proved well suited to proving all this:

**Theorem 4.** Suppose \( a_n > 0 \). Let \( s_n := \sum_{k=1}^{n} a_k \) and
\[
\tau_n := \int_0^{\infty} \prod_{k=0}^{n} \text{sinc}(a_k x) \, dx.
\]

1. Then \( 0 < \tau_n \leq \pi/(2a_0) \), with equality if \( n = 0 \), or if \( a_0 \geq s_n \) when \( n \geq 1 \)

2. If \( a_{n+1} \leq a_0 < s_n \) with \( n \geq 1 \), then
\( 0 < \tau_{n+1} \leq \tau_n < \pi/(2a_0) \) (the hard part)

3. If \( a_0 < s_{n_0} \) with \( n_0 \geq 1 \), and \( \sum_{k=0}^{\infty} a_k^2 < \infty \), then there is \( n_1 \geq n_0 \) such that
\[
\tau_n \geq \int_0^{\infty} \prod_{k=0}^{\infty} \text{sinc}(a_k x) \, dx \geq \int_0^{\infty} \prod_{k=0}^{\infty} \text{sinc}^2(a_k x) \, dx
\]
for \( n \geq n_1 \)
Harder versions led to


BJ was 18 and DB was 77 when research took place. They had a fine few weeks together at CECM—mastering some scary determinants, and volumes of polytopes.

Theorem 5. Given $b > 2$ and not a proper power, then there is no $Q$-linear Machin-type BBP arctangent formula for $\pi$

- This first paper written in DB’s 80th year uses almost every trick in the continued fraction and special function book. One of the deepest any of us has produced, it is based on **non-symmetric word analysis**:

**Theorem 6.** *In any matrix algebra if*

\[ A_nA_{n-1} \cdots A_1 \to L \]

*with \( L \) non-singular, but perhaps only conditionally convergent, then*

\[ (A_n + B_n)(A_{n-1} + B_{n-1}) \cdots (A_1 + B_1) \to M \]

*whenever*

\[ \sum_{n} \|B_n\| < \infty \]
In prior work on continued fractions of Ramanujan, Crandall and JB needed to study the dynamical system $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left( 1 - \frac{1}{n} \right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for $n$ even, odd respectively. Which we may view as a black box.

× Numerically all one sees is $t_n \to 0$ slowly

✓ Pictorially we see significantly more:
Scaling by $\sqrt{n}$, and coloring odd and even iterates, **fine structure appears**

![Attractors diagram](image)

**The attractors for various** $|a| = |b| = 1$

★ This is now fully explained with a *lot* of work—the rate of convergence in some cases by a fine *singular-value* argument
2004 DB and JB are at work on 16th joint paper

- **Euler’s reduction formula** is

\[
\zeta(s, 1) = \frac{s}{2} \zeta(s + 1) - \frac{1}{2} \sum_{k=1}^{s-2} \zeta(k + 1) \zeta(s + 1 - k)
\]

This is equivalent to is

\[
(1) \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)} = \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{n-1} \frac{1}{(m-x)}}{n(n-x)}
\]

- DB led to slick direct proof of (1) by experimentation in Maple—many vistas opened!
Boris Stoicheff’s often enthralling biography of Gerhard Herzberg* records:

*It is not knowledge, but the act of learning, not possession but the act of getting there which generates the greatest satisfaction.* (Gauss)

Fractal similarity in Gauss’ discovery of Modular Functions

*Herzberg (1903-99) fled Germany for Saskatchewan in 1935 and won the 71 Chemistry Nobel: 12 (6 math/stats) NSERC Grantees have 1st degree pre-45.*
Judith and Jonathan Borwein invite you to a **Reception**

**TO CELEBRATE DAVID BORWEIN’S EIGHTIETH BIRTHDAY**

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**Monday June 14, 2004**

8.30 - 11.00 PM

857 Bridges Street, Halifax

(902) 422-4131 or 412-1228

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A detailed map is at


Turn on Atlantic off Tower Road across from St Mary’s University and then onto Bridges.