Solutions to Selected Exercises in Chapter 4

Exercises from Section 4.1

4.1.1. (a) Suppose \( f : X \to [-\infty,+\infty] \) is lower semicontinuous. Let \( \alpha \in \mathbb{R} \), and and \( S := \{ x : f(x) \leq \alpha \} \). Suppose \( (x_n) \subset S \) and \( x_n \to \bar{x} \). Then \( \liminf f(x_n) \geq f(\bar{x}) \) because \( f \) is lower semicontinuous. Therefore, \( \bar{x} \in S \) and \( S \) is closed.

Now suppose each lower level set of \( f \) is closed. Let \( (x_n,t_n) \in \text{epi} \ f \) and suppose \( (x_n,t_n) \to (\bar{x},\bar{t}) \).

For any \( \epsilon > 0 \), the set \( \{ x : f(x) \leq \bar{t} + \epsilon \} \) is closed. Let \( x_n \in S \) for all sufficiently large \( n \), therefore \( \bar{x} \in S \). Hence \( f(\bar{x}) \leq \bar{t} + \epsilon \). Because \( \epsilon > 0 \) was arbitrary, this implies \( f(\bar{x}) \leq \bar{t} \), and so \( (\bar{x},\bar{t}) \in \text{epi} \ f \), and we conclude that \( \text{epi} \ f \) is closed.

Finally, suppose \( \text{epi} \ f \) is closed. Suppose \( x_n \to \bar{x} \). If \( \liminf f(x_n) = \infty \), then clearly \( f(\bar{x}) \leq \liminf f(x_n) \). Thus we suppose \( \liminf f(x_n) < \infty \). Let \( \alpha \) be any real number such that \( \liminf f(x_n) < \alpha \). Then \( (x_n,\alpha) \in \text{epi} \ f \) for all sufficiently large \( n \). Because \( \text{epi} \ f \) is closed, this implies \( (\bar{x},\alpha) \in \text{epi} \ f \), and \( f(\bar{x}) \leq \alpha \). Hence \( \liminf f(x_n) \leq f(\bar{x}) \). Thus \( f \) is lower semicontinuous.

(b) Suppose \( \text{epi} \ f \) is weakly closed, and suppose \( x_\alpha \to x \) weakly. In the case \( \liminf f(x_\alpha) = \infty \), it is clear \( f(x) \leq \liminf f(x_\alpha) \). So suppose \( \liminf f(x_\alpha) < \infty \). Let \( \alpha \in \mathbb{R} \) be such that \( \liminf f(x_\alpha) < \alpha \). Then \( (x_\alpha,\alpha) \) is eventually in \( \text{epi} \ f \). It then follows that \( (x,\alpha) \in \text{epi} \ f \) and \( f(x) \leq \alpha \). It follows that \( f \) is weakly-lower semicontinuous.

Conversely, suppose \( f \) is weakly-lower semicontinuous, and suppose \( (x_\lambda,t_\lambda) \in \text{epi} \ f \) satisfies \( (x_\lambda,t_\lambda) \to (\bar{x},\bar{t}) \) weakly. Then \( \liminf f(x_\lambda) \leq \liminf t_\lambda = \bar{t} \). Thus \( (\bar{x},\bar{t}) \in \text{epi} \ f \), and we deduce that \( \text{epi} \ f \) is weakly closed.

The epigraph of a convex function is convex. Therefore \( \text{epi} \ f \) is weakly closed if and only if it is (norm) closed. Consequently, by the previous parts of this exercise, \( f \) is weakly-lower semicontinuous if and only if it is lower semicontinuous.

(c) This is similar to the corresponding parts of (a) and (b).

4.1.2. (a) Clearly \( \text{cl} \ f \leq f \) and \( \text{cl} \ f \) is lower semicontinuous since it is closed. Now suppose \( g \leq f \), and \( g \) is lower semicontinuous. Then \( \text{epi} \ f \subset \text{cl} \text{epi} \ f \subset \text{epi} \ g \). Thus \( g \leq \text{cl} \ f \). Consequently, \( \text{cl} \ f = \sup \{ g : g \text{ is lower semicontinuous and } g \leq f \} \).

For any \( x_0 \in X \), it follows that

\[
\text{cl} \ f(x_0) \leq \lim_{\delta \downarrow 0} \inf_{\|x-x_0\|<\delta} f(x)
\]

(b) For (i), consider \( f(0) := -\infty \) and \( f(t) := +\infty \) when \( t \neq 0 \). This function has no continuous affine minorant. For (ii), consider \( f \) such that \( f(x) := 0 \) for all \( x \) in a dense convex subset of \( B_X \) and \( f(x) = +\infty \) otherwise. Then \( \text{cl} \ f = \delta_{B_X} \), but the suprema of affine minorants of \( f \) is identically 0 on \( X \). Part (iii) is true by the epi-point separation theorem (4.1.21).
(c) Let \( c_{00} \) be the nonclosed linear subspace of all finitely supported sequences in \( c_0 \). Consider, for example, \( f : c_0 \to (-\infty, +\infty] \) for which \( f(x) := \sum x_i \) if \( x \in c_{00} \), and \( f(x) := +\infty \) otherwise. It is not hard to show that \( \text{cl} f \equiv -\infty \).

4.1.3. See the solution to Exercise 2.1.5 which works in any normed linear space, and moreover shows the domain of a convex function is convex. \( \square \)

4.1.4. Suppose \( f : X \to (-\infty, +\infty] \) is convex. If \( f \equiv +\infty \), then \( \text{epi} f = \emptyset \) is convex. Otherwise, let \((x,t), (y,s) \in \text{epi} f \). Then for \( 0 \leq \lambda \leq 1 \) we have

\[
 f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda t + (1-\lambda)s.
\]

Therefore, \( \lambda (x,t) + (1-\lambda)(y,s) \in \text{epi} f \) as desired.

Conversely, suppose \( \text{epi} f \) is convex, if \( \text{epi} f = \emptyset \) then \( f \equiv +\infty \) is convex. Otherwise, suppose \( x, y \in \text{dom} f \). Then \((x,f(x)), (y,f(y)) \in \text{epi} f \). Thus for \( 0 \leq \lambda \leq 1 \), we have \( \lambda (x,f(x)) + (1-\lambda)(y,f(y)) \in \text{epi} f \). This implies

\[
 f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),
\]

as desired. \( \square \)

4.1.5. (a) Let \( S := \bigcap_{\alpha} C_\alpha \), where each \( C_\alpha \) is convex. Suppose \( x, y \in S \) and \( 0 \leq \lambda \leq 1 \). Then \( x, y \in C_\alpha \) for each \( \alpha \). Therefore, \( \lambda x + (1-\lambda)y \in C_\alpha \) for each \( \alpha \) because \( C_\alpha \) is convex. Then \( \lambda x + (1-\lambda)y \in S \), and thus \( S \) is convex.

(b) Given \( f := \sup\{f_\alpha : \alpha \in A\} \), we have \( \text{epi} f = \bigcap_{\alpha \in A} \text{epi} f_\alpha \). Therefore \( \text{epi} f \) is convex (resp. closed) if each \( f_\alpha \) is convex (resp. closed). Thus \( f \) is convex (resp. closed) when each \( f_\alpha \) is convex (resp. closed).

(c) Let \( g := \limsup f_\alpha \). Fix \( x, y \) and \( 0 \leq \lambda \leq 1 \). Then

\[
 g(\lambda x + (1-\lambda)y) = \limsup_{\alpha} f_\alpha(\lambda x + (1-\lambda)y) \\
 \leq \limsup_{\alpha} [\lambda f_\alpha(x) + (1-\lambda)f_\alpha(y)] \\
 \leq \lambda \limsup_{\alpha} f_\alpha(x) + (1-\lambda) \limsup_{\alpha} f_\alpha(y) \\
 = \lambda g(x) + (1-\lambda)g(y).
\]

The proof for limits is similar, but easier.

(d) Consider \( f(t) := t \) and \( g(t) := -t \). Then \( \min\{f, g\}(t) = -|t| \) is not convex. \( \square \)

4.1.8. A proof for the convexity of \( d_C \) (that works in any normed space) can be found in Fact 2.1.6. Clearly the epigraph of \( \delta_C \) is convex, so \( \delta_C \) is convex. \( \square \)

4.1.9. For \( c_0 \) let \( A = c_0^{+} \), that is \( A := \{(x_i) \in c_0 : x_i \geq 0, \ i \in \mathbb{N}\} \). Let \( B \) be the ray \( \{tx - y : t \geq 0\} \) where \( x = (4^{-i}) \) and \( y = (2^{-i}) \). Fix \( t \geq 0 \). For \( i > 2^t \), we have \( 4^{-i} - 2^{-i} < 0 \). Therefore, \( A \cap B = \emptyset \).

Now suppose \( \phi \in \ell_1 \setminus \{0\} \) is bounded below on \( A \), say \( \phi = (s_i) \). Then \( s_i \geq 0 \) for all \( i \) (otherwise if \( s_{i_0} < 0 \), then we choose \( ne_{i_0} \in c_0^{+} \) and \( \phi(ne_{i_0}) \to -\infty \) as \( n \to \infty \). Since \( \phi \neq 0 \), we know \( s_i > 0 \) for some \( i \). Therefore, \( \phi(x) > 0 \). Consequently,

\[
 \lim_{i \to \infty} \phi(tx - y) = t\phi(x) - \phi(y) = \infty.
\]
Thus $\phi$ cannot separate $A$ and $B$. The proof in the case of $\ell_p$ is similar. 

4.1.11. (a) Clearly a separating family is total since it can separate $x$ from 0 when $x \neq 0$. Conversely, if $S$ is total, and $x \neq y$, then we choose $\phi \in S$ so that $\phi(x - y) \neq 0$, and then $\phi(x) \neq \phi(y)$ so $S$ is separating.

(b) Let $\{h_n\}_{n=1}^{\infty}$ be norm dense in $S_X$. Choose $\phi_n \in S_X^*$ so that $\phi_n(x_n) = 1$. Now let $x \in X \setminus \{0\}$. Let $h = x/\|x\|$. Now choose $h_n$ such that $\|h_n - h\| < 1/2$. Then $\phi_n(h) > 1/2$, and so $\phi_n(x) > 0$.

(c) The standard basis $\{e_n\}_{n=1}^{\infty}$ of $\ell_1$ will work. In general, any countable norm-dense set in $B_X$ will work as a separating family for $X^*$. Indeed, if $\phi \in X^*$ vanishes on a norm-dense set in $B_X$, it vanishes on $B_X$ and must be the zero-functional.

4.1.13. Alaoglu’s theorem (4.1.6) ensures $B_{X^*}$ is weak*-compact. Thus, the weak*-closure of $B_X$ is contained in $B_{X^*}$. Suppose the containment is proper, say $x^{**} \in B_{X^*} \setminus C$ where $C$ is the weak*-closure of $B_X$. Notice that $C$ is convex. Thus, according to the weak*-separation theorem (4.1.22), there exists $\phi \in B_{X^*}$ such that $\phi(x^{**}) > \sup_{C} \phi = 1$ which is contraction. Thus $C = B_{X^{**}}$.

4.1.18. Observe that $f$ is convex as a maximum of two convex functions. Observe that $\text{dom } \partial f = \{(x_1,x_1) : x_1 > 0\} \cup \{(0,x_2) : |x_2| > 1\}$ which is not convex. Indeed, $f$ is continuous at each point in the first set of the union (which is the interior of the domain of $f$), while when $|x_2| \leq 1$, we have $\lim_{t \to 0^+} [f(0,x_2 + t) - f(0,x_2)]/t = -\infty$ and so $\partial f$ is empty at those points, while if $|x_2| > 1$, $\partial f(0,x_0) = \{(s,x_2/|x_2| : s \leq 0\}$.

4.1.19. (a) Let $x,y \in U$, $0 \leq \lambda \leq 1$ and let $\phi \in \partial f(\lambda x + (1-\lambda)y)$. Now let $a := f(\lambda x + (1-\lambda)y) - \phi(\lambda x + (1-\lambda)y)$. Then the subdifferential inequality implies $a + \phi(u) \leq f(u)$ for all $u \in U$, and so

$$f(\lambda x + (1-\lambda)y) = \phi(\lambda x + (1-\lambda)y) + a$$

$$= \lambda(\phi(x) + a) + (1-\lambda)(\phi(y) + a)$$

$$\leq \lambda f(x) + (1-\lambda)f(y)$$

as desired.

(b) In the case $X$ is finite-dimensional, the converse to (a) is true. Indeed, $f$ is continuous on $U$ by Theorem 2.1.12 and hence the max formula (2.1.19) ensure $\partial f(x) \neq \emptyset$ for all $x \in U$. However, when $X$ is infinite dimensional, it fails. Indeed, we let $f$ be a discontinuous linear functional on $X$. Then $\partial f(x) = \emptyset$ because $f$ is not bounded below on any open subset of $X$.

4.1.20. (b) Suppose $0 \in \text{int } C$, it is clear that $0 \in \text{core } C$. Now suppose $0 \in \text{core } C$. Given $h \in X$, there exists $\delta > 0$ such that $0 + t h \in C$ for $0 \leq t \leq \delta$. Then for $t = 1/n$ where $1/n < \delta$ we have $\frac{1}{n} h \in C$, and so $h \in nC$. Thus $\bigcup_{n=1}^{\infty} nC = X$.

Now suppose $C$ is absorbing. By the Baire category theorem, $nC$ has nonempty interior for some $n$, thus $u + rB_X \subset nC$ for some $u \in X$ and $r > 0$. Because $0 \in C$, this implies $t(u + rB_X) \subset C$ for $0 \leq t \leq 1/n$. Because $C$ is absorbing (and convex with $0 \in C$) $s(-u) \in C$ for $0 \leq s \leq 1/m$ for some $m \in N$. Now choose $k = \max\{n,m\}$. Then $\frac{1}{k}(-u) \in C$ and $\frac{1}{k}(u + rB_X) \subset C$. By the convexity of $C$,

$$\frac{r}{2k}B_X = \frac{1}{2}k(-u) + \frac{1}{2}k(u + rB_X) \subset C$$
and so \(0 \in \text{int } C\).

(c) This is straightforward. Clearly \((1/n,1/n^3) \not\in C\) for \(n = 1,2,\ldots\) and so \((0,0) \not\in \text{int } C\). We now show \((0,0) \in \text{core } C\). Indeed, let \((x,y) \in \mathbb{R}^2\). If \(y = 0\), then \(tx,y) \in C\) for all \(t \geq 0\). If \(x = 0\), again \(t(x,y) \in C\) for all \(t \geq 0\). Thus suppose \(x \neq 0\), \(y \neq 0\) and so \(y = kx\) for some \(k \neq 0\). For \(|t| \leq |k|x|\) we have

\[
t(x,y) = t(x,kx) = (tx,tkx) \in C \quad \text{since } |tkx| \geq t^2x^2.
\]

Thus, \((0,0) \in \text{core } C\). \(\square\)

4.1.21. (a) As suggested, define \(\Lambda : c_0 \rightarrow \mathbb{R}\) by \(\Lambda(x) := \sum x_i\) where \(x = (x_i)\). Then \(\Lambda\) is a linear functional, and we consider \(u_n \in c_0\), such that the \(i\)-th coordinate of \(u_n\) is \(1/n\) for \(i = 1,2,\ldots,n\), and 0 otherwise. Then \(u_n \rightarrow 0\), but \(\Lambda(u_n) = 1\) and \(\Lambda(0) = 0\) and so \(\Lambda\) is not continuous.

(b) Let \(\{e_n\}_{n=1}^{\infty}\) denote the standard coordinate (Schauder) basis of \(c_0\). Extend this to an algebraic (Hamel) basis of \(c_0\), say \(\{b_\gamma\}_{\gamma \in \Gamma}\). Then an extension of of \(\Lambda\) to \(c_0\) is \(\Lambda(\gamma) = 1\) when \(b_\gamma = e_n\) for some \(n \in \mathbb{N}\) and \(\Lambda(e_\gamma) = 0\) otherwise. \(\square\)

4.1.22. (a) \(\Rightarrow\) (c): Suppose \(X = \mathbb{R}^n\), with standard basis \(\{e_1,e_2,\ldots,e_n\}\), and \(\bigcup_{k \geq 1} kC = E\). Then there exist \(\delta_i > 0\) such that \(te_i \in C\) when \(|t| \leq \delta_i\). By the convexity of \(C\), \((x_1,x_2,\ldots,x_n) \in C\) whenever \(|x_i| \leq \delta/n\) where \(\delta := \min\{\delta_1,\delta_2,\ldots,\delta_n\}\). Thus \(0 \in \text{int } C\).

(c) \(\Rightarrow\) (b): Let \(f : X \rightarrow \mathbb{R}\) be a linear functional. Then \(\bigcup_{k \geq 1} kf^{-1}(-1,1) = \bigcup_{k \geq 1} f^{-1}(-k,k) = X\). By the hypothesis, \(f^{-1}(-1,1)\) has 0 in its interior. Thus there exists \(\delta > 0\) so that \(f(\delta B_X) \subset (-1,1)\). Therefore

\[
|f(x) - f(y)| = |f(x-y)| = \left|f\left(\delta \frac{x-y}{\|x-y\|}\right)\right| \cdot \frac{\|x-y\|}{\delta} \leq \frac{1}{\delta} \|x-y\|.
\]

That is, \(f\) has Lipschitz constant \(\frac{1}{\delta}\).

(b) \(\Rightarrow\) (a): Suppose \(X\) is not finite dimensional. Let \(\{e_\gamma\}_{\gamma \in \Gamma}\) be an algebraic basis of \(X\). Fix a countable set \(\{e_{\gamma_1},e_{\gamma_2},\ldots\}\) in this basis and define a linear functional \(f\) on \(X\) by setting \(f(e_{\gamma_k}) := k\|e_{\gamma_k}\|\). Then \(f(v_k) \rightarrow 1\) while \(v_k \rightarrow 0\) where \(v_k = e_{\gamma_k}/(k\|e_{\gamma_k}\|)\). \(\square\)

4.1.23. (a) Suppose \(C\) is a closed convex set, and suppose \(\bar{x} = \sum_{i=1}^{\infty} \lambda_ix_i\) where \(x_i \in C\) and \(\lambda_i \geq 0\), \(\sum \lambda_i = 1\). Fix \(x_0 \in C\), and write

\[
y_n := \sum_{i=1}^{n} \lambda_i x_i + \left(1 - \sum_{i=n+1}^{\infty} \lambda_i\right)x_0.
\]

Then \(y_n \rightarrow \bar{x}\), and so \(\bar{x} \in C\). Thus closed convex sets in Banach spaces are convex series closed.

Suppose \(U\) is an open convex set and suppose \(\bar{x} = \sum_{i=1}^{\infty} \lambda_ix_i\) where \(x_i \in U\) and \(\lambda_i \geq 0\), \(\sum \lambda_i = 1\). Suppose \(\bar{x} \not\in U\). By the separation theorem (4.1.17) we choose \(\phi \in X^*\) so that \(\phi(\bar{x}) > \phi(x)\) for all \(x \in U\). Then \(\phi(\bar{x} - x_i) > 0\) for \(i \in \mathbb{N}\), and therefore

\[
0 > \phi\left(\bar{x} - \sum_{i=1}^{\infty} \lambda_ix_i\right) = 0.
\]

This contradiction shows that open convex subsets of Banach spaces are convex series closed. It now follows easily that a \(G_\delta\)-set, is convex series closed as it is an intersection of convex series closed sets.
(b) We will prove this by induction on the dimension of $E$. Indeed, this is true of convex sets in $E$ where $\dim E = 1$ because those sets are intervals or singletons. Suppose we have shown the result for all Euclidean spaces of dimension less than $n$. Let $\dim E = n$ and suppose $C \subset E$ is a convex series closed set. Suppose $\bar{x} = \sum_{i=1}^{\infty} \lambda_i x_i$ where $x_i \in C$ and $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Suppose $\bar{x} \notin \text{ri} C$. Then we choose $\phi \in E$ such that

$$r = \langle \phi, \bar{x} \rangle \geq \langle \phi, x \rangle \quad \text{for all } x \in C.$$ 

This implies $\phi(x_i) = \phi(\bar{x})$ for all $i$. Consequently, $x_i \subset C \cap \phi^{-1}(r)$ for $i \in \mathbb{N}$. Now $C \cap \phi^{-1}(r)$ is a translate of a convex set in a Euclidean space of dimension less than $n$. By the induction hypothesis, $\bar{x} \subset C \cap \phi^{-1}(r)$, and so $\bar{x} \subset C$. The result now follows using mathematical induction.

(c) Suppose $S$ is a convex series closed set. If $\text{int} S = \emptyset$ there is nothing to do. By translating $S$, we assume $0 \in \text{int} S$. Thus, there exists $r > 0$ so that

$$0 \in rB_X \subset S \subset \frac{r}{2} B_X.$$ 

Multiplying the previous inclusions by $\frac{1}{2^i}$ for $i = 1, 2, \ldots$ we obtain

$$\frac{r}{2^i} B_X \subset \frac{1}{2^i} S + \frac{r}{2^{i+1}} B_X.$$ 

It then follows that

$$\frac{r}{2} B_X \subset \frac{1}{2} S + \frac{1}{4} S + \ldots + \frac{1}{2^i} S + \frac{r}{2^{i+1}} B_X.$$ 

Thus for any $u \in \frac{r}{2} B_X$, there exist $s_1, s_2, \ldots, s_i \in S$ such that

$$u \in \frac{1}{2} s_1 + \frac{1}{4} s_2 + \ldots + \frac{1}{2^i} s_i + \frac{r}{2^{i+1}} B_X.$$ 

Let $u_n := \sum_{i=1}^{n} 2^{-i} s_i$ it follows that $u_n \to u$. Because $S$ is convex series closed, it follows that $u \in S$. That is $\frac{r}{2} B_X \subset S$, or $0 \in \text{int} S$ as desired.

(d) It is easy to verify an open map is onto, so suppose $T : X \to Y$ is onto. By translation and dilation it suffices to show $T(B_X)$ has nonempty interior. Suppose $\bar{y} \in Y \setminus \{0\}$, and choose $\bar{x} \in X$ such that $T \bar{x} = \bar{y}$. Then for $0 \leq t \leq 1/\|\bar{x}\|$, we have

$$t \bar{y} = t T \bar{x} = T(t \bar{x}) \in T(B_X) \quad \text{because } \|t \bar{x}\| \leq 1.$$ 

Thus $0 \in \text{core} T(B_X) \subset \text{core} T(B_X)$, and so $0$ is in the interior of $T(B_X)$, and by part (c), $0 \in \text{int} T(B_X)$ as desired.

Further notes Clearly the proof of part (b) used finite dimensionality in a crucial fashion. This is not artificial: for example $c_{00}$ is not convex series closed in $c_0$. Moreover, this $c_{00} \subset c_0$ example confirms some additional property such as closed or open as used in (a) above is needed in infinite dimensional spaces.

4.1.28. Suppose $x, y \in X$ and $x \neq y$. Write $y = x + h$. By the three-slope inequality (2.1.1),

$$\frac{f(x + h) - f(x)}{\|h\|} \leq \limsup_{n \to \infty} \frac{f(x + nh) - f(x)}{n\|h\|} \leq \limsup_{n \to \infty} \frac{K \|x + nh\| + |f(x)|}{n\|h\|} \leq \limsup_{n \to \infty} \frac{K \|x\| + Kn\|h\| + |f(x)|}{n\|h\|} = K.$$
Thus $f(y) - f(x) \leq K\|y - x\|$. Similarly, $f(x) - f(y) \leq K\|x - y\|$ and we are done.

4.1.29. Consider $x := (1, 0)$. Then $r(1, 0) \notin C$ for all $r \geq 0$. Therefore, $\gamma_C(1, 0) = +\infty$. Now consider $x_n := (1, 1/n)$. Then $t(1, 1/n) \in C$ for all $t \geq \sqrt{n}$, and so $\gamma_C(1, 1/n) = 0$ for each $n$. Thus, $x_n \to x$, but $\liminf_{n \to \infty} \gamma_C(x_n) < \gamma_C(x)$.

4.1.30. For part (a), see the solution to Exercise 2.1.13. For part (b), suppose $x \in \text{core } C$, then $0 \in \text{core } C - x$. Thus for $u \in X$, we can write $u = th$ for some $t > 0$ and $h \in C - x$. Then $\gamma_{C-x}(u) \leq 1/t$ and the conclusion follows.

(c) Let $x \in X$ then $\Lambda(x) = \alpha$. If $\alpha \leq 1$, then $x \in C$, and if $\alpha > 1$, then $\alpha^{-1}x \in C$, and so $0 \in \text{core } C$. However, $\gamma_C$ is not continuous since there exists $x_n \to 0$ so that $\Lambda(x_n) > 1$, and hence $\gamma_C(x_n) \geq 1$. The closure of $C$ contains 0 in its interior, hence it cannot be contained in $\{x : \gamma_C(x) \leq 1\}$.

(d) Because $0 \in \text{core } C$, and $C$ is closed, there exists $r > 0$ so that $rB_X \subset C$. Then $\gamma_C(x) \leq \frac{1}{r}\|x\|$ for all $x \in X$. Consequently, $\gamma_C$ is Lipschitz with Lipschitz constant $1/r$ by Exercise 4.1.28. Clearly, $C \subset \{x \in X : \gamma_C(x) \leq 1\}$. On the other hand if $x \notin C$, because $C$ is closed it follows from the separation theorem that $\gamma_C(x) > 1$. Thus $C = \{x \in X : \gamma_C(x) \leq 1\}$. Then, clearly, $\{x \in X : \gamma_C(x) < 1\} \subset \text{int } C$ and $\{x \in X : \gamma_C(x) = 1\}$ is the boundary of $C$.

4.1.31. It is clear that $\sigma_{C_1}(\cdot) \leq \sigma_{C_2}(\cdot)$ when $C_1 \subset C_2$. Conversely, suppose $C_1 \not\subset C_2$. Then we choose $\bar{x} \in C_1 \setminus C_2$. By the basic separation theorem (4.1.12) there exists $x^* \in X^*$ such that $x^*(\bar{x}) > \sup_{x \in C_2} x^*$. Then $\sigma_{C_1}(x^*) > \sigma_{C_2}(x^*)$. By contraposition, $\sigma_{C_1}(\cdot) \leq \sigma_{C_2}(\cdot)$ implies $C_1 \subset C_2$.

Further notes. Let $C_1 := B_X$ and $C_2 := \text{int } B_X$. Then $\sigma_{C_1} \leq \sigma_{C_2}$ but $C_1 \not\subset C_2$, so we cannot remove the closure assumption. More drastically, let $C_2 := \text{span}\{e_1, e_2, e_3, \ldots\} \cap 2B_X$ and $C_1 := B_X$ where $X = e_0$. Then $\sigma_{C_1} < \sigma_{C_2}$ but $C_1 \not\subset C_2$. Convexity is crucial as well. Indeed, let $C_1 := B_X$ and $C_2 := 2S_X$. Then $\sigma_{C_1}(\cdot) < \sigma_{C_2}(\cdot)$, but $C_1 \not\subset C_2$.

4.1.32. Suppose $f$ and $g$ are proper convex functions on $X$ with $f \geq -g$ that additionally satisfy either

$$\text{dom } f \cap \text{cont } g \neq \emptyset$$

or $f$ and $g$ are both lower semicontinuous and

$$0 \in \text{core}(\text{dom } g - \text{dom } f).$$

Then there is an affine function $\alpha$, say $\alpha = x^* + r$ for some $x^* \in X^*$ and $r \in R$ such that $-g \leq \alpha \leq f$ on $X$. See Figure 1.2 for a sketch. If $\bar{x} \in X$ is such that $f(\bar{x}) = -g(\bar{x})$, then $-x^* \in \partial g(\bar{x})$ and $x^* \in f(\bar{x})$.

4.1.33. Let $C$ be a closed convex set. Then $x^* \in N_C(\bar{x})$ if and only if $x^* \in \partial \delta_C(\bar{x})$ if and only if $\langle x^*, y - \bar{x} \rangle \leq \delta_C(y) - \delta_C(\bar{x})$ for all $y \in X$ if and only if $\langle x^*, y - x \rangle \leq \delta_C(y) - \delta_C(x)$ for all $y \in C$ as desired.

4.1.34. Let $C$ be a nonempty closed convex set and let $K$ be a nonempty compact convex set that is disjoint from $C$. Consider the distance function $d_C$. Then $d_C$ is continuous and so it attains its minimum on $K$; say $d_C(x_0) = r > 0$ where $x_0 \in K$. Consider $S := K + \frac{r}{2}B_X$ and so $S := \{x + y : x \in K, y \in \frac{r}{2}B_X\}$. Then $S$ convex, has nonempty interior and is disjoint from
By the separation theorem (4.1.17) we find $\phi \in S_X$ such that $\sup_{\text{int} S} \phi \leq \inf_C \phi$. Now $\sup_K \phi + r/2 \leq \sup_{\text{int} S} \phi$ and so we are done.

**4.1.38.** Suppose $0 \in \partial f(x)$. Then $f(y) - f(x) \geq \langle 0, y - x \rangle = 0$, and so $f(y) \geq f(x)$ for all $y \in X$. Conversely, suppose $f$ has a local minimum at $x$. Then for each $y \in X$, there exists $\delta > 0$, depending on $y$, so that $f(x + th) \geq f(x)$ where $h := y - x$ and $0 \leq t \leq \delta$. In the case $\delta > 1$, we have $f(y) = f(x + h)$ and so $f(y) \geq f(x)$ as desired. In the case $\delta < 1$, we observe

$$f(x) \leq f(x + \delta h) = f((1 - \delta)x + \delta(x + h)) \leq (1 - \delta)f(x) + \delta f(y)$$

to conclude $f(y) \geq f(x)$, as desired.

**4.1.39.** Suppose $f : X \to [-\infty, +\infty]$ is a convex function, and let $x \in X$. If $\partial f(x)$ is empty, there is nothing to do. Suppose $\partial f(x)$ is not empty. Let $\phi, \Lambda \in \partial f(x)$ and suppose $\phi, \Lambda \to_{w^*} \phi$. Then for any $y \in X$,

$$\phi(y) - \phi(x) = \lim_\alpha \phi_\alpha(y) - \phi(x) \leq f(y) - f(x).$$

Thus $\partial f(x)$ is weak*-closed. Let $\phi, \Lambda \in \partial f(x)$. Suppose $0 \leq \lambda \leq 1$, then for $y \in X$,

$$(\lambda \phi + (1 - \lambda)\Lambda)(y) - (\lambda \phi + (1 - \lambda)\Lambda)(x) = \lambda \phi(y - x) + (1 - \lambda)\Lambda(y - x) \leq \lambda f(y - f(x)) + (1 - \lambda)[f(y) - f(x)] = f(y) - f(x).$$

This completes the proof.

**4.1.40.** Proposition 4.1.4 shows $f$ is continuous at $x_0$ if and only if it is Lipschitz in a neighborhood of $x_0$ and the proof of Proposition 4.1.25 shows this occurs if and only if $\partial f$ is bounded on a neighborhood of $x_0$. Exercise 4.1.39 ensures that $\partial f$ is weak*-closed. Because $\partial f(x)$ is bounded when $f$ is continuous at $x$, it follows from Alaoglu’s theorem that $\partial f$ is weak*-compact.

**4.1.45.** (a) Let $f(t) = -\sqrt{t}$ and $g(t) = -\sqrt{-t}$. Then $f + g = \delta(0)$, so $\partial(f + g)(0) = \mathbb{R}$. However, $\partial f(0) = \emptyset = \partial g(0)$.

(b) Let $g(x, y) = -\sqrt{x}$ and $A : \mathbb{R} \to \mathbb{R}^2$ be defined by $A(x, y) := (0, y)$. Then $g \circ A = 0$ ans so $\partial(g \circ A)(0) = \{0\}$, but $\partial g(0) = \emptyset$ and so $A^*\partial g(0) = \emptyset$.

**4.1.46.** (a) Follows because continuous convex functions are locally Lipschitz and directionally differentiable on the interior of their domains, and hence so is the difference of two such functions.

(b) Using the inner product,

$$f(x) = \frac{1}{2} \sup \{ \langle x, x \rangle - \langle x - y, x - y \rangle : y \in F \}$$

$$= \frac{1}{2} \sup \{ \langle x, x \rangle - \langle x, x \rangle + 2\langle x, y \rangle - \langle y, y \rangle : y \in F \}$$

$$= \sup \left\{ \langle x, y \rangle - \frac{1}{2}\|y\|^2 : y \in F \right\}.$$
Thus $f$ is convex as a supremum of convex functions.

(c) See Section 4.4 for properties of conjugate functions. Part (b) shows $f = \left(\frac{1}{2} \cdot \| \cdot \|^2 + \delta_F(\cdot) \right)^*$ and then using the fact that $f^{**}$ is the convex closure of $f$ we obtain

$$f^*(x) = \left(\frac{1}{2} \cdot \| \cdot \|^2 + \delta_F(\cdot) \right)^{**} = \text{conv} \left(\frac{1}{2} \cdot \| \cdot \|^2 + \delta_F(\cdot) \right) = \frac{1}{2} \cdot \| \cdot \|^2 + \delta_{\text{conv}(F)}(\cdot)$$

\[ \square \]

4.1.47. Let $g(t) := t \sin(1/t)$ for $t > 0$, and $g(t) := 0$ otherwise. Then $g$ is continuous but not of bounded variation in any neighborhood of $0$. Indeed, consider points $t_n := 1/(2n\pi - 3\pi/2)$ and $s_n := 1/(2n\pi - \pi/2)$. Then $|g(t_n) - g(s_n)| \geq 1/n\pi$. Given any neighborhood $U$ of $0$, there exists $N$ such that $s_n, t_n \in U$ for all $n > N$ and the claim follows. Now let $f := \int_0^x g(t) dt$. Then $f'(x) = g(x)$ for all $x$, and so $f$ is locally Lipschitz because $f'$ is continuous, but $f$ is not locally DC otherwise we would write $f = h - k$ where $h$ and $k$ are convex functions on a neighborhood of $0$. Then $f' = h'_+ - k'_-$ is of bounded variation because $h_+$ and $k_-$ are monotone.

For an example where $f$ is twice differentiable, proceed similarly with $g(t) := t^2 \sin(1/t^2)$ for $t > 0$ and $g(t) := 0$ otherwise. Then $g'(0) = 0$ and so $g$ is everywhere differentiable and $g$ is not of bounded variation as one can check with points $t_n := 1/\sqrt{2n\pi - 3\pi/2}$ and $s_n := 1/\sqrt{2n\pi - \pi/2}$. The remaining details are similar.

\[ \square \]

4.1.49. See solution to Exercise 2.4.20.

\[ \square \]

**Exercises from Section 4.2**

4.2.1. Let $C_n := \{ x \in c_0 : x_k = 0, k > n \}$. Now take a countable subset $\{ y_n \} \subset \bigcup_i C_i$ that is dense in $S_{c_0}$. Then $C$ contains the closed convex hull of $\{ n^{-2}y_n \}$. It follows as in Example 4.2.6, that $\partial f(0) = \{ 0 \}$. However, for $h := (n^{-1})$, the $(k + 1)$-th coordinate of $k^{-1}h = 1/(k+1)^2$, and so $f(k^{-1}h) = \infty$. Therefore

$$\lim_{k \to \infty} f(0 + \frac{1}{k}h) - f(0) = \infty$$

and $f$ is not Gâteaux differentiable at $0$.

(b) Let $h := f \| \cdot \|$. Then $h$ is continuous $h \geq 0$, $h(0) = 0$, and so

$$0 \in \partial h(0) \subset \partial f(0) = \{ 0 \}$$

and therefore the Gâteaux derivative of $h$ is $\nabla h(0) = 0$.

\[ \square \]

4.2.3. Let $\Lambda : X \to \mathbb{R}$ be a discontinuous linear functional. Let $f : X \to [0, +\infty]$ be defined by $f(x) := \Lambda^2(x)$ when $|\Lambda(x)| \leq 1$ and $f(x) := +\infty$ otherwise. We claim the Gâteaux derivative of $f$ at $0$ is $0$. Indeed, fix $h \in S_X$; then $\Lambda(x) = k_x$ where $k_x \in \mathbb{R}$, then

$$\lim_{t \to 0} \frac{f(0 + th) - f(0)}{t} = \lim_{t \to 0} \frac{t^2 k_x^2}{t} = \lim_{t \to 0} tk_x^2 = 0.$$

Because $\Lambda$ is not continuous, $\{ x : |\Lambda(x)| \leq 1 \}$ has empty interior.

\[ \square \]
Exercises from Section 4.3

4.3.1. The ‘if’ part was done, as suggested for the converse let \((x_i)\) be any Cauchy sequence in \(X\). Define the function \(f(x) := \lim_{n \to \infty} d(x_i, x)\). Then \(f\) is lower semicontinuous with inf \(X f = 0\). Let \(\epsilon \in (0, 1)\). Choose \(z \in X\) such that \(f(z) < \epsilon\). Applying Ekeland’s variational principle (4.3.1) with \(\lambda = 1\), there exists \(y \in X\) such that

\[
f(y) + \epsilon d(z, y) \leq f(z) \quad \text{and} \quad f(x) + \epsilon d(x, y) \geq f(y) \quad \text{for all} \quad x \in X.
\]

In particular, \(f(y) < \epsilon\) and \(f(x_n) + \epsilon d(x_n, y) \geq f(y)\). Taking the limit on the left-hand side of this equation, we get \(\epsilon f(y) \geq f(y)\). Because \(f(y) \geq 0\) this implies \(f(y) = 0\), that is, \(x_n \to y\). Thus \((X, d)\) is complete.

4.3.2. The contractivity of the mapping implies there can be at most one fixed point. As suggested, let \(f(x) := d(x, \phi(x))\). Then \(f\) is lower semicontinuous and bounded below on \(X\). Let \(z \in X\) be such that \(f(z) < \inf X f + \epsilon\). Applying Ekeland’s variational principle (4.3.1) to \(f\) with \(\lambda = 1\) and \(0 < \epsilon < 1 - k\), there exists \(y \in X\) such that \(f(x) + \epsilon d(x, y) \geq f(y)\) for all \(x \in X\). In particular, with \(x = \phi(y)\), we have

\[
d(\phi(y), \phi(\phi(y)) + \epsilon d(\phi(y), y)) \geq d(y, \phi(y)),
\]

or in other words, \(d(\phi(y), \phi(\phi(y)) \geq (1 - \epsilon)d(y, \phi(y)) > kd(y, \phi(y))\) which is a contradiction.

4.3.5. Let \(0 < \epsilon < 1\). Suppose \(x_0 \in S_X\) and \(\phi_0 \in S_{X^*}\) satisfy \(\phi_0(x_0) > 1 - \epsilon^2/2\). Then \(\phi_0 \in \partial \|x_0\|\). According to the Brøndsted–Rockafellar theorem (4.3.2) there exists \(x \in X\) with \(\|x - x_0\| \leq \epsilon/2\) and \(\phi \in \partial \|x\|\) with \(\|\phi - \phi_0\| \leq \epsilon\). Now let \(\bar{x} = x/\|x\|\) and \(\bar{\phi} = \phi\). Then \(\|\bar{x}\| = 1\), \(\|\bar{\phi}\| = 1\) and \(\bar{\phi}(\bar{x}) = 1\) as desired.

4.3.6. (a) Let \(0 < \epsilon < 1\). Fix \(x_0 \in \text{bnd}(C)\) and choose \(x_1 \in X \setminus C\) such that \(\|x_1 - x_0\| < \epsilon\). By the basic separation theorem (4.1.12) we choose \(x_0^* \in S_{X^*}\) so that \(\sigma_C(x_0^*) < \langle x_0^*, x_1 \rangle\). Because \(\|x_1 - x_0\| < \epsilon\), it follows that \(\langle x_0^*, x_0 \rangle > \sigma_C(x_0^*) - \epsilon\) and thus \(x_0^* \in \partial \epsilon f(x_0)\) where \(f := \delta_C\). Applying the Brøndsted–Rockafellar theorem (4.3.2) with \(\lambda = \sqrt{\epsilon}\) we obtain \(x \in \text{dom} f\) and \(x^* \in \partial f(x)\) such that \(\|x - x_0\| \leq \sqrt{\epsilon}\) and \(\|x^* - x_0^*\| \leq \sqrt{\epsilon}\). In particular, \(x^* \neq 0\), and \(\langle x^*, x \rangle = \sigma_C(x^*)\).

(b) Suppose \(\sigma_C(x_0^*) < \infty\) and let \(0 < \epsilon < \|x_0\|^2\). Choose \(x_0 \in C\) such that \(\langle x_0^*, x_0 \rangle > \sigma_C(x_0^*) - \epsilon\). Then \(x_0^* \in \partial \epsilon f(x_0)\) where \(f := \delta_C\). As in (a), we apply the Brøndsted–Rockafellar theorem (4.3.2) with \(\lambda = \sqrt{\epsilon}\) to find \(x \in \text{dom} f\) and \(x^* \in \partial f(x)\) such that \(\|x - x_0\| \leq \sqrt{\epsilon}\) and \(\|x^* - x_0^*\| \leq \sqrt{\epsilon}\). In particular, \(x^* \neq 0\), and \(\langle x^*, x \rangle = \sigma_C(x^*)\).

Exercises from Section 4.4

4.4.11. See solution to Exercise 2.3.12(a)(i).

4.4.12. Using the definition and subdifferential sum rule (4.1.19), we observe

\[
N_{C_1 \cap C_2}(x) = \partial (\delta_{C_1}(x) + \delta_{C_2}(x)) = \partial \delta_{C_1}(x) + \partial \delta_{C_2}(x) = N_{C_1}(x) + N_{C_2}(x).
\]
4.4.20. (a) Suppose $X$ is reflexive, then the properness and coercivity of $f$ imply that $K := \{x : f(x) \leq M\}$ is nonempty and bounded for some $M > 0$. Because $f$ is lower semicontinuous and convex, $K$ is weakly closed and convex. Therefore $K$ weakly compact. Now $f$ is bounded below on $K$, therefore, let $(x_n) \subset K$ be such that $f(x_n) \to \inf_K f$. According to the Eberlein-Šmulian theorem, $(x_n)$ converges weakly to $\bar{x}$ for some subsequence and some $\bar{x}$. Because $f$ is weakly lower semicontinuous, this implies $\liminf \ f(x_n) \geq f(\bar{x})$. Thus $f(\bar{x}) = \inf_K f$. Outside of $K$, $f(x) > M$, and so $f(\bar{x})$ is an absolute minimum for $f$.

Conversely, suppose $X$ is not reflexive. According to James’ theorem (4.1.27) there is a functional $f \in S_X^\ast$ that does not attain its norm on $B_X$. Then $f := f + \delta_{B_X}$ does not attain its minimum on $X$. Indeed, $\inf_x f = -1$, but there is no $x \in B_X$ such that $|\phi(x)| = 1$.

(b) For each $\phi \in X^\ast$, $f - \phi$ is supercoercive. Now $f - \phi$ attains its minimum at $\bar{x}$ for some $\bar{x} \in X$. Thus, $\phi \in \partial f(\bar{x})$.

(c) No. For example let $f$ be defined by $f(x) := (\sup_{n \in \mathbb{N}} |x(n) - 1|)^2 + \sum_{n=1}^{\infty} \frac{1}{2n} |x(n) - 1|^2$ where $x = (x(n)) \in c_0$. Then $f$ is a continuous (in fact bounded on bounded sets) and supercoercive convex function that does not attain its minimum. Indeed, $f(x) > 1$ for all $x \in c_0$, and for $x_n := (1, 1, \ldots, 1, 0, \ldots)$, we have $\lim_{n \to \infty} f(x_n) = 1$. Consequently, $0 \not\in \text{range } \partial f$. More generally, in any nonreflexive space, consider $f := \| \cdot \|^2$. Using James’ theorem (4.1.27), one can show that the subdifferential map is not onto.

4.4.21. (a) Because $f$ is Lipschitz, there exists $K > 0$ so that $\text{dom } f^* \subset KB_X^\ast$. According to the hypothesis, we choose $N \in \mathbb{N}$ so that $f_n^*(\phi) \leq f^*(\phi) + \epsilon$ for each $\phi \in \partial f(X) \subset KB_X^\ast$, and $n \geq N$. Now let $x \in X$, and let $\phi \in \partial f(x)$. Then for each $n \geq N$, one has

$$f_n^*(\phi) = \phi(x) - f(x) \geq f_n^*(\phi) - \epsilon \geq \phi(x) - f_n(x) - \epsilon.$$ 

Therefore, $f(x) - \epsilon \leq f_n(x) \leq f(x)$ for all $x \in X$, and all $n \geq N$.

(b) Let $x_0 \in X$. Given $\epsilon > 0$, and any number $\alpha < f(x_0)$ it suffices to show that there exists $N$ such that $f_n(x_0) > \alpha - \epsilon$ for all $n \geq N$. Now choose $\phi \in X^\ast$ such that $\phi(x) - \phi(x_0) \leq f(x) - \alpha$ for all $x \in X$. Then $f^*(\phi) \leq \phi(x_0) - \alpha$ and so $\phi \in \text{dom } f^*$. Now choose $N \in \mathbb{N}$ so that $f_n^*(\phi) \leq f^*(\phi) - \epsilon$ for $n \geq N$. Now, for all $n \geq N$,

$$f(x_0) - \alpha \geq f_n(\phi) \geq f_n^*(\phi) - \epsilon \geq \phi(x_0) - f_n(x_0) - \epsilon.$$ 

Therefore, $f_n(x_0) \geq \alpha - \epsilon$.

(c) We conclude that $f \circ \| \cdot \|^2$ converges uniformly (resp. pointwise) to $f$ provided that $f$ is Lipschitz (resp. lower semicontinuous proper) and convex because

$$(f \circ \| \cdot \|^2)^* = f^* + \frac{1}{2} \| \cdot \|_{\ast}^2,$$ 

converges uniformly (resp. pointwise) to $f^*$ (where $\| \cdot \|_{\ast}$ denotes the dual norm to $\| \cdot \|$).

4.4.22. We first prove Fact 4.4.4(b). Suppose $f$ is continuous at $x_0$, then there exist $\delta > 0$ and $M > 0$ so that $f(x) \leq M$ for $x \in x_0 + \delta B_X$ (Proposition 4.1.4). Now suppose $x^* \in X^\ast$ and $\|x^* - x_0\| \leq \delta$. According to Goldstine’s theorem (Exercise 4.1.13) there is a net $(x_\alpha) \subset \delta B_X$ with $x_\alpha \to w^*$ $(x^* - x_0)$. The weak* -lower semicontinuity of $f^*$ implies that

$$f^*(x^*) \leq \liminf \ f^*(x_0 + x_\alpha) = \liminf \ f(x_0 + x_\alpha) \leq M.$$
Thus \( f^{**} \) is bounded above on a neighborhood of \( x^{**} \), and thus it is continuous at \( x^{**} \) (Proposition 4.1.4).

Conversely, if \( f^{**} \) is continuous at \( x_0 \in X \), then so is \( f \) because \( f^{**}|_X = f \) (Proposition 4.4.2(a)). We now prove Fact 4.4.4(c). Suppose \( f \) is Fréchet differentiable at \( x_0 \). Let \( \epsilon > 0 \), the according to Proposition 4.2.7 there exists \( \delta > 0 \) so that

\[
  f(x_0 + h) + f(x_0 - h) - 2f(x_0) \leq \epsilon \|h\| \text{ if } \|h\| < \delta.
\]

Now suppose \( h \in X^{**} \) and \( \|h\| < \delta \). Then there exist \( h_\alpha \in X \) with \( \|h_\alpha\| = \|h\| < \delta \) and \( h_\alpha \rightharpoonup h \) by Goldstine’s theorem (Exercise 4.1.13). Using the weak* lower semicontinuity of \( f^{**} \) and the fact \( f^{**}|_X = f \) (Proposition 4.4.2(a)) we have

\[
  f^{**}(x_0 + h) + f^{**}(x_0 - h) - 2f^{**}(x_0) \\
  \leq \lim_{\alpha} f^{**}(x_0 + h_\alpha) + f^{**}(x_0 - h_\alpha) - 2f^{**}(x_0) \\
  = \lim_{\alpha} f(x_0 + h_\alpha) + f(x_0 - h_\alpha) - 2f(x_0) \leq \epsilon \|h\|.
\]

Applying Proposition 4.2.7 we conclude that \( f^{**} \) is Fréchet differentiable at \( x_0 \).

The converse, as in (b), follows from Proposition 4.4.2(a).

\[\Box\]

**4.4.23.** (a) (i) \( \Rightarrow \) (ii): If \( f \) is supercoercive, it is clear \( f - y^* \) is supercoercive for each \( y^* \in X^* \). The Moreau-Rockafellar theorem (4.4.11) implies the equivalence of (ii) and (iii).

We now show (ii) \( \Rightarrow \) (i) when \( X \) is finite dimensional. Indeed, suppose by what of contradiction there exists \( (x_n) \subset X \) such that \( \|x_n\| \to \infty \) but \( \limsup f(x_n)/\|x_n\| \leq K \) for some \( K > 0 \). Let \( u_n = x_n/\|x_n\| \), and by passing to a subsequence as necessary, we suppose \( u_n \to u \). Now let \( \phi \in S_X \) be such that \( \phi(u) = 1 \). Then let \( y^* = 3K\phi \). Now \( f - y^* \) is coercive, and for \( n \) sufficiently large \( y^*(u_n) > 2K \) and \( (f - y^*)(x_n) > 0 \) and so \( f(x_n) > y^*(x_n) \geq 2K\|x_n\| \) which is a contradiction.

For (b), define the conjugate function \( f^* \) by \( f^*(x) := \|x\|^2 + \sum_{i=1}^{\infty}(x_i)^2 \), \( f^* \) is a continuous convex function and supercoercive. However, \( f^* \) is not bounded on \( 2B_{\ell_2} \), since \( f(2e_n) = 2^{2n} \). Therefore, \( f = f^{**} \) cannot be supercoercive. For (c), consider \( f := \sqrt{\| \cdot \|} \).

\[\Box\]

**Exercises from Section 4.5**

**4.5.2.** (a) Choose \( (x_n) \subset C \) such that \( \|x_n - x\| \to d_C(x) \). Because \( (x_n) \) is bounded the Eberlein-Šmulian theorem ensures there is a subsequence \( (x_{n_k}) \) that converges weakly to \( \bar{x} \). Because \( C \) is closed and convex, it is weakly closed and so \( \bar{x} \in C \). The weak lower semicontinuity of the norm implies \( \liminf_{n \to \infty} \|x_n - x\| \geq \|\bar{x} - x\| \in P_C(x) \) as desired.

(b) Fix \( x \in X \), and suppose \( P_C(x) \) is not empty. If \( x \in C \), then \( P_C(x) = \{x\} \) is a singleton as desired. So, suppose \( x \notin C \), and suppose distinct \( y, z \in P_C(x) \). Then \( \frac{1}{2}(y + z) \in C \) by convexity, and the strict convexity of the norm implies

\[
  \left\| x - \frac{1}{2}(y + z) \right\| < \frac{1}{2}\|x - y\| + \frac{1}{2}\|x - z\| = P_C(x).
\]

This contradiction implies \( P_C(x) \) is a singleton.

Parts (a) and (b) together show that every closed convex subset of a strictly convex reflexive Banach space is a Čebyšev set.

\[\Box\]