Notes on the Laplace Transform of the Polygamma function

Basic identity

\[ \int_{0}^{\infty} e^{-as} \psi(x + 1) ds = L(a) - \frac{\gamma}{a} \]  

(1)

where

\[ L(a) = \frac{4}{\pi} \int_{0}^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2 e^{-a} \cos(x))} \]  

(2)

By plotting both sides, it appears that (1) is valid for \( \ln(2) \leq a \leq \infty \) so that (2) is of interest for \( 0 \leq a \leq \ln(2) \). Now, formally,

\[ L(a) = \frac{2}{\pi} \int_{0}^{\infty} ds 2^{-as} e^{-as} \int_{-\pi/2}^{\pi/2} dx \frac{x \sin(sx)}{\cos^s x} \]  

(3)

Performing the \( s \)-integral, expanding the \( x \)-integrand, after a change of variable, one is led to

\[ L(a) = \pi e^{-a} \int_{0}^{1} dt \frac{t e^{-at}}{\Gamma(1-t)} \sum_{k \neq 1} \frac{\Gamma(l-t) \Gamma(l-k-1)}{l! \Gamma(l-k)} (1 - e^{-a})^k. \]  

(4)

The sum in the integrand of (4) is kind of Appell function and it appears that one cannot get much further in general. However, for \( a = 0 \), the \( k \)-sum drops out and the \( l \)-sum can be rearranged into a \( {}_3 F_2 \), giving

\[ L(0) = 1 + \frac{1}{2} \int_{0}^{1} dt (1-t) {}_3 F_2(2-t, 1, 1; 2, 3; 1). \]  

(5)

Since, for \( 1 < z < 2 \)

\[ {}_3 F_2(z, 1, 1; 2, 3; 1) = \frac{2}{z-1} [1 - \gamma - \psi(3 - z)], \]  

(6)

\[ \int_{0}^{1} x \psi(x + 1) dx = 1 - \ln \sqrt{2\pi}, \]

We find

\[ \int_{0}^{\pi/2} \frac{x^2}{x^2 + \ln^2(2 \cos x)} dx = \frac{\pi}{8} (1 - \gamma + \ln(2\pi)) \]  

(7)

The value (7) was first found by assuming that \( L(0) = A + B \gamma + C \ln \sqrt{2\pi} \) and finding the coefficients by trial and error.