SOME BINOMIAL SERIES OBTAINED BY
THE WZ-METHOD

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2002

Abstract
Using the WZ-method we find some of the easiest Ramanujan’s for-
mulae and also some new interesting Ramanujan-like sums.

1 The WZ-method

We recall [4] that a discrete function $A(n,k)$ is hypergeometric or closed form
(CF) if the quotients

$$\frac{A(n+1,k)}{A(n,k)}$$

and

$$\frac{A(n,k+1)}{A(n,k)}$$

are both rational quotients.

And a pair of functions $F(n,k)$, $G(n,k)$ is said to be of Wilf and Zeilberger
(WZ) if $F$ and $G$ are closed forms and besides

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

In this case H.S. Wilf and D. Zeilberger [3] have proved that there exists a
rational function $C(n,k)$ such that

$$G(n,k) = C(n,k)F(n,k)$$

The rational function $C(n,k)$ is the so called certificate of the pair (F,G). We
now define

$$H(n,k) = F(n+1,n+k) + G(n,n+k)$$

Zeilberger has proved that for every WZ pair $F(n,k)$, $G(n,k)$ the following
holds

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} H(n,0)$$
In next sections we use WZ-pairs to get some Ramanujan’s formulae and also some new Ramanujan-like ones.

2 First WZ-pair

We consider the following discrete function

\[ G(n, k) = \frac{(-1)^n(-1)^k}{2^{10n+2k}} \frac{(2k)^2 (2n)^2 (4n-2k)}{(2n)_k} \frac{(n+k)}{(n)_n} (20n + 3) \]

The package EKHAD [5] allows to get the companion

\[ F(n, k) = \frac{64}{2^{10n+2k}} \frac{(-1)^n(-1)^k}{4n - 2k - 1} \frac{(2k)^2 (2n)^2 (4n-2k)}{(2n)_k} \frac{(n+k)}{(n)_n} (20n + 3) \]

We get the result

\[ \sum_{n=0}^{\infty} \frac{(-1)^n (4n)^2}{2^{10n}} (20n + 3) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)^3}{2^{10n}} (42n + 5) \]

We can extend the pair to have sense for every value of \( k \), not only integers, in the following way

\[ F(n, k) = \frac{64}{\pi^3} \frac{n^2}{4n - 2k - 1} \frac{(-1)^n \cos(\pi k) \Gamma(2n - k + 1/2) \Gamma(n + 1/2)^3 \Gamma(k + 1/2)^2}{\Gamma(n + k + 1) \Gamma(2n + 1)^2} \]

\[ G(n, k) = \frac{1}{\pi^3} (20n + 2k + 3) (-1)^n \cos(\pi k) \Gamma(2n - k + 1/2) \Gamma(n + 1/2)^3 \Gamma(k + 1/2)^2 \]

If \( k \) is an integer it is a routine to prove that \( \sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} F(n, k) \), and this implies applying Carlson’s theorem [2] that for every value of \( k \) even if \( k \) is not an integer, \( \sum_{n=0}^{\infty} = A \), where \( A \) is a constant. To determine the value of the constant observe that \( \lim_{t \to 1/2} \sum_{n=1}^{\infty} G(n, t) = 0 \). So \( A = \lim_{t \to 1/2} G(0, t) = \frac{8}{\pi} \). And we have that independently of the value of \( k \)

\[ \sum_{n=0}^{\infty} G(n, k) = \frac{8}{\pi} \]

But then we have also the sum of another family of infinite series because obviously we immediately get

\[ \sum_{n=0}^{\infty} H(n, k) = \frac{8}{\pi} \]

For \( k = 0 \) we get the following results [1]

\[ \sum_{n=0}^{\infty} G(n, 0) = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)^2 (2n)^2}{2^{10n}} (20n + 3) = \frac{8}{\pi} \]
\[ \sum_{n=0}^{\infty} H(n, 0) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n+3)}{2^{12n}} (42n + 5) = \frac{8}{\pi} \]

For other values of \( k \) we obtain also interesting results. For example, for \( k = 1/4 \) we get
\[
\frac{3\sqrt{2}}{8} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_{2n}}{(n!)^2 \left(\frac{1}{2}\right)_n 24^n} \left(\frac{40n + 7}{4n + 1}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)^2}
\]
\[
\frac{3\sqrt{2}}{8} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_{2n} \left(\frac{1}{4}\right)_n 28^n}{(n!)^2 \left(\frac{1}{2}\right)_n 28^n} \left(\frac{112n^2 + 88n + 11}{(8n + 1)(8n + 5)}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)^2}
\]

3 Second WZ-pair

we consider the following discrete function
\[ G(n, k) = \frac{(-1)^k}{216n^2\pi^2} (120n^2 + 84nk + 34n + 10k + 3) \left(\frac{2k}{n}\right)^3 \left(\frac{2n}{2n-k}\right)^4 \left(\frac{4n-2k}{2n}\right)^2 \]

The package EKHAD [5] allows to get the companion
\[ F(n, k) = 512 \frac{(-1)^k}{216n^2\pi^2} n^3 \frac{\left(\frac{2k}{n}\right)^3 \left(\frac{2n}{2n-k}\right)^4 \left(\frac{4n-2k}{2n}\right)^2}{\left(\frac{2n}{2n-k}\right)^2} \]

We have the following result
\[ \sum_{n=0}^{\infty} \frac{(2n)^4}{2^{10n}} (120n^2 + 34n + 3) = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)^5}{2^{20n}} (820n^2 + 180n + 13) \]

We can extend the pair to have sense for every value of \( k \), not only integers, in the following way
\[ F(n, k) = \frac{512}{\pi^5} \frac{n^3 \cos(\pi k)}{4n - 2k - 1} \frac{\Gamma(2n - k + 1/2)\Gamma(n + 1/2)\Gamma(k + 1/2)^3}{\Gamma(n + k + 1)^2 \Gamma(2n + 1)^3} \]
\[ G(n, k) = \frac{1}{\pi^5} (120n^2 + 84nk + 34n + 10k + 3) \frac{\cos(\pi k)\Gamma(2n - k + 1/2)\Gamma(n + 1/2)\Gamma(k + 1/2)^3}{\Gamma(n + k + 1)^2 \Gamma(2n + 1)^3} \]

if \( k \) is an integer it is a routine to prove that \( \sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} F(n, k) \), and this implies applying Carlson’s theorem [2] that for every value of \( k \) even if \( k \) is not an integer, \( \sum_{n=0}^{\infty} = A \), where \( A \) is a constant. To determine the value of the constant observe that \( \lim_{t \to 1/2} \sum_{n=1}^{\infty} G(n, t) = 0 \). So \( A = \lim_{t \to 1/2} G(0, t) = \frac{32}{\pi^2} \). And we have that independently of the value of \( k \)
\[ \sum_{n=0}^{\infty} G(n, k) = \frac{32}{\pi^2} \]
But then we have also the sum of another family of infinite series because obviously we immediately get

$$\sum_{n=0}^{\infty} H(n, k) = \frac{32}{\pi^2}$$

For $k = 0$ we get the following results

$$\sum_{n=0}^{\infty} G(n, 0) = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)^2}{2^{16n}} \cdot \frac{(2n)!}{n!} \frac{(120n^2 + 34n + 3)}{2\pi^2}$$

$$\sum_{n=0}^{\infty} H(n, 0) = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n^5)}{2^{20n}} (820n^2 + 180n + 13) = \frac{32}{\pi^2}$$

For other values of $k$ we obtain also interesting results. For example, for $k = 1/4$ we get

$$\frac{1}{8} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^3_n (\frac{3}{4})^2_n (\frac{5}{4})^{2n}}{(n!)^3 \cdot 2^{2n}} \frac{240n^2 + 110n + 11}{(4n + 1)^2} = \frac{\pi}{\Gamma(\frac{3}{4})^4}$$

$$-\frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})^3_n (\frac{3}{4})^2_n (\frac{5}{4})^{2n}}{(n!)^3 \cdot 2^{2n}} \frac{26240n^4 + 41184n^3 + 21448n^2 + 4170n + 279}{(8n + 1)^2(8n + 5)^2} = \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})^4}$$

References