Why ‘Convex’?

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Abstract This polemic makes the case for a study focussing on convex functions and their structural properties. I highlight the centrality of convexity and gives a selection of salient examples and applications.

Key words: Convex Function, Convex Set, Ubiquitous, Natural.

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1 Why a book on convex functions (and not convex analysis)?

It has been said that most of number theory devolves to the Cauchy-Schwarz inequality and the only problem is deciding ‘what to Cauchy with’. In like fashion, much mathematics is tamed once one has found the right convex ‘Green’s function’. Why convex? Well, because ....

I will give the following arguments with pictures and Chris Hamilton will help dig up examples and set up the data structures.

• topology, algebra, and geometry often coincide
• ‘convex’ = ‘easy’
• ‘differentiable’ is understood throughout the sciences ‘convex’ is not
• 3D pictures are usually accurate, while 2D are not

2 Basic Principles

Lemma 2.1. The convex functions form a convex cone. Moreover
1. \( g \) is convex iff and only if \( \text{epi}(g) \) is iff \( \iota_{\text{epi} g} \) is.

2. \( g \) is convex iff \( \nabla g \) is monotone iff \( \nabla^2 g \) is positive semidefinite.

3. \( g \circ A \) and \( m \circ g \) are convex when \( g \) is convex \( A \) is affine and \( m \) is monotone increasing and convex.

4. For \( t \geq 0 \) \((x,t) \mapsto tg(x/t)\) and \((x,t) \mapsto g(xt)/t\) are convex iff \( g \) is and in the later case if \( g(0) \geq 0 \)—add lsc details.

**Theorem 2.2.** [?] Max formula for \( f'(x,h) \).

**Theorem 2.3.** [?, ?] The following are central

- Global minima and local minima coincide
- Weak and strong topologies coincide
- Lipschitz iff continuous iff bounded above
- In finite dimensions
  - relative interior always exists
  - differentiable iff unique subgradient
  - Fréchet iff Gateaux
  - finite = ‘n + 1’ (Helly, Radon, Carathéodory)

**Theorem 2.4.** [?, ?] Fenchel duality/Hahn Banach circle

**Theorem 2.5.** Krein-Milman

**Theorem 2.6.** [?] Lyapunov Convexity/Aumann Integral

### 3 Pure Mathematical Applications

**Example 3.1.** (Optimization) Lagrange Multipliers and partially finite programmes.

**Example 3.2.** (Best Approximation) Nearest Point Theorem and non-expansive mappings

**Example 3.3.** (Inequalities and Geometry) Jensen, Holder and Minkowski Inequality and Volume of \( l_p \) balls and Blaschke-Santalo Inequalities

**Example 3.4.** (Algebra) Birkhoff’s Theorem on doubly stochastic matrices, Loewner Convexity and Toeplitz-Hausdorff Theorem on numerical range for \( B(H) \).

**Example 3.5.** (Real Analysis) Bohr-Mollerup Theorem characterizing the Gamma and Beta Function

**Example 3.6.** (Statistics and Information Theory) Kullback-Leiber Divergence, Optimal Design, Fisher Information and Surprise .

**Example 3.7.** (Complex Analysis) Gauss Theorem that the roots of the derivative lie inside the convex hull of the zeroes.

**Example 3.8.** (Combinatorics) Coupon Cutting Problem and Levy-Steiner Theorem: ask Richard Nowakowski
4 More Applied Applications

Example 4.1. (Complexity) Interior Point Methods

Example 4.2. (Calculus of Variations and Control) Lyapunov convexity Theorem and application to non-convex semi-finite programs. Hidden convexity in Brachistochrone problem $x \to \sqrt{x}$ makes the integrand jointly convex.

Example 4.3. (Spectral Analysis) Davis-Lewis Theorem characterizing convex functions of eigenvalues

Example 4.4. (Game Theory) Von Neumann and Sion Minimax Theorem

Example 4.5. (Numerical Analysis) Quasi-Newton Updates as variational objects

Example 4.6. (Entropy) Maximum Entropy Method and Magnetic Resonancing

Example 4.7. (Statistics) DAD Theorem for actuarial tables

Example 4.8. (Traveling Salesman) SDP Relaxation

Example 4.9. (Computational Geometry) ???

4.1 Other fields need mentioning

1. Physics: Snell
2. Statistical Mechanics: ask Bob Phelps
3. Electrical Engineering: Kirchhoff/check Boyd
4. Economics: ask Alejandro Iofre
5. Biology: ask Leah Edelstein
6. Chemistry: ask Gren Patey

References


