The distortion problem.

Feature Review.

This paper contains a remarkable solution of the long-standing “distortion problem”.

A Banach space \((X, \| \cdot \|)\) is said to be distortable if there exists an equivalent norm \(| \cdot |\) on \(X\) and a \(\lambda > 1\) such that for all infinite-dimensional subspaces \(Y \subseteq X\) one has \(\sup\{\|y\|/\|x\|; \ x, y \in Y, \|x\| = \|y\| = 1\} > \lambda\). Odell and Schlumprecht prove here that a separable infinite-dimensional Hilbert space is distortable. Moreover, using their proof and an earlier result of V. D. Mil’man [Uspekhi Mat. Nauk 26 (1971), no. 6(162), 73–149; MR0420226 (54 #8240); Funktsional. Anal. i Prilozhen. 3 (1969), no. 2, 67–79; MR0251513 (40 #4740)], they show that any space not containing an isomorphic copy of \(l_1\) or \(c_0\) contains a distortable subspace. (R. C. James [Ann. of Math. (2) 80 (1964), 542–550; MR0173932 (30 #4139)] proved that \(l_1\) and \(c_0\) are not distortable.)

Using a result of W. T. Gowers [European J. Combin. 13 (1992), no. 3, 141–151; MR1164759 (93g:05142)], they also prove that an infinite-dimensional Banach space \(X\) has the property that every infinite-dimensional subspace of \(X\) contains an isomorphic copy of \(c_0\) if and only if, for every real Lipschitz function \(f\) on the sphere of \(X\) and every infinite-dimensional subspace \(Y\) of \(X\), \(f\) has arbitrarily small oscillation when restricted to some infinite-dimensional subspace of \(Y\).

The fact that the Hilbert space \(l_2\) is distortable is equivalent to the existence of two separated sets in the sphere of \(l_2\) each of which intersects every infinite-dimensional closed subspace of \(l_2\). The authors actually prove a stronger version of this remarkable fact: There exists a sequence of subsets \((C_i)_{i=1}^{\infty}\) of the unit sphere of \(l_2\) such that (a) each set \(C_i\) intersects each infinite-dimensional closed subspace of \(l_2\) and (b) the sets are almost orthogonal: \(\sup\{\langle x, y \rangle; \ x \in C_i, \ y \in C_j\} \to 0\) as \(\min\{i, j\} \to \infty\).

Loosely speaking, the proof consists of two parts. In one of them the authors build a sequence of sets with properties similar to (a) and (b) above in a specific nonclassical space (built previously by Schlumprecht [Israel J. Math. 76 (1991), no. 1-2, 81–95; MR1177333 (93h:46023)], based on a construction of B. S. Tsirel’son [Funktsional. Anal. i Prilozhen. 8 (1974), no. 2, 57–60; MR0350378 (50 #2871)]). In the second they construct a uniform homeomorphism from the sphere of a general space, with an unconditional basis and nontrivial cotype, onto the sphere of \(l_2\) (generalizing the Mazur map [S. Mazur, Studia Math. 1 (1929), 83–85; Jbuch 55, 242]) and use it to transfer the sets built for the special space to ones in the sphere of \(l_2\).

Reviewed by G. Schechtman