A Survey of Convex Functions and Sequential Convergence

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ABSTRACT — Study various properties of convex functions related to sequential convergence.

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1 Introduction

The genesis of the material in this note can be found in the first author's paper [1], where, among other things, it was shown that weak Hadamard and Fréchet differentiability coincide for continuous convex functions on Asplund spaces. This was expanded upon by the first author and M. Fabian in [2] where relationships between various forms of differentiability for convex functions were connected with sequential convergence of the related topologies in the dual space. In late 1993 whilst Simon Fitzpatrick was visiting Simon Fraser University, we collaborated with him to produce [4] which among other things connected boundedness properties of convex functions with sequential convergence of related topologies in the dual space. Motivated by these previous works, shortly thereafter we wrote the closely related articles [5, 6]. A couple of years later, S. Simons [19] produced examples of continuous convex functions whose biconjugates are not continuous and asked which classes of Banach spaces admit such examples. Once again, the answer was connected to sequential convergence in dual topologies as shown in [7] which used techniques that had been developed in our collaboration with Simon Fitzpatrick while studying boundedness properties of convex functions.

The goal of this note is to survey some of the key techniques and results involving properties of convex functions and sequential convergence of dual topologies. This will be done in the next two sections. Additionally, [7] makes it clear that there is still much to be learned concerning the extension of convex functions. In the final section of this note we will present new characterizations concerning extensions of convex functions to superspaces, and we will conclude by highlighting a couple of directions for further investigation.

We now introduce some of the notation that we will use in this article. We will work in real Banach spaces X, whose unit ball and unit sphere are denoted by B_X and S_X respectively. As in [16, p.

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59] we say a *bornology* on X is a family of bounded sets whose union is all of X, which is closed under reflection through the origin and under multiplication of positive scalars, and the union of any two members of the bornology is contained in some member of the bornology. We will denote a general bornology by β, but our attention will focus on the following three bornologies: G the *Gâteaux* bornology of all finite symmetric sets; W the *weak Hadamard* bornology of weakly compact symmetric sets; and F the *Fréchet* bornology of all bounded symmetric sets. Given a bornology β on X, we will say a real-valued function is *β-differentiable* at $x \in X$, if there is a $\phi \in X^*$ such that for each β-set S, the following limit exists uniformly for $h \in S$

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = \phi(h)$$

In particular we say f is *Gâteaux differentiable* at x if β is the Gâteaux bornology. Similarly for the weak-Hadamard and Fréchet bornologies. Also, given any bornology β on X, by τ_β we mean the topology on X^* of uniform convergence on β-sets. In particular, τ_W is the Mackey topology of uniform convergence on weakly compact sets.

2 Canonical Examples

Let $\{\phi_n\} \subset S_{X^*}$. Consider the lower semicontinuous convex functions $f : X \to \mathbb{R} \cup \{\infty\}$ that are defined as follows

$$f(x) = \sup_n \phi_n(x) - \frac{1}{n} \qquad g(x) = \sup_n (\phi_n(x))^{2n} \qquad h(x) = \sum_{n=1}^{\infty} (\phi_n(x))^{2n}$$

Proposition 2.1. *(a) f is Gâteaux differentiable at 0 if and only if $\phi_n \to_{w^*} 0$.*

(b) g and h are continuous if and only if $\phi_n \to_{w^*} 0$.

(c) f is not Fréchet differentiable at 0.

(d) g and h are not bounded on some bounded set.

Proof. Outline the proof of this. □

In fact, more generally:

Proposition 2.2. *(a) f is β-differentiable at 0 if and only if $\phi_n \to_{\tau_\beta} 0$.*

(b) g and h are bounded on β-sets if and only if $\phi_n \to_{\tau_\beta} 0$.

We refer to these or similar examples as "canonical" because they capture the essence of how convex functions can behave with respect to comparing boundedness or differentiability notions as is seen in the next result.

Theorem 2.3. *Let X be a Banach space. Then the following are equivalent.*

(a) There is a sequence $\{\phi_n\} \subset S_{X^*}$ that converges weak to 0.*

(b) There is a continuous convex function that is Gâteaux differentiable but not Fréchet differentiable at some point.

(c) There is a continuous convex function that is not bounded on bounded sets.

Proof. Both (a) implies (b) and (a) implies (c) follow from the previous proposition. Now (b) implies (a) follows from the following more general observation. Let us now show that (c) implies (a). Suppose that (c) holds, and without loss of generality, suppose $f(0) = 0$. Let $\{x_n\}$ be a bounded sequence such that $f(x_n) > n$, and let $C_n = \{x : f(x) \leq n\}$. Because $f(0) = 0$ and f is continuous, it follows that 0 is in the interior of C_n. Now by the separation theorem, choose $\phi_n \in S_{X^*}$ such that $\delta \leq \sup_{C_n} \phi_n < \phi_n(x_n)$. If $\phi_n \not\to_{w^*} 0$, then there is an $x_0 \in X$ such that $\phi_n(x_n) > K$ for infinitely many n. Thus $x_0 \notin C_n$ for infinitely many n, and so $f(x_0) = \infty$ a contradiction. $\square$

Proposition 2.4. *Let X be a Banach space. Then the following are equivalent.*

(a) Every sequence in X^* that converges to 0 uniformly on weakly compact sets also converges to 0 uniformly on bounded sets.

(b) Every sequence of lower semicontinuous convex functions that converges to a continuous affine function uniformly on weakly compact sets converges uniformly on bounded sets to the affine function.

Proof. First, (b) implies (a) is trivial. The converse implication is proved as follows. Suppose $\{f_n\}$ is a sequence of lower semicontinuous convex functions that converges uniformly on weakly compact sets to some continuous affine function $A(x)$. By replacing f_n with $f_n - A$ we may assume that $A = 0$. Now suppose f_n does not converge to 0 uniformly on bounded sets. Thus there are $\{x_n\} \subset KB_X$ and $\epsilon > 0$ so that $f(x_n) > \epsilon$ for all large n (using convexity and fact $f_n(0) \to 0$). Now let $C_n = \{x : f_n(x) \leq \epsilon\}$ and choose $\phi_n \supset S_{X^*}$ such that $\sup_{C_n} \phi_n < \phi_n(x_n) \leq K$. Now if $\phi_n \not\to_{\tau_W} 0$, then there is a weakly compact set C such that $\sup_C \phi_n > K$ for infinitely many n. Thus $\{f_n\}$ does not converge uniformly to 0 on C. Hence we conclude there is a sequence $\{\phi_n\} \subset S_{X^*}$ that converges τ_W to 0. $\square$

In fact, with natural modifications to the previous proof, one can show the following more general statement.

Proposition 2.5. *Suppose $\beta_1 \subset \beta_2$ are bornologies on a Banach space X. Then the following are equivalent.*

(a) Every sequence in X^* that converges τ_{β_1} converges τ_{β_2}.

(b) Every sequence of lower semicontinuous convex functions that converges uniformly on β_1-sets to an affine function $A(x)$ converges uniformly on β_2-sets.

Putting this all together we obtain the following

Theorem 2.6. *Let X be a Banach space with bornologies $\beta_1 \subset \beta_2$. Then the following are equivalent.*

(a) τ_{β_1} *and* τ_{β_2} *agree sequentially in* X^*.

(b) β_1*-differentiability agrees with* β_2*-differentiability for continuous convex functions.*

(c) Every continuous convex function that is bounded on β_1*-sets is bounded on* β_2*-sets.*

3 Characterizations of Various Classes of Spaces

In this section we provide a listing of various classifications of Banach spaces in terms of properties of convex functions. Many of the implications in these results can be proved using the ideas found in the previous section. In any case, we will provide references to where the proofs can be found in the literature.

First, we begin with the *Schur* spaces, that is, spaces where weak and norm convergence coincide sequentially.

Theorem 3.1. *For a Banach space* X*, the following are equivalent.*

(a) X *has the Schur property.*

(b) G-differentiability and F-differentiability coincide for weak**-lsc continuous convex functions on* X^*.

(c) G-differentiability and F-differentiability coincide for dual norms on X^*.

(d) Each continuous weak**-lsc convex function on* X^* *is bounded on bounded subset of* X^*.

(e) G-differentiability and WH-differentiability agree for Lipschitz functions on X.

(f) G-differentiability and WH-differentiability coincide for differences of Lipschitz convex functions on X.

(g) Every continuous convex function on X *is weak Hadamard directionally differentiable.*

Proof. See □

Add Heinz-Patrick-Jon result on co-finite.

Next, we consider spaces where weak* and norm convergence coincide sequentially in X^*. These are the finite dimensional spaces according to the Josefson-Nissenzweig Theorem.

Theorem 3.2. *For a Banach space* X*, the following are equivalent.*

(a) X *is finite dimensional.*

(b) Weak* and norm convergence coincide sequentially in X^*.

(c) Every continuous convex function on X is bounded on bounded subsets of X.

(d) Gâteaux and Fréchet differentiability coincide for continuous convex functions on X.

Now we consider spaces where weak* and Mackey convergence coincide sequentially in X^*. As in [3], we will say that these spaces possess the *DP* Property.*

Theorem 3.3. *For a Banach space X, the following are equivalent.*

(a) X has the DP*.

(b) Gâteaux and weak Hadamard differentiability coincide for all continuous convex functions on X.

(c) Every continuous convex function on X is bounded on weakly compact subsets of X.

(d) Pointwise convergence of lsc convex functions to continuous affine functions implies uniform convergence on weakly compact sets.

Next we consider the *Grothendieck spaces*, that is spaces X where weak* and weak convergence coincide sequentially in X^*.

Theorem 3.4. *For a Banach space X, the following are equivalent.*

(a) X is a Grothendieck space.

(b) For each continuous convex function f on X, every weak*-lower semicontinuous convex extension of f to X^{**} is continuous.

(c) For each continuous convex function f on X, f^{**} is continuous on X^{**}.

(d) For each continuous convex function f on X, there is at least one weak*-lower semicontinuous convex extension of f to X^{**} that is continuous.

(e) For each Fréchet differentiable convex function f on X, there is at least one weak*-lower semicontinuous convex extension of f to X^{**} that is continuous.

Now we look at spaces X whose duals have the Schur property. That is, weak and norm convergence coincide sequentially in X^*.

Theorem 3.5. *For a Banach space X, the following are equivalent.*

(a) X^* has the Schur property.

(b) $X \not\supset \ell_1$ and X has the Dunford-Pettis property.

(c) If $f : X \to \mathbb{R}$ is a continuous convex function such that f^{**} is continuous, then f is bounded on bounded sets.

The next result covers spaces X where weak and Mackey convergence coincide sequentially in X^*.

Theorem 3.6. *For a Banach space X, the following are equivalent.*

(a) X has the Dunford-Pettis Property.

(b) Weak and Mackey convergence coincide sequentially in X^*.

(c) G-differentiability and WH-differentiability coincide for real-valued weak*-lsc convex functions on X^*.

(d) G-differentiability and WH-differentiability coincide for dual norms on X^*.

(e) Each continuous weak*-lsc convex function on X^* is bounded on weakly compact subsets of X^*.

The next result deals with spaces X where Mackey and norm convergence coincide sequentially in X^*.

Theorem 3.7. *For a Banach space X, the following are equivalent.*

(a) $X \not\supset \ell_1$.

(b) Mackey and norm convergence coincide sequentially in X^*.

(c) Weak Hadamard and Fréchet differentiability coincide for continuous convex functions on X.

(d) Every convex function on X bounded on weakly compact sets is bounded on bounded sets.

(e) If a sequence of lsc convex functions converges uniformly on weakly compact sets to a continuous affine function, then the convergence is uniform on bounded sets.

Another class to consider?

When Weak Hadamard and Fréchet differentiability coincide for dual norms, or weak*-lsc convex functions.

When the boundedness of weak*-lsc functions on weakly compact subsets of X^* implies their boundedness on bounded sets.

4 Extension of Convex Functions Questions

For this we will consider 'generalized canonical' examples which will allow us to in some respects capture the essence of all convex functions on the space.

Let $\{\phi_{n,\alpha}\}$ a bounded net. Consider the lower semicontinuous convex functions $f : X \to \mathbb{R} \cup \{\infty\}$ that are defined as follows

$$f(x) = \sup_{n,\alpha} \phi_n(x) - a_{n,\alpha} \qquad g(x) = \sup_{n,\alpha} n(\phi_{n,\alpha}(x))^{2n}$$

where $b_n \leq a_{n,\alpha} \leq c_n$ and $b_n \downarrow 0$, $c_n \downarrow 0$. Then

Proposition 4.1. *(a) f is β-differentiable at 0 if and only if $\phi_{n,\alpha} \to_{\tau_\beta} 0$.*

(b) g is bounded on β-sets if and only if $\phi_{n,\alpha} \to_{\tau_\beta} 0$.

Lemma 4.2. *Let Y be a closed subspace of X, and suppose $f : Y \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous convex functions such that $f \leq g|_Y$. Then f can be extended to a continuous convex function $\tilde{f} : X \to \mathbb{R}$.*

Proof. Let $\tilde{g}$ be a continuous convex extension of g to X. Use favorite version of the separation theorem to prove this even though it seems intuitively clear? □

Theorem 4.3. *Suppose Y is a closed subspace of a Banach space X. Then the following are equivalent.*

(a) Every continuous convex function $f : Y \to \mathbb{R}$ can be extended to a continuous convex function $\tilde{f} : X \to \mathbb{R}$.

(b) Every bounded net $\{\phi_{n,\alpha}\} \subset Y^*$ (lexographical index) that converges weak to 0 can be extended to a bounded net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges weak* to 0.*

Proof. (a) ⇒ (b). Suppose $\{\phi_{n,\alpha}\}$ is a bounded net in Y^* that converges weak* to 0, and without loss of generality suppose $\|\phi_{n,\alpha}\| \leq 1$ for all n, α. Now define

$$f(y) = \sup(\phi_{n,\alpha}(y))^{2n}.$$

Then $f : Y \to \mathbb{R}$ is a continuous convex function, so we extend it to a continuous convex function $\tilde{f} : X \to \mathbb{R}$. Now let $C_n = \{x \in X : \tilde{f}(x) \leq 2^{2n}\}$. Then

$$B_X \subset C_n \qquad \text{and} \qquad C_n \cap Y \subset \{x : \phi_{n,\alpha}(x) \leq 2\}$$

Define the sublinear function $p_n(x) = 2\mu_{C_n}$. Then $\phi_{n,\alpha}(y) \leq p_n(y)$ for all $y \in Y$. By the Hahn-Banach theorem, extend $\phi_{n,\alpha}$ to $\tilde{\phi}_{n,\alpha}$ so that $\phi_{n,\alpha}(x) \leq p_n(x)$ for all $x \in X$. Then $\|\tilde{\phi}_{n,\alpha}\| \leq 2$. Now let us suppose that $\{\tilde{\phi}_{n,\alpha}\}$ does not converge weak* to 0. Then we can find $x_0 \in X$ such that $\phi_{n,\alpha}(x_0) > 2$ for infinitely many n. Thus $x_0 \notin C_n$ for infinitely many n, and so $f(x_0) > 2^{2n}$ for infinitely n. Thus $f(x_0) = \infty$ which contradicts the continuity of f.

(b) ⇒ (a): Suppose $f : Y \to \mathbb{R}$ is a continuous convex function. Without loss of generality we may assume $f(0) = 0$. Now define $C_n = \{y \in Y : f(y) \leq n\}$. Because f is continuous there is a $\delta > 0$ such that $\delta B_Y \subset C_n$ for each $n \in \mathbb{N}$. Thus we can write

$$C_n = \cap_\alpha \{y \in Y : \phi_{n,\alpha}(y) \leq 1\}$$

where $\|\phi_{n,\alpha}\| \leq 1/\delta$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$ (A_n an index set A with cardinality the density of Y). Also, $\{\phi_{n,\alpha}\}$ converges weak* to 0, otherwise there would be a $y_0 \in Y$ such that $\phi_{n,\alpha}(y_0) > 1$ for infinitely many n, and hence $y_0 \notin C_n$ for infinitely many n, and so $f(y_0) = \infty$. By the hypothesis of (b), $\{\phi_{n,\alpha}\}$ extends to a net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges weak* to 0. Thus define

$$g(x) = \sup n(\tilde{\phi}_{n,\alpha}(x))^{2n} + 1$$

Then $g : X \to \mathbb{R}$ is a continuous convex function. Moreover, $g(y) \geq f(y)$ for all $y \in Y$ because $g(x) \geq 0$ for all $x \in X$, and if $n - 1 \leq f(y) < n$, then $g(y) \geq n$. According to Lemma 4.2, there is a continuous convex extension $\tilde{f} : X \to \mathbb{R}$ of f. □

From the proof above, one can derive the following natural generalization of the preceding theorem.

Theorem 4.4. *Suppose Y is a closed subspace of a Banach space X. Then the following are equivalent.*

(a) Every continuous convex function $f : Y \to \mathbb{R}$ bounded on β-sets can be extended to a continuous convex function $\tilde{f} : X \to \mathbb{R}$ that is bounded on β-sets in X.

(b) Every bounded net $\{\phi_{n,\alpha}\} \subset Y^*$ (lexographical index) that converges τ_β to 0 can be extended to a bounded net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges τ_beta to 0.

When Y is a complemented subspace of X the natural method of extending $f : Y \to \mathbb{R}$ is simply $\tilde{f}(x) = f(P(x))$. Again, this doesn't extend to quasicomplements because c_0 is quasicomplemented in ℓ_∞; see [11, Theorem 11.42]. At the present, the following is the only other general extension result of which we are aware.

Proposition 4.5. *Suppose Y is a $C(K)$ Grothendieck space. Then any continuous convex function $f : Y \to \mathbb{R}$ can be extended to a continuous convex function $f : X \to \mathbb{R}$ where X is any superspace of Y.*

Proof. Write $Y \subset X$. Then $Y^{**} \cong Y^{\perp\perp} \subset X^{**}$. According to [7, Theorem 2.1], f can be extended to a continuous convex function on Y^{**}. Now, Y^{**} as the bidual of a $C(K)$ space is isomorphic to a $C(K)$ spaces where K is compact Stonean; see [18, p. 121]. Therefore, Y^{**} is complemented by a norm one projection in any superspace; see [18, Theorem 7.10, p. 110]. Hence f can be extended to $Y^{\perp\perp}$ and then to X^{**} which contains X. □

Observe that the previous proposition doesn't work for general $C(K)$ spaces, e.g. $c_0 \subset \ell_\infty$, and doesn't work for reflexive Grothendieck, e.g. $\ell_2 \subset \ell_\infty$. More significantly, using some very deep results in Banach space theory one can conclude that the above proposition applies to some noncomplemented cases.

Remark 4.6. *There are Grothendieck $C(K)$ spaces that are not complemented in every superspace.*

Proof. Consider X as Haydon's Grothendieck $C(K)$ space that does not contain ℓ_∞ [13]. Because $X \not\supset \ell_\infty$, X is not complemented in every superspace according to the work of Rosenthal [17], see also notes in [10, p. 178]. □

Similarly, I think we can consider extensions that preserve a given point of differentiability. Again, recall examples of $f : c_0 \to \ell_\infty$ where Gâteaux differentiability cannot be preserved upon extension of f to ℓ_∞.

Theorem 4.7. *Suppose Y is a closed subspace of a Banach space X. Then the following are equivalent.*

(a) Every Lipschitz convex function $f : Y \to \mathbb{R}$ that is Gâteaux differentiable at some $y_0 \in Y$ can be extended to a Lipschitz convex function $\tilde{f} : X \to \mathbb{R}$ that is Gâteaux differentiable at y_0.

(b) Every bounded net $\{\phi_{n,\alpha}\} \subset Y^*$ (lexographical index) that converges weak to 0 can be extended to a bounded net $\{\tilde{\phi}_{n,\alpha}\} \subset X^*$ that converges weak* to 0.*

Proof. (proposed idea)

(a) $\Rightarrow$ (b): Let $\{\phi_{n,\alpha}\} \subset Y^*$ be a bounded net that converges weak* to 0. Define $f(x) = \sup \phi_{n,\alpha}(x) - \frac{1}{n}$. Extend f preserving Gâteaux differentiability at 0. Use extended function to extend the $\phi_{n,\alpha}$.

(b) $\Rightarrow$ (a): Need only to consider case where $f'(0) = 0$. Write $f(y) = \sup\{\phi_u(y) - a_u\}$ where $\phi_u \in \partial f(u)$. Rewrite this as

$$f(y) = \sup\{\phi_{n,\alpha}(y) - a_{n,\alpha}\}$$

where indexing is done so that $\frac{1}{n} \le a_{n,\alpha} < \frac{1}{n-1}$. (will need to do something for case when $a_u = 0$ write as $a_{u,n} \to 0$ for example). Now Gâteaux differentiability of f at 0 forces $\phi_{n,\alpha} \to_{w^*} 0$. Then extend $\phi_{n,\alpha}$ as in (b) and use this to extend f. □

Note, also, we may be able to write every f as needed in Theorem 4.4 as a 'generalized canonical' example rather than just as one dominated by a generalized canonical example. This would be attempted by normalizing subgradients–recall Simon Fitzpatrick's original method of getting a JN-sequence from a convex function unbounded on a bounded set and then playing with the indexing—just as one needs to play with it in the above. While this doesn't explicitly help us to extend functions, it may clarify general behavior of convex functions.

Questions 1. General extension question for continuous convex functions, even on separable spaces seems to be open.

2. The question from [2] about sets that are not too big or not too small. To me, it is interesting not because of another way to produce JN-functions, but because as we are noting in our book it is directly tied to our ability to produce a lsc convex function whose subdifferential is a singleton, but such that the function is not Gâteaux differentiable.

References

[1] J.M. Borwein, Asplund spaces are 'sequentially reflexive', University of Waterloo Technical Report, 1991.

[2] J. Borwein and M. Fabian, On convex functions having points of Gateaux differentiability which are not points of Fréchet differentiability, *Canad. J. Math.* **45** (1993), 1121–1134.

[3] J. Borwein, M. Fabian and J. Vanderwerff, Characterizations of Banach spaces via convex and other locally Lipschitz functions, *Acta. Vietnamica* **22** (1997), 53–69.

[4] J. Borwein, S. Fitzpatrick and J. Vanderwerff, Examples of convex functions and classifications of normed spaces, *J. Convex Anal.* **1** (1994), 61–73.

[5] J. Borwein and J. Vanderwerff, Epigraphical and uniform convergence of convex functions, *Trans. Amer. Math. Soc.* **348** (1996), 1617–1631.

[6] J. Borwein and J. Vanderwerff, Convex functions on Banach spaces not containing ℓ_1, *Canad. Math. Bull.* **40** (1997), 10–18.

[7] J. Borwein and J. Vanderwerff, On the continuity of biconjugate convex functions, *Proc. Amer. Math. Soc.* **130** (2001), 1797–1803.

[8] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64**, 1993.

[9] J. Diestel, *Sequences and Series in Banach spaces*, Graduate Texts in Mathematics **92**, Springer–Verlag, Berlin–New York–Tokyo, 1984.

[10] J. Diestel and J.J. Uhl, Jr, *Vector Measures*, AMS Mathematical Surveys **15**, Providence, 1977.

[11] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelan and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books in Mathematics **8**, Springer-Verlag, 2001.

[12] G. Godefroy, Prolongement de fonctions convexes définies sur un espace de Banach E au bidual E'', *C. R. Acad. Sci. Paris*, **292** (1981), 371–374.

[13] R. Haydon, A non-reflexive Grothendieck space that does not contain ℓ_∞, *Israel J. Math.* **40** (1981), 65–73.

[14] R.B. Holmes, *Geometric Functional Analysis and its Applications*, Graduate Texts in Mathematics, **24**, Springer–Verlag, New York Heildelberg Berlin, 1975.

[15] P. Ørno, On J. Borwein's concept of sequentially reflexive Banach spaces, unpublished manuscript, 1992.

[16] R.R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Mathematics **1364**, Springer-Verlag, 1993.

[17] H.P. Rosenthal, On injective Banach spaces and the spaces $L_\infty(\mu)$ for finite measures μ, *Acta Math.* **124** (1970), 205–248.

[18] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer–Verlag, New York-Heidelberg-Berlin, 1974.

[19] S. Simons, Maximal monotone multifunctions of Bronsdsted–Rockafellar type, *Set-Valued Analysis* **7** (1999), 255–294.

[20] V.E. Zizler, Smooth extension of norms and complementability of subspaces, *Arch. Math.* **53** (1989), 585–589.