The Optimality of James’s Distortion Theorems

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Abstract. A renorming of $\ell_1$, explored here in detail, shows that the copies of $\ell_1$ produced in the proof of the Kadec-Pelczyński theorem inside nonreflexive subspaces of $L_1[0,1]$ cannot be produced inside general nonreflexive spaces that contain copies of $\ell_1$. Put differently, James’s distortion theorem producing one-plus-epsilon-isomorphic copies of $\ell_1$ inside any isomorphic copy of $\ell_1$ is, in a certain sense, optimal. A similar renorming of $c_0$ shows that James’s distortion theorem for $c_0$ is likewise optimal.

James’s distortion theorems for $\ell_1$, the space of absolutely summable sequences of scalars, and $c_0$, the space of null sequences of scalars, are well-known [J]. The former states that, whenever a Banach space contains a subspace isomorphic to $\ell_1$, the Banach space contains subspaces that are almost isometric to $\ell_1$. Several of the authors of this article, individually and in concert, have tried to use this feature of $\ell_1$ to determine if all (equivalent) renormings of $\ell_1$ fail to have the fixed point property for nonexpansive mappings (the FPP); i.e. if, in any renorming of $\ell_1$, there exist a nonempty, closed, bounded and convex subset $C$ and a nonexpansive self-map $T$ of $C$ without a fixed point. The basis of these attempts was to use the fact that $\ell_1$ in its usual norm fails to have the fixed point property and, since each renorming of $\ell_1$ contains subspaces almost isometric to $\ell_1$, a perturbation of the usual example would hopefully produce a nonexpansive self-map of a nonempty, closed, bounded, convex set in any renorming of $\ell_1$. Similar attempts in $c_0$ were also made. What appeared to be needed in these attempts were strengthened versions of James’s distortion theorems.

To be specific, James’s theorem for $\ell_1$ states that if a Banach space $X$ with norm $\| \cdot \|$ contains an isomorphic copy of $\ell_1$, then, for each $\epsilon > 0$, there exists a sequence $(x_k)$ in the unit sphere of $X$ such that $(1 - \epsilon) \sum_{k=1}^\infty |t_k| \leq \| \sum_{k=1}^\infty t_k x_k \| \leq \sum_{k=1}^\infty |t_k|$ for all $(t_k) \in \ell_1$. The proof of the theorem shows even more than the statement indicates. The sequence $(x_k)$ may be chosen to have the additional property that, if $(\epsilon_n)$ is a sequence of positive numbers decreasing to 0, then for each $n$, $(1 - \epsilon_n) \sum_{k=n}^\infty |t_k| \leq \| \sum_{k=n}^\infty t_k x_k \| \leq \sum_{k=n}^\infty |t_k|$ for all $(t_k) \in \ell_1$. That is, for each $\delta > 0$, by ignoring a finite number of terms at the beginning of the sequence $(x_k)$, one obtains copies of $\ell_1$ which are $(1 + \delta)$-isomorphic to $\ell_1$. This

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leads one to ask if James’s distortion theorem can be strengthened in the following sense:

**Question.** If $X$ is a Banach space that contains an isomorphic copy of $\ell_1$ and $(c_\nu)$ is a sequence of positive numbers that decreases to 0, does there exist a sequence $(x_n)$ in the unit sphere of $X$ such that $\sum_{k=1}^{\infty} (1 - c_\nu) |t_k| \leq \| \sum_{k=1}^{\infty} t_k x_k \| \leq \sum_{k=1}^{\infty} |t_k|$ for all $(t_k) \in \ell_1$?

The closed linear span of such a sequence $(x_n)$ in the above question is called an *asymptotically isometric copy* of $\ell_1$. As noted in [DL], the proof of the Kadec-Pełczyński theorem [KP] shows that nonreflexive subspaces of $L_1[0,1]$ contain such “good” copies of $\ell_1$ and, in this case, there exist nonexpansive self-maps on closed, bounded and convex sets without fixed points. (This provides a converse to a theorem of Maurey [M] that every reflexive subspace of $L_1[0,1]$ has the FPP.)

Thus, if every renorming of $\ell_1$ were to contain an asymptotically isometric copy of $\ell_1$, then every renorming of $\ell_1$ would fail the fixed point property. One purpose of this article is to present a renorming of $\ell_1$ which contains no asymptotically isometric copy of $\ell_1$. Thus James’s distortion theorem for $\ell_1$ is, in this sense, optimal and the question of whether $\ell_1$ can be given an equivalent norm with the fixed point property remains open. Using the predual of this renorming of $\ell_1$, it will be seen that James’s distortion theorem for $c_0$ is similarly optimal and the question as to whether $c_0$ can be given an equivalent norm with the fixed point property likewise remains open.

Recent papers ([CDL, DLT]) have extended the classes of spaces known to contain asymptotically isometric copies of $\ell_1$. In a related paper, Smyth [S] showed that the dual of every space $C(\Omega)$, where $\Omega$ is an infinite compact Hausdorff space, fails the weak-star fixed point property with an affine contraction.

In the ensuing discussion, $K$ will denote the scalar field (the real or the complex numbers) and $\mathbb{N}$ will denote the positive integers. The Banach space $\ell_1$ is as usual the space of absolutely summable scalar sequences with its usual norm $\|x\|_1 := \sum_{n=1}^{\infty} |x_n|$, for all $x = (x_n) \in \ell_1$. More generally, for $p \geq 1$, the Banach space of $p$-summable sequences of scalars is denoted by $\ell_p$, and is normed by $\|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ for all $x = (x_n) \in \ell_p$. The sequence $(c_\nu)$ will always denote the canonical unit vector basis in $\ell_p$. Recall that $\|x\|_p \leq \|x\|_1$ for $p \geq 1$ and $x \in \ell_1$.

The space to be defined is, on the surface, quite simple. It is a countable sum of $K$’s and is akin to the classical $\ell_p$-spaces. There are two significant features to notice: the varying values of the exponents (similar to spaces of Nakano) and the placement of the parentheses in defining the norm. Fix a sequence $p = (p_\nu)$ of real numbers in $(1, \infty)$ converging to 1. Then the space we wish to define is:

$$K \oplus_{p_1} (K \oplus_{p_2} (K \oplus_{p_3} \ldots)) .$$

Let $X := K^\mathbb{N}$. For $x = (x_\nu) \in X$, define:

$$\nu_1(p,x) := |x|_1 ,$$

$$\nu_2(p,x) := (|x_1|^{p_1} + |x_2|^{p_1})^{1/p_1} ,$$

$$\nu_3(p,x) := (|x_1|^{p_1} + (|x_2|^{p_2} + |x_3|^{p_2})^{p_1/p_2})^{1/p_1} .$$
\[ \nu_4(p, x) := \left( |\xi_1|^{p_1} + \left( |\xi_2|^{p_2} + (|\xi_3|^{p_3} + |\xi_4|^{p_3}/p_3)\right)^{p_1/p_2} \right)^{1/p_1}. \]

To proceed further with this inductive construction, some notation is useful. Define the shift operator \( S : X \to X \) by \( Sz := (z_2, z_3, \ldots, z_k, \ldots) \) for all \( z \in X \). For \( p \) and \( x \) as above define, for each \( n \in \mathbb{N} \),

\[ \nu_{n+1}(p, x) := (|\xi_1|^{p_1} + \nu_n(Sp, Sx)^{p_1})^{1/p_1}. \]

Each \( \nu_n(p, \cdot) \) is a seminorm on \( X \) and, for each \( x \in X \), the sequence \( (\nu_n(p, x))_{n=1}^\infty \) increases to a limit \( \nu_p(x) \). Clearly, for all \( x \in X \), \( \nu_n(p, x) \leq |\xi_1| + \cdots + |\xi_n| \) for every \( n \). Thus \( \nu_p(x) \leq \|x\|_1 \) for each \( x \in \ell_1 \).

In seeking lower estimates for \( \nu_p(x) \), first note that since all two-dimensional normed linear spaces are equivalent, two-dimensional \( \ell_2^q \) is equivalent to \( \ell_1^2 \) and in fact, for \( q \geq 1 \),

\[ \| (\xi_1, \xi_2) \|_q \geq 2^{-1+1/q}(\|\xi_1\|_1, \|\xi_2\|_1). \]

Then, with \( K_j = 2^{-1+1/p_j} \),

\[ \nu_n(p, x) = (|\xi_1|^{p_1} + \nu_{n-1}(Sp, Sx)^{p_1})^{1/p_1} \]
\[ \geq K_1 (|\xi_1| + \nu_{n-1}(Sp, Sx)) \]
\[ = K_1 \left( |\xi_1| + (|\xi_2|^{p_2} + \nu_{n-2}(S^2p, S^2x)^{p_2})^{1/p_2} \right) \]
\[ \geq K_1 \left( |\xi_1| + K_2 \left( |\xi_2| + \nu_{n-2}(S^2p, S^2x) \right) \right) \]
\[ \geq K_1 K_2 \left( |\xi_1| + |\xi_2| + \nu_{n-2}(S^2p, S^2x) \right) \]
\[ \vdots \]
\[ \geq K_1 K_2 \cdots K_n \sum_{j=1}^n |\xi_j|. \]

Specializing to the sequence \( p = (p_j) \) where \( p_j = 2^{j}/2^{j-1} \) yields:

\[ \frac{1}{2} \|x\|_1 \leq \nu_p(x) \leq \|x\|_1 \quad \text{for all } x \in \ell_1. \]

Thus, for this specific choice of \( p \), \( \nu_p(\cdot) \) is an equivalent norm on \( \ell_1 \).

It is clear that \( \nu_p(\cdot) \) is equivalent to the \( \ell_1 \) norm whenever \( p_n \) converges to 1 sufficiently quickly. Moreover, it is easy to determine for which sequences \( p \) the norm \( \nu_p(\cdot) \) is equivalent to the \( \ell_1 \) norm. The characterization is in terms of the dual norm \( \nu_q(\cdot) \), where \( q = (q_1, q_2, \ldots) \) satisfies \( \frac{1}{p_n} + \frac{1}{q_n} = 1 \) for each \( n \in \mathbb{N} \).

**Proposition 1.** Let \( p \) be a sequence in \((1, \infty)\) and let \( q \) be the sequence of conjugate exponents of \( p \). Then the following are equivalent.

(i) \( \nu_p(\cdot) \) is equivalent to the \( \ell_1 \) norm.

(ii) \( \nu_q(\cdot) \) is equivalent to the \( \ell_0 \) norm.

(iii) \( \lim_{n \to \infty} \nu_q(1_{[1,n]}) < \infty \).

(iv) There exists \( \delta > 0 \) so that for all \( n \), \( q_n^\# \geq \delta \log n \), where \( (q_n^\#) \) is the increasing rearrangement of \( q \).
Proof. We will use the notation \( 1_E \) to denote the characteristic function of a subset \( E \) of \( \mathbb{N} \). The equivalence of the first three conditions is well-known in a general context. That implication (iii) implies (iv) is straightforward. Indeed, since for each \( n \), there are at least \( n \) values of \( k \) for which \( q_k \leq q^n \), we have for sufficiently large \( N \) that

\[
\nu_q(1_{[1,N]}) \geq n^{1/q^n}.
\]

So, if we set \( C := \lim_{N \to \infty} \nu_q(1_{[1,N]}) \), then for all \( n \),

\[
q^n \geq \frac{\log n}{\log C}.
\]

For (iv) implies (iii), let \( C > 3^{1/s} \). Then

\[
\sum_{n=1}^{\infty} C^{-q^n} = \sum_{n=1}^{\infty} C^{-q^n} < \infty.
\]

Now for \( x > 0 \), \( s > 1 \), \((1 + x^{-1})^{1/s} \leq 1 + s^{-1}x^{-1} \), and therefore \((1 + x^s)^{1/s} \leq x + s^{-1}x^{-s} \). Hence, for each \( k \leq N \)

\[
\nu_q(1_{[k,N]}) \leq \nu_q(1_{[k+1,N]}) + q_k^{-1} \nu_q(1_{[k+1,N]})^{1-q_k}.
\]

If \( \nu_q(1_{[1,N]}) \leq C \) for all \( N \in \mathbb{N} \), then we are done. If \( \nu_q(1_{[1,N]}) > C \), choose \( m \) so that \( \nu_q(1_{[m,N]}) \geq C > \nu_q(1_{[m+1,N]}) \). Then

\[
\nu_q(1_{[1,N]}) \leq C + 1 + \sum_{k=1}^{m-1} q_k^{-1} \nu_q(1_{[k+1,N]})^{1-q_k} \leq C + 1 + C \sum_{k=1}^{\infty} q_k^{-1} C^{-q_k} < \infty.
\]

The next result shows the optimality of James’s theorem by proving that the above renormings of \( \ell_1 \) fail to contain any asymptotically isometric copies of \( \ell_1 \).

**Theorem 1.** Let \( p = (p_n) \) be a sequence in \((1, \infty)\), converging to \( 1 \) and such that \( \nu_p \) is an equivalent norm on \( \ell_1 \); and let \((\epsilon_n)\) be a null sequence in \((0,1)\). Then there does not exist a \( \nu_p \)-normalized sequence \((x_k)\) in \( \ell_1 \) such that, for all \( t = (t_j) \in \ell_1 \),

\[
\sum_{j=1}^{\infty} (1 - \epsilon_j) |t_j| \leq \nu_p \left( \sum_{j=1}^{\infty} t_j x_j \right) \leq \sum_{j=1}^{\infty} |t_j|.
\]

Proof. Without loss of generality, assume \( p \) strictly decreases to \( 1 \). In order to obtain a contradiction, assume that there exists a null sequence \((\epsilon_n)\) in \((0,1)\) and a \( \nu_p \)-normalized sequence \((x_k)\) in \( \ell_1 \) such that

\[
(*) \quad \sum_{j=1}^{\infty} (1 - \epsilon_j) |t_j| \leq \nu_p \left( \sum_{j=1}^{\infty} t_j x_j \right) \leq \sum_{j=1}^{\infty} |t_j| \quad \text{for all } t = (t_j) \in \ell_1 .
\]

By passing to a subsequence of \((x_n)\) if necessary, there is no loss of generality in assuming that

\[
(**) \quad \sum_{n=1}^{\infty} \epsilon_n < 1 .
\]
Note also that there is no loss of generality in assuming additionally that the sequence \((x_n)\) is disjointly supported, i.e., that the support of \(x_m\) is disjoint from the support of \(x_n\) if \(m \neq n\). Indeed this is a classical gliding hump argument. Since the closed unit ball of \(\ell_1\) is weak-star sequentially compact with respect to the predual \(c_0\), by passing to a subsequence, we may suppose that \((x_n)\) converges weak-star (and so pointwise with respect to the usual basis \((e_n)\) of \(\ell_1\)) to some \(y \in \ell_1\). By replacing \((x_n)\) by the \(\nu_p\)-normalization of the sequence \(\left(\frac{x_{2j} - x_{2j+1}}{2}\right)\), we may assume that \(y = 0\). As in the proof of the Bessaga-Pelczyński theorem [BP] (or see, for example [D]), by passing to a subsequence of \((x_n)\) which is essentially disjointly supported, truncating to obtain a disjointly supported sequence, and then normalizing, yield a block basis \((b_k)\) of \((e_n)\) which satisfies (*) Consequently, we henceforth assume that \((x_n)\) is disjointly supported.

Let \((m(k))_{k=0}^{\infty}\) be a strictly increasing sequence in \(\mathbb{N} \cup \{0\}\) with \(m(0) = 0\) and \((\xi_j)_{j=1}^{\infty}\) a sequence of scalars such that, for each \(k \in \mathbb{N}\),

\[
x_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j.
\]

Let \(N\) be in \(\mathbb{N}\) and, in (*), set \(t_j = 1\) for \(j = 1, \cdots, N\) and 0 otherwise. Then, for \(N \geq m(1)\):

\[
N - \sum_{j=1}^{N} \xi_j \leq \nu_p \left( \sum_{k=1}^{N} x_k \right)
= \left( |\xi_1|^{p_1} + \nu_p \left( \sum_{j=2}^{m(N)} \xi_j e_j \right)^{p_1/p_1} \right)^{1/p_1}
\leq \left( |\xi_1|^{p_2} + \nu_p \left( \sum_{j=2}^{m(N)} \xi_j e_j \right)^{p_2/p_2} \right)^{1/p_2}
= \left( |\xi_1|^{p_2} + |\xi_2|^{p_2} + \nu_p \left( \sum_{j=3}^{m(N)} \xi_j e_j \right)^{p_2/p_2} \right)^{1/p_2}
\leq \left( |\xi_1|^{p_3} + |\xi_2|^{p_3} + \nu_p \left( \sum_{j=3}^{m(N)} \xi_j e_j \right)^{p_3/p_3} \right)^{1/p_3}
\vdots
\leq \left( |\xi_1|^{p_{m(1)}} + \cdots + |\xi_{m(1)}|^{p_{m(1)}} + \nu_p \left( \sum_{j=m(1)+1}^{m(N)} \xi_j e_j \right)^{p_{m(1)}/p_{m(1)}} \right)^{1/p_{m(1)}}
= \left( \|x_1\|^{p_{m(1)}} + \nu_p \left( \sum_{k=2}^{N} x_k \right)^{p_{m(1)}/p_{m(1)}} \right)^{1/p_{m(1)}}
\leq (\|x_1\|^{p_{m(1)}} + (N - 1)^{p_{m(1)}})^{1/p_{m(1)}}.
\]
Thus, for \( N \geq m(1) \),
\[
\left( N - \sum_{j=1}^{N} \epsilon_j \right)^{p_{m(1)}} - (N - 1)^{p_{m(1)}} \leq \|x_1\|^{p_{m(1)}}.
\]
By (**), the left-hand side of the inequality tends to \( \infty \) with \( N \). This yields a contradiction which finishes the proof. \( \square \)

In the previous proof, choosing vectors of the form \( x_1 + Mx_N \), instead of \( x_1 + \cdots + x_N \), also leads to a contradiction (by letting \( N \) and then \( M \) become arbitrarily large).

We note here that the proof of James’s distortion theorem for \( c_0 \) gives us that if a Banach space \((X, \| \cdot \|)\) contains an isomorphic copy of \( c_0 \), then for each sequence \((\epsilon_n)\) of positive numbers decreasing to 0, there exists a sequence \((x_n)\) in the unit sphere of \( X \) such that for each \( n \), \((1 - \epsilon_n) \max_{k \geq n} |t_k| \leq \| \sum_{k=n}^{\infty} t_k x_k \| \leq (1 + \epsilon_n) \max_{k \geq n} |t_k|\), for all \((t_k) \in c_0 \). In order to show that James’s distortion theorem for \( c_0 \) is also optimal, the construction introduced for \( \ell_1 \) can be used as long as the sequence \( p = (p_j) \) is chosen to increase sufficiently quickly to infinity. For example, with \( p_j = 2^j \),
\[
\|x\|_\infty \leq \nu_p(x) \leq 2 \|x\|_\infty \quad \text{for all } x \in c_0.
\]
Thus, with this choice of \( p, \nu_p(\cdot) \) is an equivalent norm on \( c_0 \). (For other choices of \( p \), we may apply Proposition 1.)

A Banach space is said to contain an asymptotically isometric copy of \( c_0 \) if, for every sequence of positive numbers \((\epsilon_n)\) decreasing to 0, there exists a sequence \((x_n)\) in the Banach space such that \( \max_{n \in F}(1 - \epsilon_n)|\alpha_n| \leq \| \sum_{n \in F} \alpha_n x_n \| \leq \max_{n \in F}(1 + \epsilon_n)|\alpha_n| \) for all choices of scalars \((\alpha_n)\) and for all finite subsets \( F \) of natural numbers. (Note that \((1 + \epsilon_n)\) may be replaced by 1 in this definition.) The next result provides a useful connection between the two asymptotically isometric properties.

**Theorem 2.** Let \((X, \| \cdot \|)\) be a Banach space that contains an asymptotically isometric copy of \( c_0 \). Then \( X^* \), with the dual norm, contains an asymptotically isometric copy of \( \ell_1 \).

**Proof.** By hypothesis, given any null sequence \((\epsilon_n)\) in \((0, 1)\), there is a sequence \((x_n)\) in \( X \) such that for all finite sequences of scalars \((\alpha_n)_{n=1}^{N} \),
\[
\max_{1 \leq n \leq N} (1 - \epsilon_n)|\alpha_n| \leq \left\| \sum_{n=1}^{N} \alpha_n x_n \right\| \leq \max_{1 \leq n \leq N} |\alpha_n|.
\]
Let \((x^*_n)\) be a sequence of Hahn-Banach extensions to elements of \( X^* \) of the linear functionals on the span of \((x_n)\) that are biorthogonal to \((x_n)\). Consider \( x^*_m \), for some \( m \in \mathbb{N} \). Then, for all vectors \( x \) of the form \( \sum_{n=1}^{N} \alpha_n x_n \) with \( N \geq m \), we have
\[
|x^*_m(x)| = |\alpha_m| = (1 - \epsilon_m)^{-1} (1 - \epsilon_m) |\alpha_m| \leq (1 - \epsilon_m)^{-1} \max_{1 \leq n \leq N} (1 - \epsilon_n) |\alpha_n| \leq (1 - \epsilon_m)^{-1} \|x\|;
\]
and hence it follows that \( \|x^*_m\| \leq (1 - \epsilon_m)^{-1} \). Set \( x^*_n := |x^*_n|^{-1} x^*_n \) for each \( n \in \mathbb{N} \). Fix a sequence \((\alpha_n)_{n=1}^{N} \) of scalars and let \( \beta_n = \text{sign } \alpha_n \) for all \( n \). Then, since \( \|\sum_{n=1}^{N} \beta_n x_n\| \leq \max_{1 \leq n \leq N} |\beta_n| = 1 \), we have
Proof. It is enough to apply Theorems 1 and 2, after noting that the dual of \( (C,D,L) \) contains an asymptotically isometric copy of \( \ell_1 \) in a closed, bounded and convex set \( T \), generality, assume \( (\nu_q) \) is an equivalent norm on \( c_0 \); and let \( (\epsilon_n) \) be a null sequence in \( (0,1) \). Then there does not exist a sequence \( (x_k) \) in \( c_0 \) such that, for all \( \alpha = (\alpha_j) \in c_0 \),

\[
\max_{n \in F} (1 - \epsilon_n) |\alpha_n| \leq \nu_q \left( \sum_{n \in F} \alpha_n x_n \right) \leq \max_{n \in F} (1 + \epsilon_n) |\alpha_n|
\]

for all finite subsets \( F \) of natural numbers.

Proof. It is enough to apply Theorems 1 and 2, after noting that the dual of \( (c_0,\nu_q) \) is \( (\ell_1,\nu_p) \), where \( p \) is the sequence of conjugate exponents of \( q \).

In closing, note that other renormings of \( \ell_1 \) exist that fail to contain asymptotically isometric copies of \( \ell_1 \). One such norm is:

\[
\|x\| := \sup_{n \in \mathbb{N}} \sum_{k=n}^{\infty} |\xi_k|, \quad \text{for all } x = (\xi_n) \in \ell_1,
\]

where \( (\gamma_n) \) is a fixed sequence in \( (0,1) \) that strictly increases to 1. The details needed to show that \( \ell_1 \) with this norm fails to contain asymptotically isometric copies of \( \ell_1 \) are similar to those given for the \( \nu_p \)-norm. (Although, when checking the analogue of the proof of Theorem 1 for the \( ||\cdot||' \)-norm, (after assuming, without loss of generality, that \( (\epsilon_n) \) decreased to 0 sufficiently fast), we used the sequence \( (x_1 + N x_n) \) instead of \( (x_1 + \cdots + x_N) \).

Whether \( \ell_1 \) endowed with either of the norms \( ||\cdot||' \) or \( \nu_q \) has the fixed point property is unknown. The norm \( \| \cdot \|' \), suggested to us by the referee of another paper, is interesting because of its link to the strengthening of James’s distortion theorem described earlier.

Finally, let us consider \( c_0 \). It is shown in [DLT] that whenever a Banach space \( (X,\|\cdot\|) \) contains an asymptotically isometric copy of \( c_0 \), it must fail the FPP. We remark that the spaces \( (c_0,\nu_q) \) of Theorem 3 also fail the FPP. Indeed, without loss of generality, assume \( (q_n) \) increases to \( \infty \). Then a fixed point free \( \nu_q \)-nonexpansive map \( T \) on a closed, bounded and convex set \( C \) is provided by the usual \( c_0 \) example: i.e. let \( C := \{ x = (\xi_n) \in c_0 : 0 \leq \xi_n \leq 1 \text{ for all } n \} \) and define \( T(x) := (1, \xi_1, \xi_2, \xi_3, \ldots) \).

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