Computing the bifurcation points and superstable orbits of the logistic map

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Abstract

The computational complexity of the bifurcation points of the logistic map is investigated and compared to that of the superstable periodic orbits using symbolic computation and symbolic dynamics techniques. A conjectural degree for the minimal irreducible polynomial of the fifth bifurcation point \(B_5\) of the logistic map is obtained.

1 Introduction

The discovery that simple deterministic systems can exhibit a vast richness of behaviors in response to variations of initial conditions and/or control parameters, has been at the origin of an intense interdisciplinary activity during the last two decades \([1, 2, 3, 4]\). One paradigm of complex system are the unimodal maps whose study leads us to important achievements.

In particular, one can develop a whole machinery for the construction and the classification in an ordered way of the “patterns” of the superstable periodic orbits of unimodal maps of any period. This task has been accomplished, first in \([5]\) for finite limit sets and later completed in \([6]\) in order to include infinite limit sets. Further progress has been achieved in \([7, 8, 9]\).

2 The Metropolis-Stein-Stein algorithm

Let \(P\) be the pattern associated with the m-period (many different patterns may correspond to the same period). By definition, the (first) harmonic of \(P\) is the pattern, \(\hat{H}(P) = P\mu P\), where \(\mu = L\) if \(P\) contains an odd number of \(R^r\)'s, and \(\mu = R\) otherwise. The procedure can be iterated, so that one may speak of the second, third, ..., \(j\)-th harmonic, etc, hereafter denoted as \(\hat{H}_j(P)\). Metropolis, Stein, Stein (MSS) also introduce the \(\hat{H}\)-extension of a pattern \(P\) as the pattern generated by iterating the harmonic construction applied \(j\) times to \(P\), when \(j\) increases indefinitely. In the sequel, following \([7]\), we shall rather adopt the notation \(\hat{H}_\infty(P)\), for this asymptotic pattern.

MSS \([5]\) prove that, if \(P\) is allowed, \(\hat{H}(P)\) is allowed too (their theorem 1). Furthermore, in their universal ordering, we have \(P < \hat{H}(P)\) and the harmonics are adjacent, that is, no allowed sequence exists between \(P\) and \(\hat{H}(P)\).

\[\forall j, \quad P < \hat{H}(P) < \hat{H}_2^2(P) < ... < \hat{H}_\infty(P).\] (1)

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For reasons of completeness and for later use, we list hereafter the first few harmonics associated with the $2^k$ and $3 \cdot 2^k$ supercycles.

- $P \equiv R (2 \text{ period}) \rightarrow$

  \[
  \hat{H}(R) = RLR (4 \text{ period}) \rightarrow \hat{H}^2(R) = RLR RLR = RLR^3 LR (8 \text{ period}) \rightarrow \\
  \hat{H}^3(R) = RLR RRLRL R = RLR^3 LRLRL^3 LR (16 \text{ period}) \rightarrow \\
  \hat{H}^4(R) = RLR RRLRL RRLRLRLRLRLRL = RLR^3 LRLRL^3 LRLRL^3 LR (32 \text{ period})
  \]

- $P \equiv RL (3 \text{ period}) \rightarrow$

  \[
  \hat{H}(RL) = RL RRL (6 \text{ period}) \rightarrow \hat{H}^2(RL) = RLR RRLRL = RLR^2 L^2 RL (12 \text{ period}) \rightarrow \\
  \hat{H}^3(RL) = RLR RRLRLRLRLRLRL = RLR^2 L^2 RL^2 L^2 RL (24 \text{ period})
  \]

One example of unimodal maps is obtained when one considers the quadratic nonlinearity. The archetype example of quadratic nonlinearity is the logistic map. Usually, one writes the logistic map in one of the following two forms

\[ y_{n+1} = \lambda y_n(1 - y_n), \quad y_n \in (0, 1), \quad \lambda \in (0, 4) \quad (2) \]

or

\[ x_{n+1} = 1 - \mu x_n^2, \quad x_n \in (-1, 1), \quad \mu \in (0, 2). \quad (3) \]

There is an easy transformation which connects these two forms, see [4], in particular:

\[ \lambda = 1 \pm \sqrt{1 + 4\mu}. \]

Consider now the logistic map written as in (3). Then the location of the superstable periodic orbits is specified by polynomial equations in one unknown.

## 3 Word-lifting technique

There is an easy way to pass from symbolic dynamics of the superstable periodic orbits to an algebraic equation for the control parameter value, involving square roots. For the purposes of illustration we detail this construction in the case of one of the three superstable periodic orbits of period 5, namely the orbit $RLR^2$.

This periodic orbit leads to the equation of composition of functions

\[ R \circ L \circ R \circ R(0) = 1 \quad (4) \]

where

\[ R(y) = \sqrt{(1 - y)/\mu}, \quad L(y) = -\sqrt{(1 - y)/\mu}. \quad (5) \]
Equation (4), using definitions (5) gives:

\[ \sqrt{\left( 1 + \sqrt{\left( 1 - \sqrt{\frac{1 - \sqrt{\mu^{-1}}}{\mu}} \right) \mu^{-1}} \right) \mu^{-1}} = 1. \]

Upon multiplying by \( \mu \) on both sides and inserting the factor \( \mu \) under the square root sign successively, we obtain the standard form of the equation describing the periodic orbit of period 5, \( RLR^2 \):

\[ \mu = \sqrt{\mu + \sqrt{\mu - \sqrt{\mu - \sqrt{\mu}}}}. \]

Using the patterns \( \tilde{H}^2(R) \) and \( \tilde{H}^3(R) \) from the word-lifting technique above, we construct similar equations with square roots for periods 4, 8 and 16 orbits. Then we write the general form of the equations for the period \( 2^k \) orbits. In all these cases, we transform the equations involving square roots, into equations involving squares. This has the advantage that we no longer need the succession of plus-minus signs required by the word-lifting technique, but rather, a succession of minus signs exclusively.

**Superstable 4 = 2^2 periodic orbit**

\[ \mu = \sqrt{\mu + \sqrt{\mu - \sqrt{\mu}}} \]

Obliterating the radicals we obtain the following polynomial equation:

\[ \left( (\mu^2 - \mu)^2 - \mu \right)^2 - \mu \]

The degree of the above equation is \( 2^3 = 8 \) and this constitutes an upper bound for the degree of the minimal polynomial of the point specifying the superstable 4 periodic orbit. The factorization of the polynomial on the left-hand-side of equation (6) exhibits the following structure:

\[ \mu \left( \mu - 1 \right) \left( \mu^6 - 3\mu^5 + 3\mu^4 - 3\mu^3 + 3\mu^2 - 1 \right). \]

**Superstable 8 = 2^3 periodic orbit**

\[ \mu = \sqrt{\mu + \sqrt{\mu - \sqrt{\mu + \sqrt{\mu - \sqrt{\mu}}}}} \]

Obliterating the radicals we obtain the following polynomial equation:

\[ \left( \left( \left( (\mu^2 - \mu)^2 - \mu \right)^2 - \mu \right)^2 - \mu \right)^2 - \mu = 0 \]
The degree of the above equation is $2^7 = 128$ and this constitutes an upper bound for the degree of the minimal polynomial of the point specifying the superstable 8 periodic orbit. The factorization of the polynomial on the left-hand-side of equation (7) exhibits the following structure:

$$
\mu (\mu - 1) (\mu^6 - 3\mu^5 + 3\mu^4 - 3\mu^3 + 2\mu^2 + 1) P_{120}(\mu)
$$

where $P_{120}(\mu)$ stands for an irreducible polynomial of degree 120.

**Superstable $16 = 2^4$ periodic orbit**

\[
\mu = \mu + \mu - \mu + \mu + \mu - \mu + \mu - \mu + \mu + \mu - \mu + \mu - \mu
\]

Obliterating the radicals we obtain the following polynomial equation:

$$
\left(\left(\cdots \left( (\mu^2 - \mu)^2 - \mu \right)^2 - \mu \right)^2 \cdots \right)^2 - \mu = 0
$$

The degree of the above equation is $2^{15} = 32768$ and this constitutes an upper bound for the degree of the minimal polynomial of the point specifying the superstable 16 periodic orbit. The factorization of the polynomial on the left-hand-side of the above equation has been reported by Bruno Salvy (INRIA Rocquencourt, France) in November 2004. The factorization was done by Eric Schost (Ecole Polytechnique, Paris, France) using Magma compiled for 32 bit platforms on a Medicis AMD 64 3500+ machine. The factorization is available on-line at [http://www.cargo.wlu.ca/kk/expmath/](http://www.cargo.wlu.ca/kk/expmath/) in a 61.6M compressed text file. The structure of the factorization is the following:

$$
\mu (\mu - 1) (\mu^6 - 3\mu^5 + 3\mu^4 - 3\mu^3 + 2\mu^2 + 1) P_{32640}(\mu)
$$

where $P_{32640}(\mu)$ is the same polynomial of degree 120 appearing in the factorization of the superstable 8 periodic orbit and $P_{32640}(\mu)$ stands for an irreducible polynomial of degree 32640.

**Superstable $2^k$ periodic orbit** The radical equation will be composed of $2^k - 1$ radicals as specified by the word-lifting technique:

$$
\mu = \sqrt{\mu \pm \sqrt{\mu \pm \ldots \sqrt{\mu \pm \sqrt{\mu}}}} \quad (8)
$$

Obliterating the radicals we obtain the following polynomial equation with the corresponding $2^k - 1$ powers of 2:

$$
\left(\cdots \left( (\mu^2 - \mu)^2 - \mu \right)^2 - \mu \right)^2 - \mu = 0 \quad (9)
$$
The degree of the above equation is $2^{k-1}$ and this constitutes an upper bound for the degree of the minimal polynomial of the point specifying the superstable $2^k$ periodic orbit.

Even though equation (8) contains plus and minus signs according to the word-lifting technique, the equivalent equation (9) contains minus signs in front of $\mu$ exclusively. This allows us to conclude immediately that the degree of the polynomial in the left-hand side of equation (9) is equal to $2^{k-1}$.

4 Exact computation of the bifurcation points of the logistic map

We will show that the quantity $B_k(B_k - 2)$ is the natural quantity to consider, in the computation of the bifurcation point $B_k$ of the logistic map. This puts into perspective the relevant second Bailey-Broadhurst conjecture, see [11]. Moreover, it allows us to render the computation of the bifurcation point $B_4$ accessible in about half an hour.

4.1 Algebraic systems for the bifurcation points of the logistic map

The determination of the bifurcation points of the logistic map:

$$x_{n+1} = f(x_n) = \mu x_n(1 - x_n), \quad \mu \in (0, 4]$$

is specified by the following system of polynomial equations:

$$
\begin{align*}
  x_2 &= \mu x_1 (1 - x_1) \\
  x_3 &= \mu x_2 (1 - x_2) \\
  &\vdots \\
  x_n &= \mu x_{n-1} (1 - x_{n-1}) \\
  x_1 &= \mu x_n (1 - x_n)
\end{align*}
$$

(10)

together with the condition

$$\left| \prod_{i=1}^{n} \mu (1 - 2x_i) \right| = 1$$

(11)

expressing the fact that the bifurcation point is the right endpoint of the interval for $\mu$ for which the period $n$ solution is stable. Equation (11) can in turn be replaced by the two equations:

$$\prod_{i=1}^{n} \mu (1 - 2x_i) = 1, \quad \text{or} \quad \prod_{i=1}^{n} \mu (1 - 2x_i) = -1.$$

(12)

Equations (10) together with one of equations (12) constitute a system of nonlinear polynomial equations specifying fully the successive bifurcation points of the logistic map.
5 Transformation of the system

Equations (12) suggest a way to simplify the system of equations that specifies the bifurcation points of the logistic map, in two steps as follows:

- Set \( y_i = 1 - 2x_i \) for \( i = 1, \ldots, n \). This step simplifies the stability equation (12) without introducing additional complexity to equations (10).
- Set \( z_i = \mu y_i \) for \( i = 1, \ldots, n \). This step distributes the factor \( \mu^n \) of the stability equation (12) evenly into the \( n \) terms created by step 1.

The determination of the bifurcation points of the logistic map: is now specified by the equivalent system of equations:

\[
\begin{align*}
z_2^2 & - 2z_1 + u \\
z_3^2 & - 2z_2 + u \\
\vdots \\
z_n^2 & - 2z_{n-1} + u \\
z_1^2 & - 2z_0 + u
\end{align*}
\]

where \( u = 2\mu - \mu^2 = \mu(2 - \mu) \), together with the condition

\[
\left( \prod_{i=1}^{n} z_i \right) + 1 = 0.
\]

Therefore, we see that \( u = 2\mu - \mu^2 \) is a natural quantity to compute. Of course, when \( u = 2\mu - \mu^2 \) is known, \( \mu \) is obtained by a classical computational construction, called the resultant. The resultant of two polynomials of degrees \( n \) and \( m \) is defined as the determinant of an \( n + m \times n + m \) matrix formed by placing the coefficients of the polynomials in successive rows. The resultant has been used traditionally in elimination theory. See [12] for an exposition of the theory of resultants.

5.1 Rapid computation of \( B_4 \) (4th bifurcation point of the logistic map)

In [11] the authors conjecture that \( B_4 \) might satisfy a 240-degree polynomial and that \( B_4(B_4 - 2) \) might satisfy a 120-degree polynomial. These conjectures were proved recently in [10].

Taking \( n = 8 \) in equations (13) and (14) we obtain the following system:

\[
\begin{align*}
z_2^2 & - 2z_1 + u \\
z_3^2 & - 2z_2 + u \\
\vdots \\
z_8^2 & - 2z_7 + u \\
z_1^2 & - 2z_8 + u
\end{align*}
\]
where \( u = 2\mu - \mu^2 \), together with the condition
\[
\left( \prod_{i=1}^{8} z_i \right) + 1 = 0. \tag{16}
\]

The reduced lexicographical Gröbner basis of the above system has 12 elements and is computed in Magma in approximately 38 minutes. The univariate polynomial in \( u \) is of degree 144 and factorizes in 4 factors:
\[
Q_{120}(u)(u^4 + 16 u^3 + 96 u^2 + 256 u + 257)(u^8 + 2 u^4 - 64 u^3 + 320 u^2 - 512 u + 257)
\]
(\( 1^{12} + 24 u^{11} + 240 u^{10} + 1536 u^9 + 7392 u^8 + 30752 u^7 + 94192 u^6 + 294016 u^5 + 650499 u^4 + 1548296 u^3 + 4177792 u^2 + 16974593 \))

The polynomial \( Q_{120}(u) \) gives rise to the polynomial of degree 240 satisfied by \( B_4 \) via the resultant computation \( \text{Resultant}(Q_{120}(u), u - 2\mu + \mu^2, u) \) in Maple for instance. For reference, here is the previous system, used in [10].

\[
\begin{align*}
x_2 - \mu x_1 (1 - x_1) &= 0 \\
x_3 - \mu x_2 (1 - x_2) &= 0 \\
x_4 - x_3 (1 - x_3) &= 0 \\
x_5 - x_4 (1 - x_4) &= 0 \\
x_6 - x_5 (1 - x_5) &= 0 \\
x_7 - x_6 (1 - x_6) &= 0 \\
x_8 - x_7 (1 - x_7) &= 0 \\
x_9 - x_8 (1 - x_8) &= 0 \\
\mu^8 (1 - 2 x_1) (1 - 2 x_2) (1 - 2 x_3) (1 - 2 x_4) (1 - 2 x_5) (1 - 2 x_6) (1 - 2 x_7) (1 - 2 x_8) + 1 &= 0
\end{align*}
\]

6 Computational complexities of the bifurcation points and superstable periodic orbits of the logistic map

In the table below, we summarize the computational results obtained with the methods of [10] and with the word-lifting technique as laid out in the previous paragraphs.

<table>
<thead>
<tr>
<th>( k )</th>
<th>degree of minimal polynomial of ( B_k(B_k - 2) )</th>
<th>degree of minimal polynomial of ( 2^{k-1} ) superstable periodic orbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 1 \rightarrow 1 )</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>( 12 \rightarrow 6 )</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>( 240 \rightarrow 120 )</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>( ? )</td>
<td>32640</td>
</tr>
</tbody>
</table>

The above table shows that the degrees of the minimal polynomials of \( B_k(B_k - 2) \) and of the \( 2^{k-1} \) superstable periodic orbits are equal, for \( k = 2..4 \).
6.1 On the degree of the minimal polynomial of the bifurcation point $B_5$ of the logistic map

The remark of the previous paragraph on the relationship between the computational complexities of the bifurcation points and superstable periodic orbits of the logistic map, implies an upper bound on the degree of the bifurcation point $B_5$. In particular, since the $2^5$ superstable periodic orbit is given by a polynomial of degree $2^{15} = 32768$, this means that an upper bound for the degree of the minimal polynomial of $B_5$ is 32768. Moreover, since the degrees of the minimal polynomials of $B_k(B_k - 2)$ and of the $2^{k-1}$ superstable periodic orbits are equal, for $k = 2..4$, it seems reasonable to conjecture that the degree of $B_5(B_5 - 2)$ is equal to 32640.

7 Conclusion

The computation of the bifurcation points of the logistic map can be accelerated via a transformation of the original system of nonlinear polynomial equations. This transformation exemplifies the fact that $B_k(B_k - 2)$ is a natural quantity to compute, since it possesses a lower degree minimal polynomial than $B_k$. The computation of the superstable periodic orbits of the logistic map via the world-lifting technique, reveals the fact that the degrees of the corresponding minimal polynomials are equal to the degrees of the minimal polynomials of the bifurcation points of the logistic map. This has the important consequence that the computational complexities of these two sequences of points are essentially the same. Finally, we conjecture the degree of the minimal polynomial of $B_5$, the fifth bifurcation point of the logistic map.

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