Solutions to Selected Exercises in Chapter 6

Exercises from Section 6.2

6.2.3. Observe that $x \in \liminf A_k$ if and only if there exists $x_k \in A_k$ such that $x_k \to x$ if and only if $\limsup_{k \to \infty} d_{A_k}(x) = 0$. Also, $x \in \limsup A_k$ if and only if there exist $x_n \in A_n$ such that $x_n \to x$ if and only if $\liminf d_{A_k}(x) = 0$.

6.2.4. Suppose $\text{epi} f_k$ converges Kuratowski-Painlevé to $\text{epi} f$. First consider $f(x) = \infty$. Then $\limsup_{k \to \infty} f_k(x_k) \leq f(x)$ for any sequence $x_k \to x$. Suppose by way of contradiction that $\liminf_{k \to \infty} f_k(x_k) < \alpha < \infty$ for some sequence $x_k \to x$. Then $(x_n, \alpha) \in \text{epi} f_{n_k}$ and $(x_n, \alpha) \to (x, \alpha)$ which contradicts that $(x, \alpha) \not\in \text{epi} f$. Thus $\liminf_{k \to \infty} f_k(x_k) = \infty$ for any sequence $x_k \to x$. (In the case $\text{epi} f = \emptyset$, then $f := +\infty$, this argument may be applied for all $x$).

Suppose now $f(x) < \infty$. Consider $(x_k, t_k) \in \text{epi} f_k$ such that $(x_k, t_k) \to (x, f(x))$. Then $\limsup f_k(x_k) = \lim t_k = f(x)$. Now consider $x_k \to x$ and $\alpha := \liminf f_k(x_k)$. Then $(x_n, t_n) \to (x, \alpha)$ where $(x_n, t_n) \in \text{epi} f_{n_k}$ implies $(x, \alpha) \in \text{epi} f$. Then $f(x) \leq \alpha = \liminf f_k(x_k)$.

For the converse, we first show $\text{epi} f \subseteq \liminf \text{epi} f_n$. For this, let $(x, t) \in \text{epi} f$. Choose $x_n \to x$ such that $\limsup f_n(x_n) \leq f(x)$. Thus we choose $t_n$ so that $(x_n, t_n) \in \text{epi} f_n$ with $t_n \to t$. Then $(x, t) \in \liminf \text{epi} f_n$. Lastly, we show $\limsup \text{epi} f_n \subset \text{epi} f$. Indeed, suppose $(x_{n_k}, t_{n_k}) \in \text{epi} f_{n_k}$ such that $(x_{n_k}, t_{n_k}) \to (x, t)$. From this, we easily have a sequence $x_k \to x$ such that $\liminf f_k(x_k) \leq t$. Thus $f(x) \leq t$ and $(x, t) \in \text{epi} f$ as desired.

In the case $f := +\infty$. Let $(x_n)$ be a convergent sequence. Then $\liminf f_n(x_n) = +\infty$ and thus when $(x_n, t_n) \in \text{epi} f_n$, one has $t_n \to +\infty$ and so $(x_n, t_n)$ does not converge in $X \times \mathbb{R}$. Thus $\liminf \text{epi} f_n = \emptyset = \text{epi} f$ as desired.

Exercises from Section 6.6

6.6.5. Given the bounded subset $A \subseteq X^*$, we let

$$S(x, A, \epsilon/n) := \left\{ x^* \in A : \langle x, x^* \rangle \geq \sup_A x - \frac{\epsilon}{n} \right\}.$$

This notation for a slice is assumed but not explained in the text. Given $x_n^* \in S(x, A, \epsilon/n)$ we have $x_n^*(x) \geq p(x) - \epsilon/n$ and clearly $x_n^*(y) \leq p(y)$ for all $y \in X$. Therefore,

$$x_n^*(y) - x_n^*(x) \leq p(y) - p(x) + \epsilon/n \quad \text{for all } y \in X.$$

Thus $x_n^* \in \partial_{\epsilon/n} p(x)$. The argument for $y_n \in S(x, A, \epsilon/n)$ is identical.

\[\square\]