On Rockafellar’s conjecture on the maximality of the sum of two maximal monotone operators in general Banach spaces

Jonathan M. Borwein * Radu I. Boț † Ernő R. Csetnek ‡
Andrew C. Eberhard §
February 25, 2008

Abstract

By using two recently published results concerning monotone operators we prove the maximality of the sum of two maximal monotone operators in general Banach spaces under a weak regularity condition. This provides an affirmative answer to Rockafellar’s celebrated conjecture stated in 1970.

Key Words. maximal monotone operator, Fitzpatrick function, representative function

AMS subject classification. 47H05, 46N10, 42A50

1 Preliminaries

Let $X$ be a nonzero Banach space and $X^*$ its topological dual space. By $\langle \cdot, \cdot \rangle$ we denote the duality products in both $X \times X^*$ and $X^* \times X^{**}$, i.e. for $x \in X$, $x^* \in X^*$ and $x^{**} \in X^{**}$ we have $\langle x, x^* \rangle := x^*(x)$ and $\langle x^*, x^{**} \rangle := x^{**}(x^*)$, respectively. The canonical embedding of $X$ into $X^{**}$ is defined by $\hat{\cdot} : X \to X^{**}$, $\langle x^*, \hat{x} \rangle := \langle x, x^* \rangle$ for all $x \in X$ and $x^* \in X^*$.

A set-valued operator $S : X \rightrightarrows X^*$ is said to be monotone if $\langle y - x, y^* - x^* \rangle \geq 0$, whenever $x^* \in S(x)$ and $y^* \in S(y)$. We denote by $G(S) = \{(x, x^*) : x^* \in S(x)\} \subseteq X \times X^*$ the graph and by $D(S) = \{x \in X : S(x) \neq \emptyset\}$ the domain of $S$, respectively.

The monotone operator $S$ is called maximal monotone if $G(S)$ is not properly contained in the graph of any other monotone operator $S' : X \rightrightarrows X^*$. The classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function (see [17]). However, there exist maximal monotone operators which are not subdifferentials (see [19]).

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*Faculty of Computer Science, Dalhousie University, Halifax, Nova Scotia, Canada, e-mail adress: jborwein@cs.dal.ca
†Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail adress: radu.bot@mathematik.tu-chemnitz.de
‡Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail adress: robert.csetnek@mathematik.tu-chemnitz.de. Research supported by a Graduate Fellowship of the Free State Saxony, Germany
§Department of Mathematics, Royal Melbourne University of Technology, Melbourne, Australia 3001, e-mail adress: andy.eb@rmit.edu.au
The sum of two maximal monotone operators is monotone, but not always maximal monotone. Rockafellar proved in [18] that, whenever $X$ is a reflexive Banach space and $S, T : X \rightharpoonup X^*$ are two maximal monotone operators such that $\text{int}(D(S)) \cap D(T) \neq \emptyset$, then $S + T$ is maximal monotone.

In the literature different regularity conditions, weaker than the one in [18], have been given for guaranteeing the maximality of $S + T$, but in reflexive Banach spaces (see, for instance, [1, 2, 4–6, 8, 14–16, 20, 25]). On the other hand, in the last years, an increasing number of characterizations of the maximality of monotone operators as well as different sufficient conditions for the maximality of the sum of two maximal monotone operators in general Banach spaces have been given (see, for instance, [3, 13, 19, 22–24]). The consequence of this intensive research is a number of strong results on which the main statement of this paper is based. In this paper we give a weak regularity conditions which guarantees the maximality of the sum of two maximal monotone operators in general Banach spaces. This result implies that Rockafellar’s result is valid also in this more general framework.

In order to make the paper self-contained we introduce some preliminary notions and results. For a subset $C$ of $X$ we denote by $\text{int}(C)$, $\text{cl}(C)$, $\text{co}(C)$ and $\text{core}(C)$ its interior, closure, convex hull and algebraic interior, respectively. Let us note that if $C$ is a convex set, then an element $x \in X$ belongs to $\text{core}(C)$ if and only if $\bigcup_{\lambda > 0} \lambda (C - x) = X$.

The normal cone of $C$ at $x \in C$ is defined as $N_C(x) = \{ x^* \in X^*: \langle y - x, x^* \rangle \leq 0 \text{ for all } y \in C \}$. If $x \notin C$, we take $N_C(x) = \emptyset$. We also consider the indicator function of the set $C$, denoted by $\delta_C$, which is zero for $x \in X$ and $+\infty$ otherwise.

For a function $f : X \to \mathbb{R}$ we denote by $\text{dom} f = \{ x \in X : f(x) < +\infty \}$ its domain and call $f$ proper if $\text{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The Fenchel-Moreau conjugate of $f$ is the function $f^*: X^* \to \mathbb{R}$ defined by $f^*(x^*) = \sup_{x \in X} \{ \langle x, x^* \rangle - f(x) \}$ for all $x^* \in X^*$. Having a function $h : X \times X^* \to \mathbb{R}$ we denote by $\widehat{h}^*: X^* \times X^* \to \mathbb{R}$ its conjugate function and by $h^*: X^* \times X \to \mathbb{R}$, $h^*(x^*, x) = \widehat{h}^*(x^*, \hat{x})$ its canonical embedding to $X^* \times X$.

Having $f, g : X \to \mathbb{R}$ two proper functions we consider their infimal convolution, namely the function denoted by $f \square g : X \to \mathbb{R}$, $f \square g(x) = \inf_{u \in X} \{ f(u) + g(x - u) \}$ for all $x \in X$. For a function $f : A \times B \to \mathbb{R}$, where $A$ and $B$ are nonempty sets, we denote by $f^\top$ the transpose of $f$, namely the function $f^\top : B \times A \to \mathbb{R}, f^\top(b, a) = f(a, b)$ for all $(b, a) \in B \times A$. We consider also the projection operator $\text{pr}_A : A \times B \to A$, $\text{pr}_A(a, b) = a$ for all $(a, b) \in A \times B$. When an infimum or a supremum is attained we write min, respectively max instead of inf, respectively sup.

We introduce now further notions and results concerning monotone operators. Having a monotone operator $S : X \rightharpoonup X^*$ one can associate to it the so-called Fitzpatrick function $\varphi_S : X \times X^* \to \mathbb{R}$, defined by

$$\varphi_S(x, x^*) = \sup \{ \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle : y^* \in S(y) \},$$

which is obviously convex and norm-weak* lower semicontinuous. Introduced by Fitzpatrick (see [11]), it proved to be very important in the theory of maximal monotone operators, revealing some connections between convex analysis and monotone operators (see [2, 4–6, 15, 16, 19, 20, 24] and the references therein). Considering the function $c : X \times X^* \to \mathbb{R}$, $c(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$, we get the equality $\varphi_S(x, x^*) = (c + \delta_{G(S)})^\top(x^*, x)$ for all $(x, x^*) \in X \times X^*$.

**Lemma 1** (cf. [11]) Let $S$ be a maximal monotone operator. Then
(i) $\varphi_S(x,x^*) \geq \langle x,x^* \rangle$ for all $(x,x^*) \in X \times X^*$;

(ii) $G(S) = \{(x,x^*) \in X \times X^* : \varphi_S(x,x^*) = \langle x,x^* \rangle\}$.

Motivated by these properties of the Fitzpatrick function, the notion of representative function of a monotone operator was introduced and studied in the literature. For $S : X \rightrightarrows X^*$ a monotone operator, we call representative function of $S$ a convex and norms-weak* lower semicontinuous function $h_S : X \times X^* \to \overline{\mathbb{R}}$ fulfilling

$$h_S \geq c \text{ and } G(S) \subseteq \{(x,x^*) \in X \times X^* : h_S(x,x^*) = \langle x,x^* \rangle\}.$$  

**Remark 1** The above definition is in the sense considered by J.M. Borwein in [3]. In the case of maximal monotone operators it coincides with the usual definition for representative functions considered, for instance, in [4,16] (see Proposition 2 below).

We observe that if $G(S) \neq \emptyset$ (in particular if $S$ is maximal monotone), then every representative function of $S$ is proper. It follows immediately from Lemma 1, that the Fitzpatrick function associated to a maximal monotone operator is a representative function of the operator. The next result is a direct consequence of [3, Proposition 2 and Corollary 4].

**Proposition 2** Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and $h_S$ be a representative function of $S$. Then:

(i) $\varphi_S \leq h_S \leq \varphi_S^\ast$\top;  

(ii) $\varphi_S^\ast$\top is also a representative function of $S$;  

(iii) $\{(x,x^*) \in X \times X^* : h_S(x,x^*) = \langle x,x^* \rangle\} = \{(x,x^*) \in X \times X^* : \varphi_S^\ast(\overline{x},\overline{x}^*) = \langle \overline{x},\overline{x}^* \rangle\} \subseteq G(S)$.

**Remark 2** These properties of representative functions are well-known in the framework of reflexive Banach spaces (see [16]). It is shown in [3] that these characterizations hold also in a general Banach space. For more on the properties of representative functions we refer to [2–4,16] and the references therein.

In the following we recall two results recently introduced in the literature, which are the pillars that sustain our main theorem. We start with a result proved by A.C. Eberhard and J.M. Borwein in [9].

**Theorem 3** (cf. [9, Theorem 37]) For any maximal monotone operator $S : X \rightrightarrows X^*$ it holds $\varphi_S^\ast(\overline{x},\overline{x}^*) \geq \langle \overline{x},\overline{x}^* \rangle$ for all $(\overline{x},\overline{x}^*) \in X^* \times X^{**}$.

The next theorem is a part of a result given by M. Marques Alves and B.F. Svaiter in [13], which generalizes to general Banach spaces some results given in [7,16].

**Theorem 4** (cf. [13, Theorem 4.2]) Suppose that $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper, convex and norm-norm lower semicontinuous function such that $h(x,x^*) \geq \langle x,x^* \rangle$ for all $(x,x^*) \in X \times X^*$ and $h^\ast(x^*,x^{**}) \geq \langle x^*,x^{**} \rangle$ for all $(x^*,x^{**}) \in X^* \times X^{**}$. Define $S : X \rightrightarrows X^*$ by $G(S) = \{(x,x^*) \in X \times X^* : h(x,x^*) = \langle x,x^* \rangle\}$. Then $G(S) = \{(x,x^*) \in X \times X^* : h^\ast(\overline{x},\overline{x}^*) = \langle \overline{x},\overline{x}^* \rangle\}$ and $S$ is maximal monotone.
We close this section with the following result.

**Theorem 5** Suppose that $S, T : X \rightrightarrows X^*$ are two maximal monotone operators with representative functions $h_S$ and $h_T$, respectively, fulfilling

$$0 \in \text{core}(\text{pr}_X(\text{dom } h_S^\diamond) - \text{pr}_X(\text{dom } h_T^\diamond)).$$

Then the function $h : X \times X^* \to \mathbb{R}$ defined by $h(x, x^*) = (h_S \boxplus_2 h_T)(x, x^*) := \inf \{h_S(x, u^*) + h_T(x, v^*): u^*, v^* \in X^*, u^* + v^* = x^*\}$ is convex and norm-norm lower semicontinuous. Further, for all $(x, x^*) \in X \times X^*$ we have $h(x, x^*) \geq \langle x, x^* \rangle$ and the infimum in the definition of $h$ is attained. The function $h$ is proper if and only if $\text{pr}_X(\text{dom } h_S) \cap \text{pr}_X(\text{dom } h_T) \neq \emptyset$. Moreover, $G(S + T) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x, x^* \rangle\}$.

**Proof.** The convexity of $h$ is trivial.

For $K, M$ and $r$ real numbers let us define the set

$$H(K, M, r) := \{(x_1^*, x_2^*) \in X^* \times X^*: \exists x \in X, \|x\| \leq M \text{ such that } \|x_1^* + x_2^*\| \leq r \text{ and } h_S(x, x_1^*) + h_T(x, x_2^*) \leq K\}.$$

We show that for all $u, v \in X$ there exists $C(K, M, r, u, v)$ real number such that for all $(x^*, y^*) \in H(K, M, r)$ we have $\langle u, x^* \rangle + \langle v, y^* \rangle \leq C(K, M, r, u, v)$.

Take $u, v \in X$. The hypotheses imply that there exist $\lambda > 0$, $(a^*, a) \in \text{dom } h_S^\diamond$ and $(b^*, b) \in \text{dom } h_T^\diamond$ fulfilling $u - v = \lambda(a - b)$. Then using the Young-Fenchel inequality we obtain

$$\langle u, x^* \rangle + \langle v, y^* \rangle = \lambda \langle a, x^* \rangle + \lambda \langle b, y^* \rangle + \langle v - \lambda b, x^* + y^* \rangle = \lambda((a, x^*) + (b, y^*)) + \lambda(\langle v - \lambda b, x^* + y^* \rangle - \lambda(x, a^* + b^*)) \leq \lambda(h_S^\diamond(a, a^*) + h_T^\diamond(b, b^* + h_T(x, y^*)) + \|x^* + y^*\| \parallel v - \lambda b \parallel + \lambda M \|a^* + b^*\| \leq \lambda(K + h_S^\diamond(a^*, a) + h_T^\diamond(b^*, b) + M \|a^* + b^*\|) + r \parallel v - \lambda b \parallel := C(K, M, r, u, v).$$

By the uniform boundedness principle we get that $H(K, M, r)$ is a bounded set.

We now show that the level sets of $h_S \boxplus_2 h_T$ are norm-norm closed. Take $K \in \mathbb{R}$ and a sequence $(x_n, x_n^*) \in \{(x, x^*) : (h_S \boxplus_2 h_T)(x, x^*) \leq K\}$ such that $(x_n, x_n^*)$ converges (in the norm-norm topology) to $(x, x^*)$. Let $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ and $r > 0$ be such that $\max\{\varepsilon_n, \|x_n^*\|\} \leq r$ for all $n \geq 1$. Take $M := 2\|x\|$. W.l.o.g we have $\|x_n\| \leq M$ for all $n \geq 1$. We obtain easily the existence of the sequences $u_n^*, v_n^* \in X^*$ such that $u_n^* + v_n^* = x_n^*$ and $(u_n^*, v_n^*) \in H(K + \varepsilon_n, M, r, x_n, x_n^*) \in H(K + \varepsilon_n, M, r, n \geq 1).$ By the previous observation, we have that $(u_n^*, v_n^*)$ is a bounded sequence, hence, by the celebrated Banach-Alaoglu Theorem we can suppose, passing eventually to a subsequence, that $(u_n^*, v_n^*) \rightrightarrows (u^*, v^*)$ for some $u^*, v^* \in X^*$ with $u^* + v^* = x^*$. Employing the norm-weak* lower semicontinuity of the functions $h_S$ and $h_T$ we get

$$h_S(x, x^* - v^*) + h_T(x, v^*) \leq \liminf_n h_S(x_n, x_n^* - v_n^*) + \liminf_n h_T(x_n, v_n^*) \leq \liminf_n \left(h_S(x_n, x_n^* - v_n^*) + h_T(x_n, v_n^*)\right) \leq \liminf_n (K + \varepsilon_n) = K.$$
The inequality \( h(x, x^*) \geq \langle x, x^* \rangle \) for all \((x, x^*) \in X \times X^*\) follows from the definition of the function \( h \) and thus the statement regarding the properness of this function is obvious.

Next, to show that for \((x, x^*) \in X \times X^*\) the infimum in the definition of \( h(x, x^*) \) is attained, we take \( K := h(x, x^*) \) and \( \varepsilon_n \) with \( 0 < \varepsilon_n < 1 \), \( n \geq 1 \), and \( \varepsilon_n \to 0 \). As above, there exists \( u_n^*, v_n^* \in X^* \) such that \( u_n^* + v_n^* = x^* \) and \( h_S(x, u_n^*) + h_T(x, v_n^*) \leq K + \varepsilon_n, n \geq 1 \). Thus for all \( n \geq 1 \) one has \((u_n^*, v_n^*) \in H(K + \varepsilon_n, \|x\|, \|x^*\|) \subseteq H(K + 1, \|x\|, \|x^*\|) \). Since \((u_n^*, v_n^*)\) is bounded, on taking \( w^*\)-convergent subsequences we may suppose \((u_n^*, v_n^*) \rightharpoonup (u^*, v^*)\) for some \( u^*, v^* \in X^* \) with \( u^* + v^* = x^* \). This guarantees that \( h(x, x^*) = h_S(x, x^* - v^*) + h_T(x, v^*) \).

Finally,

\[
\{(x, x^*) : h(x, x^*) = \langle x, x^* \rangle \}
\]

\[
\{(x, x^*) : \exists v^* \in X^* \text{ such that } h_S(x, x^* - v^*) + h_T(x, v^*) = \langle x, x^* \rangle \}
\]

\[
\{(x, x^*) : \exists v^* \in X^* \text{ such that } h_S(x, x^* - v^*) - \langle x, x^* - v^* \rangle + h_T(x, v^*) - \langle x, v^* \rangle = 0 \}
\]

\[
\{(x, x^*) : \exists v^* \in X^* \text{ such that } h_S(x, x^* - v^*) = \langle x, x^* - v^* \rangle \text{ and } h_T(x, v^*) = \langle x, v^* \rangle \}
\]

\[
\{(x, x^*) : \exists v^* \in X^* \text{ such that } x^* - v^* \in S(x) \text{ and } v^* \in T(x) \}
\]

\[
\{(x, x^*) : x^* \in (S + T)(x) \} = G(S + T).
\]

\[\square\]

2 The main result and some consequences

We start by giving a sufficient condition for the maximality of the sum of two maximal monotone operators defined on a Banach space \(X\).

**Theorem 6** Let \( S, T : X \rightrightarrows X^* \) be two maximal monotone operators such that

\[0 \in \operatorname{core} \left( \operatorname{pr}_X (\operatorname{dom} \varphi_S^\circ) - \operatorname{pr}_X (\operatorname{dom} \varphi_T^\circ) \right).\]

Then \( S + T \) is a maximal monotone operator.

**Proof.** Since from Proposition 2, we have \( \operatorname{pr}_X (\operatorname{dom} \varphi_S^\circ) \cap \operatorname{pr}_X (\operatorname{dom} \varphi_T^\circ) \subseteq \operatorname{pr}_X (\operatorname{dom} \varphi_S) \cap \operatorname{pr}_X (\operatorname{dom} \varphi_T), \) Theorem 5 guarantees that the function \( h : X \times X^* \rightarrow \mathbb{R}, h := \varphi_S \square \varphi_T \) is proper, convex, norm-norm lower semicontinuous, \( h(x, x^*) \geq \langle x, x^* \rangle \) for all \((x, x^*) \in X \times X^*\) and

\[G(S + T) = \{(x, x^*) : h(x, x^*) = \langle x, x^* \rangle \}.
\]

In view of Theorem 4, it remains to prove that \( \hat{h}^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle \) for all \((x^*, x^{**}) \in X^* \times X^{**}\).

Take an arbitrary \((x^*, x^{**}) \in X^* \times X^{**}\). It follows by the definition of the function \( h \) that

\[\hat{h}^*(x^*, x^{**}) = \sup_{u \in X, x_1^*, x_2^* \in X^*} \{(u, x^*) + \langle x_1^*, x^{**} \rangle + \langle x_2^*, x^{**} \rangle - \varphi_S(u, x_1^*) - \varphi_T(u, x_2^*)\}.
\]

Consider now the functions \( F, G : X \times X^* \times X^* \rightarrow \mathbb{R}, \) defined by \( F(u, x_1^*, x_2^*) = \varphi_S(u, x_1^*) \) and \( G(u, x_1^*, x_2^*) = \varphi_T(u, x_2^*) \) for all \((u, x_1^*, x_2^*) \in X \times X^* \times X^*\). Then

\[\hat{h}^*(x^*, x^{**}) = (F + G)^*(x^*, x^{**}, x^{**}).\]
One can deduce that \( \text{dom } F - \text{dom } G = ( \text{pr}_X(\text{dom } \varphi_S) - \text{pr}_X(\text{dom } \varphi_T)) \times X^* \times X^* \). From Proposition 2 it follows \( \text{pr}_X(\text{dom } \varphi_S^*) - \text{pr}_X(\text{dom } \varphi_T^*) \subseteq \text{pr}_X(\text{dom } \varphi_S) - \text{pr}_X(\text{dom } \varphi_T) \), which combined with the hypotheses implies \( 0 \in \text{core } (\text{dom } F - \text{dom } G) \). The latter condition ensures \( (\hat{F}^* \square \hat{G}^*)(x^*, x^{**}, x^{**}) = (\hat{F}^* \square \hat{G}^*)(x^*, x^{**}, x^{**}) \) (see [25, Theorem 2.8.7]).

One can show that for all \((u^*, x_1^*, x_2^*)\), the conjugate functions \(\hat{F}^*, \hat{G}^* : X^* \times X^{**} \times X^{**} \to \mathbb{R}\) have the following forms

\[
\hat{F}^*(u^*, x_1^{**}, x_2^{**}) = \begin{cases} 
\hat{\varphi}_S^*(u^*, x_1^{**}), & \text{if } x_2^{**} = 0, \\
+\infty, & \text{otherwise}
\end{cases}
\]

and

\[
\hat{G}^*(u^*, x_1^{**}, x_2^{**}) = \begin{cases} 
\hat{\varphi}_T^*(u^*, x_2^{**}), & \text{if } x_1^{**} = 0, \\
+\infty, & \text{otherwise}
\end{cases}
\]

respectively. Further we have \((\hat{F}^* \square \hat{G}^*)(x^*, x^{**}, x^{**}) = \inf_{u^* \in X^*} \{\hat{\varphi}_S^*(u^*, x^{**}) + \hat{\varphi}_T^*(x^* - u^*, x^{**})\}\). Hence, employing Theorem 3 we obtain \(\hat{h}^*(x^*, x^{**}) = \inf_{u^* \in X^*} \{\hat{\varphi}_S^*(u^*, x^{**}) + \hat{\varphi}_T^*(x^* - u^*, x^{**})\} \geq \inf_{u^* \in X^*} \{\langle u^*, x^{**} \rangle + \langle x^* - u^*, x^{**} \rangle\} = \langle x^*, x^{**} \rangle\), so the proof is complete.

Since \(\text{co}(D(S) - D(T)) \subseteq \text{pr}_X(\text{dom } \varphi_S^*) - \text{pr}_X(\text{dom } \varphi_T^*)\) we have also the following result.

**Corollary 7** Let \(S, T : X \rightrightarrows X^*\) be two maximal monotone operators such that

\[
0 \in \text{core } \text{co} \left( D(S) - D(T) \right).
\]

Then \(S + T\) is a maximal monotone operator.

As the condition \(\text{int}(D(S)) \cap D(T) \neq \emptyset\) obviously implies \(0 \in \text{core } \text{co} \left( D(S) - D(T) \right)\), we get a positive answer regarding Rockafellar’s conjecture [18].

**Corollary 8** Let \(S, T : X \rightrightarrows X^*\) be two maximal monotone operators defined on a Banach space \(X\) such that \(\text{int}(D(S)) \cap D(T) \neq \emptyset\). Then \(S + T\) is a maximal monotone operator.

It is worth giving the following particular case, since it leads to a positive answer concerning different open problems in the theory of maximal monotone operators.

**Corollary 9** Let \(S : X \rightrightarrows X^*\) be a maximal monotone operator defined on a Banach space \(S\) and \(C\) be a nonempty, convex and closed subset of \(X\) such that

\[
\text{int}(D(S)) \cap C \neq \emptyset \text{ or } \text{int}(C) \cap D(S) \neq \emptyset.
\]

Then \(S + N_C\) is a maximal monotone operator.

Recall that a monotone operator \(S : X \rightrightarrows X^*\) is said to be of type (FPV) or, maximal monotone locally, if for any open convex subset \(U\) of \(X\) and \((x, x^*) \in U \times X^*\) such that \(U \cap D(S) \neq \emptyset\) and

\[
\langle y - x, y^* - x^* \rangle \geq 0 \text{ for all } (y, y^*) \in G(S) \text{ such that } y \in U,
\]

we have \((x, x^*) \in G(S)\) (see [19, Definition 36.7]). The operators of type (FPV) were introduced by Fitzpatrick-Phelps in [12, p. 65] and Verona-Verona [21, p. 268]. Taking
$U := X$, we see that every monotone operator of type (FPV) is maximally monotone. Whether the converse is true or not in case of nonreflexive Banach spaces was not yet known (see [19, Problem 36.8]). Combining Corollary 9 and [19, Theorem 44.1 and Theorem 44.2] we obtain the following result.

**Corollary 10** Every maximal monotone operator $S : X \rightrightarrows X^*$ defined on a Banach space $X$ is of type (FPV) and $\text{cl}(D(S))$ is a convex set.

In view of the above result, the open questions Problem 28.3, Problem 31.3 and Problem 36.8 from [19] receive positive answers.

**References**


