SOME PROBLEMS IN NUMBER THEORY I: THE CIRCLE PROBLEM

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Introduction

This paper begins a series on the estimation within satisfactory tolerances of the number of character weighted lattice points in natural curved regions of Euclidean spaces and applications of the results and methods to the study of zeta- and L-functions. The study of the lattice in a circle represents a model for the subsequent treatment of the Dirichlet and other divisor problems.

C.F. Gauss [Gaus1] [Gaus2] showed that the number of lattice points in circles is the area, \( \pi t \), up to a tolerance of order at most the circumference, i.e. the square root, \( O(\sqrt{t}) \). Sierpinski in 1904, following work of Voronoi [Vor] on the Dirichlet Divisor Problem, lowered the exponent in Gauss’s result to 1/3 [Sierp1] [Sierp2]. Further improvements that have been obtained will be described below. This paper provides a proof of the conjectured optimal error bound, \( O(t^{1/4 + \epsilon}) \).

The present approach begins with a new asymptotic expression (9) for the number of lattice points in a circle. To obtain such an expression, one begins by embedding the integer lattice within a lattice refined by a factor \( Q \) that can be adjusted depending on the size of the circle. A convex polygon with vertices in the refined lattice approximates the circle. This number of lattice points can then be rewritten as a sum of standard trigonometric functions over the points of the refined lattice that lie within this polygon. An elementary generalized Euler-Maclaurin formula (the shortest possible such formula) for lattice sums in lattice polygons then provides an estimate of this sum in terms of certain integrals over the polygon. The lattice polytope structure permits such a formula in terms of Lebesgue measure. The integrals can in turn be approximated by integrals over circular regions; classically, such integrals yield Bessel functions. This procedure thus leads to an expression for the number of lattice points in a circle in terms of sums of Bessel functions or exponential sums. The expressions considered by Hardy, Landau and others have a similar form, but (9) differs in important features; for example, the sum is taken over lattice points in a square, whereas classically the summation pattern in asymptotic formulae for the lattice points in a circle retains the circular symmetry.

The classical approach to any such sum next calls for application of Poisson summation, followed by lots of estimates, with the goal of arriving at a situation in which the
method of stationary phase applies to transform the original problem into a new one. It turns out that only a portion of the summation has relevance to proving the conjectured error estimate; the rest contributes less than the desired error. The expression $E(t)$ of (10) contains a form of the crucial terms. The nature of the integrals to be summed over $(m, n)$ suggests dividing the domain of summation into regions near and far from the circle $m^2 + n^2 = t$. In this paper, (10) is studied directly without reference to (9), but the method for (9) and what precedes it plays a key role in the analysis of the sum over points near the circle, for example in the proof of Prop. 2.5. The final result, (17), is a complete analytic expression for $E(t)$ for a given division of the domain into near and far regions. Upon comparison of different choices of this division depending on a small enough parameter $\eta > 0$, some terms drop out and what remains is an estimate in the error term of the circle problem improved by a factor of $t^{-\eta}$ in comparison with a given previous estimate. After a sufficient number of iterations of this argument, the desired result follows.

Our efforts related to these problems began in the early '90's with results on general Euler-Maclaurin expansions for lattice convex polytopes, [CS4] [CS2] [S], and their relations to certain reciprocity laws for some generalized Dedekind sums, extending known connections between lattice problems and certain Dedekind sums [Mo][P]. The possibility of such applications provided an impetus for investigations of Euler-Maclaurin expansions and possible number theory connections. Our approach generalized methods we had earlier used to compute all coefficients of the Hilbert-Ehrhardt polynomials that count the number of lattice points of convex lattice polytopes [CS1]. These methods relied on our computations of the Todd classes of toric varieties, in turn based on our general formulae to compute (many kinds of) characteristic classes of singular varieties in various topological and algebraic-geometric settings [CS1] [CS3] (see also [CMS1,2], [CLMS]). Higher dimensional analogues of the Euler-MacLaurin formula will appear in subsequent papers in this series. Some other valuable modern contributions on lattice points and Euler-MacLaurin formulae from differing perspectives include [BV] [DR][K][Kh][KK][KSW][M].

A summary of earlier work: A shorter proof of the Sierpinski estimate was given by Landau [Lan2] [Lan3] in 1912, after making rigorous a heuristic geometrical method of Pfeiffer [Pfe] of 1886. Hardy [H1915] [H1916a] and Landau showed in 1915 that the error bound could not be lowered to $O(t^{1/4})$. The Hardy-Landau formula [L] for lattice points in circular regions constituted the basis of subsequent efforts. New methods of estimating exponential sums were introduced by Hermann Weyl, and by Van der Corput [VdC1] [VdC2], who in 1923 gave exponents slightly below 1/3 for Gauss’s circle problem. His estimate of 33/106 was subsequently improved in work of his student L.W. Nieland [Nie] to 27/82= 0.329268... , which Van der Corput himself had obtained for the Dirichlet Divisor Problem. Littlewood and Walfisz [LW] shortly thereafter lowered the estimate, initially obtaining 37/112 = 0.33035..., followed by further small improvements [Wa1] [Wa2] [Wa3]. Further improvements were obtained by Vinogradov [V] in 1932 to 34/106 = 0.32075... , and by Titchmarsh in 1934 [Ti]. Other results obtained include Hua Loo Keng [Keng1] in 1942 of 13/40 = 0.325; Yin Wen Lin [Liu] in 1962 and Chen Jing Run [Run] in 1963 of 12/37 = 0.324324...; and Nowak [Now] in 1984
of \(35/108 = 0.324074...\). The theory of exponential pairs gave a systematic account of some approaches to exponential sums; see [GK]. G. Kolesnik [Kol] obtained the exponent bound of \(139/429 = 0.324009324...\). Using methods introduced by E. Bombieri and H. Iwaniec [BI] and further development, Iwaniec and C.J. Mazzochi [IM] obtained in 1988 the then best bound of \(7/22 = 0.3181818...\). M. Huxley [Hu1990] improved this estimate to \(46/146 = .31507...,\) and then slightly further in [Hu1996][Hu2001].

The present paper represents a considerable simplification and shortening of a version of several years ago. We benefitted from comments from colleagues at various points, including Enrico Bombieri for surveying for us the theory of exponential pairs, Peter Sarnak on several occasions, David Kazhdan, M. Huxley, V. Guillemin and S. Sternberg, Steve Shatz, and Edward Y. Miller for a careful reading of this version in a slightly longer form.

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1. An Approximate Euler MacLaurin Formula in the Plane

Let \(\mathbb{Z}^2 \subset \mathbb{R}^2\) be the lattice of integer points in the plane. Let

\[
\psi_1(t) = \begin{cases} 
    t - \lfloor t \rfloor - \frac{1}{2} & x \notin \mathbb{Z} \\
    0 & x \in \mathbb{Z}
\end{cases}
\]

Let \(\pi_i, i = 1, 2\) be projection on the first and second coordinates. If \(v = (v_1, v_2)\), let \(v^i = (v_2, v_1)\).

**Theorem.** Let \(\Delta\) be a convex polytope in \(\mathbb{R}^2\) with vertex points in \(\mathbb{Z}^2\). Let \(E_1, \ldots, E_k\) be the edges of \(\Delta\), going counter-clockwise around the boundary, and let \(n_j \in \mathbb{Z}^2\) be the minimal integral outward vector normal to \(E_j\). Let \(E_j\) have normalized Lebesgue measure so that the interval between two adjacent lattice point has measure one. Let \(f\) be a \(C^2\) function on \(\mathbb{R}^2\). Assume that \(f(x, y) = f(-y, x)\), and that \(\Delta\) is symmetric with respect to rotation through \(\frac{\pi}{2}\).

Let

\[
T(\Delta, f) = \int_{\Delta} f + \int_{\Delta} f_x \cdot (\psi_1 \circ \pi_1) + \int_{\Delta} f_y \cdot (\psi_1 \circ \pi_2) + \int_{\Delta} f_{xy} \cdot (\psi_1 \circ \pi_1)(\psi_1 \circ \pi_2) - \sum_{j=1}^{k} \pi_1 n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_1) - \sum_{j=1}^{k} \pi_2 n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_2) - \frac{1}{2} \sum_{j=1}^{k} \int_{E_j} D_{n_j} f \cdot (\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2). \]

Then

\[
\left| \sum_{(x,y) \in \mathbb{Z}^2 \cap \Delta} f(x, y) - T(\Delta, f) \right| \leq \sum_{(x,y) \in \mathbb{Z}^2 \cap \partial \Delta} |f(x, y)|.
\]
To prove this, let $\delta$ be the unit distribution on the line, concentrated (symmetrically) at integer points. For example, $\delta$ may be thought of as the formal limit $\delta = \lim_{\epsilon \to 0^+} \delta_\epsilon$, where $\delta_\epsilon(t) = \epsilon^{-1}$ for $t$ within $\frac{\epsilon}{2}$ of an integer and zero otherwise. Let

$$w : \mathbf{Z}^2 \cap \Delta \to (0, 1]$$

be a weighting of the lattice points in the polytope defined as follows: If $(m,n)$ is in the interior of $\Delta$, then $w(m,n) = 1$; if $(m,n)$ lies on an edge but is not a vertex, then $w(m,n) = \frac{1}{2}$; and if $(m,n)$ is a vertex, then $w(m,n)$ is the proportion of the area of a small rectilinear square $D$ centered at $(m,n)$ that lies within $\Delta$. (Because of integrality this is the same as the area of $\Delta \cap \bar{D}$ if $\bar{D}$ is a unit square.) Then

$$\sum_{(x,y) \in \mathbf{Z}^2 \cap \Delta} w(x,y) f(x,y) = \int_{\Delta} f \cdot (\delta \circ \pi_1) \cdot (\delta \circ \pi_2).$$

In this and the rest of this proof, integration in the plane refers to Lebesgue integration in the plane with the unit square having unit measure. We may write

$$\int_{\Delta} f \cdot (\delta \circ \pi_1) \cdot (\delta \circ \pi_2) = \int_{\Delta} f - \int_{\Delta} f \cdot (1 - \delta \circ \pi_1) - \int_{\Delta} f \cdot (1 - \delta \circ \pi_2) + \int_{\Delta} f \cdot (1 - \delta \circ \pi_1) \cdot (1 - \delta \circ \pi_2).$$

Then

$$\int_{\Delta} f \cdot (1 - \delta \circ \pi_1) = - \int_{\Delta} f_x \cdot (\psi_1 \circ \pi_1) + \sum_{j=1}^k \pi_1 n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_1).$$

Similarly,

$$\int_{\Delta} f \cdot (1 - \delta \circ \pi_2) = - \int_{\Delta} f_y \cdot (\psi_1 \circ \pi_2) + \sum_{j=1}^k \pi_2 n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_2).$$

Similarly, by averaging the effects of integrating with respect to $x$ and $y$,

$$\int_{\Delta} f \cdot (1 - \delta \circ \pi_1) \cdot (1 - \delta \circ \pi_2) =$$

$$\int_{\Delta} f_{xy} \cdot (\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2) - \frac{1}{2} \sum_{j=1}^k \int_{E_j} D_{n_j} f \cdot (\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2)$$

$$- \sum_{j=1}^k \pi_1 n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_1) \cdot (1 - \delta \circ \pi_2) - \sum_{j=1}^k \pi_2 n_j \int_{E_j} f \cdot (1 - \delta \circ \pi_1) \cdot (\psi_1 \circ \pi_2).$$

The last two terms can be combined as
\[ \sum_{j=1}^{k} \pi_1 n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_1) \cdot (1 - \delta \circ \pi_2) + \sum_{j=1}^{k} \pi_2 n_j \int_{E_j} f \cdot (1 - \delta \circ \pi_1) \cdot (\psi_1 \circ \pi_2) \]

\[ = \sum_{j=1}^{k} \int_{E_j} f \cdot D_{n_j}((\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2)). \]

Since \( \psi_1(-y)\psi_1(x) = -\psi_1(x)\psi_1(y), \)

\[ D_{\nu}((\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2))(-y, x) = -D_{\nu}((\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2))(x, y). \]

Hence, by the symmetry of \( f \) and the polytope, this sum vanishes. This completes the proof of (1.1)

**Notes:**
1. Although an integral polytope structure is crucial in (1.1), convexity is not being used in any essential way.
2. Clearly there is an exact formula

\[ \sum_{(x, y) \in \mathbb{Z}^2 \cap \Delta} w(x, y)f(x, y) = T(\Delta, f) \]

for the weighted sum introduced in the proof. This is the first, and shortest, of a family or Euler-MacLaurin formulae that can be obtained from repeated applications of the same argument and, essentially, integration by parts, all to appear in [CS 4]. Our earlier approaches used our study of the topology of toric varieties (see [CS 2][S]).

2. Lattice Points in the Circle

Let \( P(t) \) denote the number of integral points \((m, n)\) in the disk \(D_t\) in the plane of radius \(\sqrt{t}\), i.e. the number of lattice points satisfying

\[ m^2 + n^2 \leq t. \]

**2.1 Theorem.** For any \( \epsilon > 0 \),

\[ P(t) = \pi t + O(t^{\frac{3}{2}+\epsilon}). \]

It is known [L] that it is not true that

\[ |P(t) - \pi t| \leq O(t^{\frac{3}{2}}). \]

To prove

\[ P(t) \leq \pi t + O(t^{\frac{3}{2}+\epsilon}), \]

we will begin with a new formula for the number of lattice points in a circle.
Let \( Q > 1 \) be an odd integer, and let \( \Delta = \Delta_{t,Q} \) be the polytope spanned by the integral points satisfying
\[
\left( \frac{m}{Q} \right)^2 + \left( \frac{n}{Q} \right)^2 \leq t.
\]

Let \( R = \frac{Q-1}{2} \) and let
\[
f(x, y) = \frac{1}{Q^2} \sum_{p=-R}^{R} \sum_{q=-R}^{R} \exp \frac{2\pi i}{Q}(px + qy),
\]
Then
\[
P(t) = \sum_{\Delta \cap \mathbb{Z}^2} f(x, y).
\]

It is well known that a circle of radius at most \( \sqrt{t} \) that has any lattice points on it has less than \( O(t^{1/2}) \) lattice points. Furthermore, for \( n \) a positive integer,
\[
\sqrt{n} - \sqrt{n-1} \geq \frac{1}{2\sqrt{n}},
\]
and it is easy to see that the boundary of \( \Delta \) lies between the circle of radius \( Q\sqrt{t} - 1 \) and the circle of radius \( Q\sqrt{t} \). (For example, if \( (a, b) \) is a lattice point in the interior of the first quadrant and at most \( Q\sqrt{t} - 1 \) units from the origin, then the triangle with vertices \( (a, b), (a+1, b)(a, b+1) \) and the three unit squares with \( (a, b) \) as a vertex and not containing this triangle all lie within \( \Delta \).) Hence, by (1.1),
\[
|P(t) - T(\Delta, f)| \ll \frac{t^{1/2} + \epsilon}{Q}.
\]
(3)

(In the sequel we use the notation \( f \ll g \) and \( |f| \leq O(g) \) interchangeably.)

A non-horizontal edge \( E_j \) has the description \( E_j = \{ (L_jy, y) | c_j \leq y \leq d_j \} \), \( L_j \) a linear function, and a non-vertical edge the description \( E_j = \{ (x, M_jx) | a_j \leq y \leq b_j \} \). Then

\[
\sum_{j=1}^{k} \pi_1n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_1) = \sum_{j} \frac{1}{Q^2} \sum_{p,q=-R}^{R} \frac{\pi_1n_j}{\pi_2n_j} \int_{c_j}^{d_j} e\left(\frac{pL_jy + qy}{Q}\right)\psi_1(L_jy) \, dy,
\]
the sum on the right being over the non-horizontal edges. Here and in the sequel \( e(t) = \exp(2\pi it) \). Similarly

\[
\sum_{j=1}^{k} \pi_2n_j \int_{E_j} f \cdot (\psi_1 \circ \pi_2) = \sum_{j} \frac{1}{Q^2} \sum_{p,q=-R}^{R} \frac{\pi_2n_j}{\pi_2n_j} \int_{a_j}^{b_j} e\left(\frac{px + qM_jx}{Q}\right)\psi_1(M_jx) \, dx,
\]
over the non-vertical edges, and

\[
\sum_{j=1}^{k} \int_{E_j} D_{n_j} f \cdot (\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2) = \\
\sum_{j=1}^{k} \int_{E_j} D_{n_j} f \cdot (\psi_1 \circ \pi_1) \cdot (\psi_1 \circ \pi_2)
\]

\[
= \sum_j \frac{1}{Q^2} \int_{c_{\pi n_j}} \int_{a_{\pi n_j}} e\left(\frac{p\pi n_j y + qy}{Q}\right) \psi_1(y) \psi_1(L_j y) dy
\]

\[
+ \sum_j \frac{1}{Q^2} \int_{c_{\pi n_j}} \int_{a_{\pi n_j}} e\left(\frac{px + q\pi n_j x}{Q}\right) \psi_1(x) \psi_1(M_j x) dx,
\]

the first sum also over the non-horizontal edges and the second over the non-vertical ones.

Let \( \Lambda = \Lambda_{t, Q} = Q^{-1} \Delta \) be the polytope spanned by the points of

\[
Z[1/Q] \oplus Z[1/Q] \cap \{x^2 + y^2 \leq t\}.
\]

Then, with \( L_j y = Q^{-1} L_j y \) and \( M_j y = Q^{-1} M_j x \),

\[
T(\Delta, f) = \sum_{p,q} \left[ \int_{\Lambda} e(px + qy) + \frac{2\pi i p}{Q} \int_{\Lambda} e(px + qy) \psi_1(Qx) \\
+ \frac{2\pi i q}{Q} \int_{\Lambda} e(px + qy) \psi_1(Qy) - \frac{4\pi^2 pq}{Q^2} \int_{\Lambda} e(px + qy) \psi_1(Qx) \psi_1(Qy) \\
- \sum_j \frac{\pi n_j}{Q |\pi n_j|} \int_{c_{\pi n_j}} \int_{a_{\pi n_j}} \frac{e(p\pi n_j y + qy) \psi_1(Q\pi n_j y)}{Q} dy \\
- \sum_j \frac{\pi n_j}{Q |\pi n_j|} \int_{c_{\pi n_j}} \int_{a_{\pi n_j}} \frac{e(px + q\pi n_j x) \psi_1(Q\pi n_j x)}{Q} dx \\
- \sum_j \frac{\pi n_j}{Q^2 |\pi n_j|} \int_{c_{\pi n_j}} \int_{a_{\pi n_j}} \frac{e(p\pi n_j y + qy) \psi_1(Q\pi n_j y) \psi_1(Qy)}{Q^2} dy \\
- \sum_j \frac{\pi n_j}{Q^2 |\pi n_j|} \int_{c_{\pi n_j}} \int_{a_{\pi n_j}} \frac{e(px + q\pi n_j x) \psi_1(Qx) \psi_1(Q\pi n_j x)}{Q^2} dx \right].
\]

Let \( F(t, Q) \) be given by the same formula, but with the domain of integration replaced by the disk \( D_t \) of radius \( t \) and the boundary with the circle of this radius, i.e.
\[
F(t, Q) = \sum_{p,q} \left[ \int_{D_t} e(px + qy) + \frac{2\pi ip}{Q} \int_{D_t} e(px + qy)\psi_1(Qx) \right. \\
+ \frac{2\pi iq}{Q} \int_{D_t} e(px + qy)\psi_1(Qy) - \frac{4\pi^2 pq}{Q^2} \int_{D_t} e(px + qy)\psi_1(Qx)\psi_1(Qy) \\
- \frac{1}{Q} \int_{-\sqrt{t}}^{\sqrt{t}} e(p\sqrt{t-y^2} + qy)\psi_1(Q\sqrt{t-y^2})dy \\
+ \frac{1}{Q} \int_{-\sqrt{t}}^{\sqrt{t}} e(-p\sqrt{t-y^2} + qy)\psi_1(-Q\sqrt{t-y^2})dy \\
- \frac{1}{Q} \int_{-\sqrt{t}}^{\sqrt{t}} e(px + q\sqrt{t-x^2})\psi_1(Q\sqrt{t-x^2})dx \\
+ \frac{1}{Q} \int_{-\sqrt{t}}^{\sqrt{t}} e(px - q\sqrt{t-x^2})\psi_1(-Q\sqrt{t-x^2})dx \\
- \frac{\pi i q}{Q^2} \int_{-\sqrt{t}}^{\sqrt{t}} e(p\sqrt{t-y^2} + qy)\psi_1(Qy)\psi_1(Q\sqrt{t-y^2})dy \\
+ \frac{\pi i q}{Q^2} \int_{-\sqrt{t}}^{\sqrt{t}} e(-p\sqrt{t-y^2} + qy)\psi_1(Qy)\psi_1(-Q\sqrt{t-y^2})dy \\
- \frac{\pi i p}{Q^2} \int_{-\sqrt{t}}^{\sqrt{t}} e(px + q\sqrt{t-x^2})\psi_1(Qx)\psi_1(Q\sqrt{t-x^2})dx \\
+ \frac{\pi i p}{Q^2} \int_{-\sqrt{t}}^{\sqrt{t}} e(px - q\sqrt{t-x^2})\psi_1(Qx)\psi_1(-Q\sqrt{t-x^2})dx \right].
\]

\[(2.2)\text{PROPOSITION. For any given } \epsilon > 0\]

\[| T(\Delta, f) - F(t, Q) | \ll \left( \frac{t^{1/2+\epsilon}}{Q} + t^{1/4+\epsilon} \right) \ln^2 Q. \]

\textbf{Proof of (2.2).} We start with the area integrals. For these, it will suffice to show that modulus of the functions obtained from the area integrals on the right side of (4) or (5), but with the region of integration replaced by the compact region $C$ between $\Lambda$ on the inside and the boundary of $D_t$ on the outside, is bounded as in the statement of the proposition.

In general,

\[
\frac{1}{Q^j} \left| \sum_{p=-R}^{R} p^j \exp(2\pi ipx) \right| \ll \min\{Q, \|x\|^{-1}\},
\]

where $\|x\|$ is the distance from $x$ to the nearest integer. As is well-known, this holds for $j = 0$ by summation of a geometric series. For $j > 0$, it is proven inductively using
\[
\frac{1}{2\pi i} \frac{d}{dx} \sum_{1}^{N} \exp(2\pi i px) = \sum_{j=0}^{N-1} \exp(2\pi i jx) \sum_{p=1}^{N-j} \exp(2\pi ipx) .
\]

Further, \( C \) is contained in the annulus \( A \) with thickness \( \sqrt{2}/Q \) and outer radius \( \sqrt{t} \). Hence it will be enough to show that

\[
\int_{A} \min\{Q, \|x\|^{-1}\} \min\{Q, \|y\|^{-1}\} \ll \left( \frac{t^{1/2+\epsilon}}{Q} + t^{1/4+\epsilon} \right) \ln^2 Q . \tag{*}
\]

Note that for \( M \) an integer and \( M \leq a < b \leq M + 1 \),

\[
\int_{a}^{b} \min\{Q, \|x\|^{-1}\} dx \leq 2 \int_{M}^{M+1/Q} Q dx + 2 \int_{M+1/2}^{M+1/Q} \frac{dx}{x-M} \ll \ln Q .
\]

From this it is not difficult to obtain, for \( B \) a unit square with lattice point vertices, the estimate

\[
\int_{A \cap B} \min\{Q, \|x\|^{-1}\} \min\{Q, \|y\|^{-1}\} \ll \ln^2 Q .
\]

Let \( \beta \) be the collection of all unit squares having a vertex within distance \( t^{-1/4} \) of the annulus \( A \). By the same considerations as for (2) above, the number of lattice points within this distance of \( A \) is at most

\[
O \left( \frac{t^{1/2+\epsilon}}{Q} + \frac{(\sqrt{t} + t^{-1/4})^{1+2\epsilon}}{t^{1/4}} \right) = O \left( \frac{t^{1/2+\epsilon}}{Q} + t^{1/4+\epsilon} \right) .
\]

Therefore, if

\[
C_0 = A \cap \bigcup_{\beta} B ,
\]

then

\[
\int_{C_0} \min\{Q, \|x\|^{-1}\} \min\{Q, \|y\|^{-1}\} \ll \left( \frac{t^{1/2+\epsilon}}{Q} + t^{1/4+\epsilon} \right) \ln^2 Q .
\]

On the other hand, for \( B \) a unit square with lattice point vertices, such that \( A \cap B \) stays at least a distance \( a^{-1} \) from the vertices, similar considerations, using polar coordinates, yield

\[
\int_{A \cap B} \min\{Q, \|x\|^{-1}\} \min\{Q, \|y\|^{-1}\} \ll \frac{a}{Q} \ln Q .
\]

For \( a > b \geq 1 \), the number of lattice points whose distance from either the inner edge or the outer edge of the annulus \( A \) is in the interval \([a^{-1}, b^{-1}]\) is at most \( O(t^{1/2+\epsilon}(b^{-1} - a^{-1})) \). Hence, if \( \beta(a, b) \) is the collection of unit squares with lattice point vertices with the closest vertex to \( A \) in this set of lattice points, and
\[ C(a, b) = A \cap \bigcup_{\beta(a, b)} B, \]

then

\[
\int_{C(a, b)} \min\{Q, \|x\|^{-1}\} \min\{Q, \|y\|^{-1}\} \ll \ln Q \left( \frac{t^{1/2+\epsilon}}{Q} \left( \frac{a}{b} - 1 \right) \right).
\]

Let \( N = N(\epsilon) \) be the largest integer such that \( 1/4 - N\epsilon > 0 \). Then

\[
A = C_0 \cup \bigcup_{1}^{N} C(t^{1/4-(k-1)\epsilon}, t^{1/4-k\epsilon}) \cup C(t^{1/4-N\epsilon}, 1).
\]

Hence

\[
\int_{A} \min\{Q, \|x\|^{-1}\} \min\{Q, \|y\|^{-1}\} \ll \left( N(\epsilon) \frac{t^{1/2+2\epsilon}}{Q} + t^{1/4+\epsilon} \right) \ln^2 Q
\]

which clearly implies (*), with \( \epsilon \) replaced by \( 2\epsilon \), thereby completing the proof of (2.2) for the area integrals.

Next we consider the boundary integrals. Let

\[
\psi_N^*(x) = \sum_{|n|=1}^{N} \gamma_n e(nx)
\]

be Vaaler’s approximation of \( \psi_1(x) \) by an exponential polynomial; see the final appendix of [GK] or [V]. The coefficients satisfy \( |\gamma_n| \ll n^{-1} \) and

\[ |\psi_N^*(x) - \psi_1(x)| \ll N^{-1} \]

for \( x \notin \mathbb{Z} \). This means that we can replace \( \psi_1(\pm Q\sqrt{t-y^2}) \) with \( \psi_N^*(\pm Q\sqrt{t-y^2}) \) and \( \psi_1(\pm Q\sqrt{t-x^2}) \) with \( \psi_N^*(\pm Q\sqrt{t-x^2}) \), in the integrals we are considering, with an error no greater that \( \frac{\sqrt{t}}{Q} \), provided \( N \geq Q^2 \). (This follows by estimating the exponential times \( \psi_1(\pm Qx) \) or \( \psi_1(\pm Qy) \) by unity; by instead summing up a geometric series we could reduce \( N \) to \( Q \) at the cost of at this point multiplying the error by \( \ln Q \).)

Let \( E_j \) be a non-horizontal edge. \( E_j \) will necessarily lie entirely on one side of the \( y \)-axis, say on the positive side. Considering one of the boundary terms, for example, we have

\[
\int_{\frac{d_j}{Q}}^{\frac{d_j}{Q}} \frac{dx}{\sqrt{t-x^2}} \psi_1(Qy)dy = \int_{\frac{d_j}{Q}}^{\frac{d_j}{Q}} e((p + nQ)\sqrt{t-y^2} + qy)\psi_1(Qy)dy
\]

\[
= \int_{\frac{d_j}{Q}}^{\frac{d_j}{Q}} \int_{L_jy} 2\pi i(p + nQ)e((p + nQ)x + qy)\psi_1(Qy)dxdy.
\]
Therefore,
\[
\int_{-\bar{c}}^{\bar{c}} e(p\sqrt{t - y^2} + qy)\psi_1(Q\sqrt{t - y^2})\psi_1(Qy)dy \\
- \sum_j \int_{\frac{d_j}{Q}}\psi_1(Q\hat{L}_j y + qy)\psi_1(Qy)dy \\
= \sum_{n=1}^{Q^2} \gamma_n \int_{C_+} 2\pi i (p + nQ)e((p + nQ)x + qy)\psi_1(Qy) + O(\sqrt{t}).
\]
Here the sum is over edges to the right of the y-axis. Here \(\bar{c}\) is the greatest rational number that is at most \(\sqrt{t}\) and has denominator dividing \(Q\), and \(C_+\) is the part of \(C\) to the right of the y-axis and with \(-\bar{c} \leq y \leq \bar{c}\). Notice that the error in replacing \(\sqrt{t}\) with \(\bar{c}\) is trivially at most \(O(1)\).

The argument for the desired estimate is now the same as for the area integrals above, except for the following modification. As above, let \(M \leq a \leq b \leq M + 1\). Write
\[
\int_a^b \sum_{p=-R}^{R} \frac{p+nQ}{Q} e(p+nQ)dx = \int_a^b \sum_{p=-R}^{R} \frac{p}{Q} e(nQx)dx + \int_a^b \sum_{p=-R}^{R} ne(nQ)e(px)dx.
\]
As above,
\[
\left| \int_a^b \sum_{p=-R}^{R} \frac{p}{Q} e(nQx)dx \right| \ll \int_a^b \min\{Q, \|x\|^{-1}\} dx \ll \ln Q.
\]
For the second integral,
\[
\int_a^b \sum_{p=-R}^{R} ne(nQ)e(px)dx = \int_a^b ne\left(\frac{(2n-1)Q+1}{2} x\right) e(Qx) - 1\ e(x) - 1\ dx.
\]
The function \(\frac{e(Qx)-1}{e(x)-1}\) within a unit interval has total variation \(O(Q\ln Q)\). Hence, by integration by parts
\[
\left| \int_a^b \sum_{p=-R}^{R} ne(nQ)e(px)dx \right| \ll \frac{2nQ}{(2n-1)Q + 1}\ln Q \ll \ln Q
\]
as well. The rest of the argument proceeds as before, and similarly for the other edges and the other terms on the boundary; note that the summation over \(n\) leads to another factor of \(\ln Q\), as \(|\gamma_n| \ll n^{-1}\). This completes the argument for (2.2).

The Fourier series
\[
\psi_1(t) = -\frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\sin 2\pi \nu t}{\nu} = -\frac{1}{2\pi i} \sum_{\nu=1}^{\infty} \frac{e(\nu t) - e(-\nu t)}{\nu}
\]
can be substituted for $\psi_1$ in (5). Then the explicit integration formula
\[
\int_{-\sqrt{t}}^{\sqrt{t}} e(p\sqrt{t - y^2} + qy) dy - \int_{-\sqrt{t}}^{\sqrt{t}} e(-p\sqrt{t - y^2} + qy) dy = 2\pi ip \int_{D_t} e(px + qy) dxdy
\]
and the same thing with respect to $y$ can be used to replace the boundary terms by area integrals. Some terms cancel, so that what results is the formula
\[
F(t, Q) = \sum_{p,q} \left[ \int_{D_t} e(px + qy) - \frac{1}{2Q} \sum_{\mu \neq 0, \nu \neq 0} \left( \frac{p}{\mu} + \frac{q}{\nu} \right) \int_{D_t} e((p + \mu Q)x + (q + \nu Q)y) \right. \\
\left. + \sum_{\mu \neq 0} \int_{D_t} e((p + \mu Q)x + qy) + \sum_{\nu \neq 0} \int_{D_t} e(px + (q + \nu Q)y) \right].
\]
From symmetry considerations, the third and fourth term are equal when summed over $p$ and $q$, so that
\[
F(t, Q) = \sum_{p,q} \left[ \int_{D_t} e(px + qy) + 2 \sum_{\mu \neq 0} \int_{D_t} e((p + \mu Q)x + qy) - \frac{1}{2Q} \sum_{\mu \neq 0, \nu \neq 0} \left( \frac{p}{\mu} + \frac{q}{\nu} \right) \int_{D_t} e((p + \mu Q)x + (q + \nu Q)y) \right].
\]
We recall a little from the theory of Bessel functions.
\[
\int_{D_t} e(ax + by) = \int_{0}^{\sqrt{t}} r \int_{0}^{2\pi} e(ar \cos \theta + br \sin \theta) d\theta dr = \int_{0}^{\sqrt{t}} r \int_{0}^{2\pi} e(r \sqrt{a^2 + b^2} \sin(\theta + \tau)) d\theta dr,
\]
with $\sin \tau = \frac{a}{\sqrt{a^2 + b^2}}$ and $\cos \tau = \frac{b}{\sqrt{a^2 + b^2}}$. Hence, by [W, 2.2 (p 20), (7)], the evenness of $J_0$ (e.g. [W,2.2,(2)]), and [W, 2.12, (5)],
\[
\int_{D_t} e(ax + by) = 2\pi \int_{0}^{\sqrt{t}} r J_0(2\pi r \sqrt{a^2 + b^2}) dr = \sqrt{\frac{t}{a^2 + b^2}} J_1(2\pi \sqrt{a^2 + b^2}).
\]
Thus
\[ F(t, Q) = \sum_{p,q=-R}^{R} \sqrt{\frac{t}{p^2 + q^2}} J_1(2\pi \sqrt{t(p^2 + q^2)}) \]

\[ + 2 \sum_{\mu \neq 0} \sqrt{\frac{t}{(p + \mu Q)^2 + q^2}} J_1(2\pi \sqrt{t((p + \mu Q)^2 + q^2)}) \]

\[ - \frac{1}{2Q} \sum_{\mu, \nu \neq 0, \mu \neq \nu} \left( \frac{p}{\mu} + \frac{q}{\nu} \right) \sqrt{\frac{t}{(p + \mu Q)^2 + (q + \nu Q)^2}} J_1(2\pi \sqrt{t((p + \mu Q)^2 + (q + \nu Q)^2)}) \]

(7)

This expression can be simplified further. The term in (7) corresponding to \( p = q = 0 \) is

\[ \lim_{a \to 0} \sqrt{\frac{t}{a}} J_1(2\pi \sqrt{ta}) = \pi t. \]

Further, it is well known that \( |J_1(z)| \ll |z|^{-\frac{1}{2}} \), so that

\[ \left| \sum_{p} \frac{\sqrt{t}}{p} J_1(p\sqrt{t}) \right| \ll t^{\frac{1}{4}}. \]

Therefore, we can rewrite (7) as

\[ F(t, Q) = \pi t + 4 \sum_{p,q=1}^{R} \sqrt{\frac{t}{p^2 + q^2}} J_1(2\pi \sqrt{t(p^2 + q^2)}) \]

\[ + 2 \sum_{\mu \geq 1} \sqrt{\frac{t}{(p + \mu R)^2 + q^2}} J_1(2\pi \sqrt{t((p + \mu R)^2 + q^2)}) \]

\[ - \frac{1}{2Q} \sum_{\mu, \nu \geq 1} \left( \frac{p}{\mu} + \frac{q}{\nu} \right) \sqrt{\frac{t}{(p + \mu R)^2 + (q + \nu R)^2}} J_1(2\pi \sqrt{t((p + \mu R)^2 + (q + \nu R)^2)}) \]

\[ + O(t^{\frac{1}{4}}). \]

(8)

**Note:** In the sum over \( \mu \) inside the brackets, we can restrict to \( \mu \leq Q \), (or \( \leq R \)) without increasing the overall error beyond \( O(t^{\frac{1}{4}}) \). For the last sum in the brackets, restricting to \( \mu, \nu \leq Q \) adds an error of \( O(t^{\frac{1}{4}} \ln t) \). This will be important later. To prove the second statement, for example, from the estimate on \( J_1(z) \) above, it follows that this error will be at most a constant multiple of

\[ t^{\frac{1}{4}}R^{\frac{1}{2}} \sum_{\mu \text{ or } \nu > Q} \left( \frac{1}{\mu} + \frac{1}{\nu} \right) \frac{1}{(\mu^2 + \nu^2)^{\frac{1}{4}}}. \]

To estimate the sum, it suffices to consider the case \( \nu \geq \mu \).
\[ \sum_{\nu \geq \mu \geq Q} \frac{1}{\mu} + \frac{1}{\nu} \frac{1}{(\mu^2 + \nu^2)^{\frac{3}{4}}} \]

\[ = \sum_{\nu > Q \geq \mu} \frac{1}{\mu} + \frac{1}{\nu} \frac{1}{(\mu^2 + \nu^2)^{\frac{3}{4}}} + \sum_{\mu, \nu > Q} \frac{1}{\mu} + \frac{1}{\nu} \frac{1}{(\mu^2 + \nu^2)^{\frac{3}{4}}} \]

\[ \ll \sum_{\nu > Q \geq \mu} \frac{1}{\mu} \frac{1}{\nu^2} + \sum_{\mu, \nu > Q} \frac{1}{\mu} \frac{1}{\nu^2} \ll \frac{1}{\sqrt{Q}} + \frac{\ln Q}{\sqrt{Q}}. \]

which implies the desired estimate.

Thus, we obtain the result:

(2.3) Proposition.

\[ P(t) = \pi t + 4 \sum_{p,q=1}^{R} \sqrt{\frac{t}{p^2 + q^2}} J_1(2\pi \sqrt{t(p^2 + q^2)}) \]

\[ + 2 \sum_{\mu \geq 1} \sqrt{\frac{t}{(p + \mu R)^2 + q^2}} J_1(2\pi \sqrt{t((p + \mu R)^2 + q^2)}) \]

\[ - \frac{1}{2Q} \sum_{1 \leq \mu, \nu \leq R} \frac{p + q}{\mu} \frac{1}{\nu} \sqrt{\frac{t}{(p + \mu R)^2 + (q + \nu R)^2}} J_1(2\pi \sqrt{t((p + \mu R)^2 + (q + \nu R)^2)}) \]

\[ + O\left(t^{\frac{1}{4}} \ln t + \frac{t^{\frac{4}{1} + \epsilon}}{Q}\right). \]

(9)

**Remark.** From just the above, classical estimates on exponential sums lead quickly to the corresponding classical results on the circle problem. For example, the trivial estimates yield upon setting \(Q = t^{1/6}\) the result

\[ P(t) = \pi t + O(t^{\frac{5}{4} + \epsilon}). \]

In general for \((k, l)\) an exponential pair (see [GK]), we get

\[ P(t) = \pi t + O(t^{\frac{k + l}{2(k + l + 1)} + \epsilon}). \]

If \((k, l, k, l)\) is an exponential quadruple (conjecturally all pairs give such) then a two-dimensional dyadic subdivision argument gives the improvement

\[ P(t) = \pi t + O(t^{\frac{2k + 2l - 1}{4l + 2} + \epsilon}). \]

For example, the known pair \((1/30, 26/30)\) leads by the two methods to the exponents \(27/82 + \epsilon\) and \(12/37 + \epsilon\) respectively, in the error term. However, in the sequel we do not use deep results on exponential pairs.
We will begin the actual proof of the main result by considering a certain expression. Let $J$ denote the set of integers $(m, n)$ with

$$1 \leq m \leq \sqrt{\frac{t}{2}}, \quad \sqrt{\frac{t}{2}} \leq n \leq t.$$  

The expression is

$$E(t) = \sum_{(m,n) \in J} \frac{t}{mn} \int_{1}^{R} \int_{0}^{\pi} r^{-\frac{1}{2}} v(\theta)(\sin \theta)^{\frac{1}{2}} \cos \theta \cos 2\pi (r(\sqrt{t} \csc \theta - m \cot \theta - n) + \frac{3}{8}) \, d\theta \, dr. \quad (10)$$

Here $v$ is a bump function that is unity for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ and zero for $0 \leq \theta \leq \frac{\pi}{8}$ and $\frac{3\pi}{4} \leq \theta \leq \pi$. Most results to follow will be true for general $Q$ and $R$ but in applications it will be assumed $R = \left[ \frac{t^{\frac{1}{2}} + \epsilon}{2} \right]$.

We will expand the inner integral $I = I(r, m, n)$ using the stationary phase result as given in [H, 7.7.5]. The simplest way to do this is to set

$$\csc \theta = \cosh x \quad \cot \theta = \sinh x.$$  

Then $d\theta = \text{sech} x \, dx$ and

$$I = \int_{-\infty}^{\infty} v_1(x) \tanh x (\text{sech})^{\frac{3}{2}}(x) \cos 2\pi (r(\sqrt{t} \cosh(x - A) - n) + \frac{3}{8}) \, dx.$$  

In this expression, $v_1(x) = v(\theta)$ and

$$e^A = \frac{\sqrt{t + m}}{\sqrt{t - m^2}}.$$  

Clearly

$$I = \int_{-\infty}^{\infty} v_1(x + A) \tanh(x + A) (\text{sech})^{\frac{3}{2}}(x + A) \cos 2\pi (r(\sqrt{t} \cosh x - n) + \frac{3}{8}) \, dx.$$  

From [H, 7.7.5], $I$ will be the real part $I = \Re I_1$ of

$$I_1 = \frac{e(r(\sqrt{t} - m^2) - n)}{r(t - m^2)\frac{1}{4}} \sum_{j=0}^{N-1} r^{-j} (t - m^2)^{-\frac{j}{4}} \sum_{\nu + j = j, 2\nu + 3\mu \geq 3\mu} i^{-j} 2^{-\nu} D^{2\nu - 3\mu} (\tanh \cdot (\text{sech})^{\frac{3}{2}})(-A)$$

$$+ O(r^{-N+\frac{1}{2}} (t - m^2)^{-\frac{N}{2}}). \quad (11)$$

In particular,
\[ I = \frac{m\sqrt{t - m^2}}{rt\frac{t}{4}} \cos 2\pi r(\sqrt{t - m^2} - n) \]
\[ + r^{-2}\left(\frac{m(t - m^2)^{\frac{3}{2}}}{4t\frac{t}{4}} + \frac{9m^3(t - m^2)^{\frac{1}{2}}}{t\frac{t}{4}}\right) \sin 2\pi r(\sqrt{t - m^2} - n) + O(r^{-\frac{5}{4}}(t - m^2)). \]

(12)

As a consequence
\[ I = \frac{m\sqrt{t - m^2}}{rt\frac{t}{4}} \cos 2\pi r(\sqrt{t - m^2} - n) + O(r^{-2}(t - m^2)^{\frac{3}{4}}). \]

(13)

Note similarly that by using one more term the remainder in (11) could be improved to \(O(r^{-3}(t - m^2)^{\frac{5}{4}})).\)

We will need some summation results related to the expression that appears in (13).

(2.5) **Proposition.** Let \(0 < \eta < \epsilon'.\) Let \(\alpha\) be an integer with \(\sqrt{\frac{t}{2}} - t^{\frac{1}{4}} - 1 \leq \alpha \leq \sqrt{\frac{t}{2}}.\) Let \(L = L(t, \alpha, \eta)\) denote the be the region in the plane with \(0 \leq x \leq \alpha\) and
\[ |y - \sqrt{t - x^2}| \leq \left\lfloor \eta t \right\rfloor = \tau. \]

Then
\[ \sum_{(m,n) \in L} \frac{\sqrt{t - m^2}}{t^{\frac{t}{4}} n} \int_1^R \cos 2\pi r(\sqrt{t - m^2} - n) \frac{dr}{r} = \]
\[ \frac{2\alpha}{t^{\frac{t}{4}} \pi} \int_1^R \sin 2\pi \left\lfloor \eta t \right\rfloor \frac{dr}{r^2} + t^{-\frac{1}{4}} L(R, t) + O(t^{\eta + \epsilon'}). \]

**Notes:** 1. The result is stated and proved for \(R = \left\lfloor t^{\frac{t}{4} + \epsilon'} \right\rfloor.\) In general, the error would have an extra factor of \(t^{\frac{1}{4} + \epsilon'}.\)
2. The essential feature of the term \(L(R, t)\) is its independence of \(\eta.\) However, we will actually obtain the following interesting expression for this term:
\[ L(R, t) = \pi \sum_{m=1}^{\left\lfloor \sqrt{\frac{t}{2}} \right\rfloor} \sum_{q=1}^{R} \frac{\cos 2\pi \sqrt{t - m^2}}{q} + O(R + t^{\frac{1}{4}} \ln R). \]

It can be shown (and even derived from arguments in this paper) that up to the same type of error
\[ P(t) - \pi t \approx -8 \sum_{m=1}^{\left\lfloor \sqrt{\frac{t}{2}} \right\rfloor} \sum_{q=1}^{R} \frac{\sin 2\pi \sqrt{t - m^2}}{q}, \]
which is essentially a known result. It will follow from our final result that both of these sums are \(O(t^{\frac{1}{4} + \epsilon'}).\)
The proof of (2.5) will involve the application of the following form of the classical Euler-MacLaurin formula:

$$\sum_{a < p \leq b} \phi(p) = \int_{a}^{b} \phi(z)dz + \int_{a}^{b} \phi(z)\psi(z)dz + \psi(a)\phi(a) - \psi(b)\phi(b).$$

Here

$$\phi(t) = t - [t] - \frac{1}{2}.$$

The first integral on the right will be referred to as the principal term, the second as the remainder term, and last two expressions as the endpoint terms.

Let $L'$ differ from $L$ by removal of the curve $y = \sqrt{t - x^2} - \tau$. Let

$$\Sigma = \sum_{(m,n) \in L'} \int_{1}^{R} \cos(2\pi rF(m, n)) \frac{dr}{r},$$

with

$$F(x, y) = y - \sqrt{t - x^2}.$$

Then it is not difficult to see that $t^{-\frac{1}{4}}\Sigma$ differs from the sum of (2.5) that we are studying by at most $O(t^\eta)$.

More generally, let $-[\lceil t^\eta \rceil] - 1 \leq d_1 \leq d_2 \leq \lceil t^\eta \rceil + 1$ and let $L(d_1, d_2)$ be the set of points $(x, y)$ in the plane satisfying the above condition on $x$ and

$$d_1 < y - \sqrt{t - x^2} \leq d_2.$$

Let

$$\Sigma(d_1, d_2) = \sum_{(m,n) \in L(d_1, d_2)} \int_{1}^{R} \cos(2\pi rF(m, n)) \frac{dr}{r}.$$

We now apply the above form of the classical Euler-MacLaurin formula to $y$-summation in this expression. The endpoint terms obtained in this way and summed over $m$ provide a contribution to $\Sigma(d_1, d_2)$ of

$$\sum_{m} \psi(y) \int_{1}^{R} \cos(2\pi rF(m, y)) \frac{dr}{r} \sqrt{t - m^2 + d_1} \sqrt{t - m^2 + d_2}. $$

The principal term in $y$-summation produces a contribution

$$\sum_{m} \int_{\sqrt{t - m^2 + d_1}}^{R} \int_{1}^{\sqrt{t - m^2 + d_2}} \cos(2\pi rF(m, y)) \frac{dr}{r} dy.$$

The endpoint terms for this $x$-summation will amount to no more than $O(t^\epsilon(d_2 - d_1))$, and the principal term will be
\[ \int_{L(d_1,d_2)}^{R} \int_{1}^{d} \cos(2\pi r F(x,y)) \frac{dr}{r} dxdy . \]

We will now look at the remainder term, which will be (with a change of order of integration)

\[ 2\pi \int_{1}^{R} \int_{L(d_1,d_2)}^{\psi(x) F_x(x,y) \sin(2\pi r F(x,y))} dxdydr . \]

Let \( K \) be described in polar co-ordinates as the points \((\rho, \theta)\) with

\[ \tan^{-1} \frac{\sqrt{t - \alpha^2}}{\alpha} = \theta_0 \leq \theta \leq \frac{\pi}{2} \]

and

\[ \sqrt{t} + d_1 \sin \theta \leq \rho \leq \sqrt{t} + d_2 \sin \theta . \]

Then it is not difficult to see that replacement of \( L(d_1,d_2) \) by \( K \) affects the integral by at most an error a constant multiple of \( R(d_2 - d_1) \). (At this point, \( R(d_2 - d_1) \) would suffice for our purposes.) Therefore, passing to the complexification for the moment, it suffices to consider instead

\[ \int_{1}^{R} \int_{\theta_0}^{\frac{\pi}{2}} \int_{\sqrt{t} + d_1 \sin \theta}^{\sqrt{t} + d_2 \sin \theta} \psi(\rho \cos \theta)(\cot \theta)(\frac{\sqrt{t}}{\rho} - 2) e(r \csc \theta(\sqrt{t} - \rho)) \rho d\rho d\theta dr. \]

Up to an error of at most a constant multiple of \( R(d_2 - d_1) \), this is the same as

\[ -\sqrt{t} \int_{1}^{R} \int_{\theta_0}^{\frac{\pi}{2}} \int_{\sqrt{t} + d_1 \sin \theta}^{\sqrt{t} + d_2 \sin \theta} \psi((\sqrt{t} + z) \cos \theta)(\cot \theta)(1 + \frac{z}{\sqrt{t} + z}) e(r z \csc \theta dz d\theta dr. \]

To estimate this integral, we plug in the Fourier expansion

\[ \psi_1(x) = -\frac{1}{2\pi i} \sum_{p=1}^{\infty} \frac{e(px) - e(-px)}{p} \]

for \( \psi \), which equals \( \psi_1 \) except at integer points. Therefore, we have to look at

\[ \sum_{p=1}^{\infty} \frac{\sqrt{t}}{2\pi i p} \int_{1}^{R} \int_{\theta_0}^{\frac{\pi}{2}} \int_{\sqrt{t} + d_1 \sin \theta}^{\sqrt{t} + d_2 \sin \theta} (\cot \theta)(1 + \frac{z}{\sqrt{t} + z}) e(r z \csc \theta \pm p(z + \sqrt{t}) \cos \theta) dz d\theta dr. \]

However, the \( \theta \)-derivative of the phase function is large; i.e.

\[ | - rz \csc \theta \cot \theta \mp p(\sqrt{t} + z) \sin \theta| \gg p \sqrt{t} . \]
Therefore, by integration by parts with respect to \( \theta \), followed by trivial estimates with respect to the other variables, this sum will be at most a constant multiple of

\[
R(d_2 - d_1) \sum_{p=1}^{\infty} \frac{1}{p^2}.
\]

Hence the remainder term for \( y \)-summation of the principal terms for \( x \)-summation is estimated as at most \( O(R(d_2 - d_1)) \).

Next, since \( F_y \equiv 1 \), the remainder term from \( y \)-summation will be

\[
-2\pi \sum_{m} \int_{1}^{R} \frac{\psi(y) \sin(2\pi r F(m, y))}{\sqrt{t - m^2 + d_1}} dr dy.
\]

Summing up, so far we have

\[
\Sigma(d_1, d_2) = \sum_{m} \psi(y) \int_{1}^{R} \cos(2\pi r F(m, y)) \frac{dr}{r} \frac{\sqrt{t - m^2 + d_1}}{\sqrt{t - m^2 + d_2}} + \int_{L(d_1, d_2)} \int_{1}^{R} \frac{\cos(2\pi r F(x, y))}{r} dx dy + 2\pi \sum_{m} \int_{d_1}^{d_2} \frac{\psi(\sqrt{t - m^2 + z})}{z} (\cos 2\pi Rz - \cos 2\pi z) dz + O(R(d_2 - d_1)).
\]

Strictly speaking, the last term has an apparent singularity if zero is in the interval between \( d_1 \) and \( d_2 \). However, the number of lattice points in \( L(d_1, d_2) \) is at most \( O(t^{\frac{1}{2} + \eta'}(d_2 - d_1)) \) for any \( \eta' > 0 \), and it follows readily that

\[
|\Sigma(d_1, d_2)| \ll t^{\frac{1}{2} + \eta'}(d_2 - d_1)
\]

and

\[
\left| \sum_{m} \psi(y) \int_{1}^{R} \cos(2\pi r F(m, y)) \frac{dr}{r} \frac{\sqrt{t - m^2 + d_1}}{\sqrt{t - m^2 + d_2}} \right| \ll t^{\frac{1}{2} + \eta'}(d_2 - d_1).
\]

Clearly,

\[
\left| \int_{L(d_1, d_2)} \int_{1}^{R} \cos(2\pi r F(x, y)) \frac{dr}{r} dx dy \right| \ll t^{\frac{1}{2} + \eta'}(d_2 - d_1)
\]
as well. Therefore, the last term is not only defined even when zero is in the interval, but also satisfies
\[
\left| \sum_m \int_{d_1}^{d_2} \psi(\sqrt{t - m^2} + z) \frac{(\cos 2\pi Rz - \cos 2\pi z) dz}{z} \right| \ll t^{\frac{1}{2} + \eta'(d_2 - d_1)}.
\]
These estimates will be needed in the continuation.
To analyze the last term, the remainder term from \( y \)-summation, let
\[
A(z) = \int_1^{\alpha} (\sqrt{t - x^2} + z) dx
\]
be the area of the region \( A(z) \) under the curve \( y = \sqrt{t - x^2} + z \) and let \( L(z) \) be the number of lattice points in the region bounded by this curve, the \( x \)-axis, and the lines \( x = 0 \) and \( x = \alpha \). Then, by trapezoidal approximation,
\[
\sum_m \psi(\sqrt{t - m^2} + z) = A(z) - L(z) + \frac{1}{2}(-\alpha + \sqrt{t} + \sqrt{t - \alpha^2} + 2z) + O(1).
\]
We now use the methods of the beginning of this paper to find a useful expression for the number of lattice points \( L(z) \). For the sake of simplicity, we will derive this expression assuming \( z \geq 0 \) and leave the case of negative \( z \) to the reader. First of all, let \( L_1(z) \) be the number of lattice points in \( L(z) \) that lie in the closed subregion \( A_1(z) \subset A(z) \) of points on or above the line \( y = x \) and \( L_2(z) \) the number in the triangle with integral vertices \( A_2(z) \subset A(z) \) of points on or below this line. Then
\[
L(z) = L_1(z) + L_2(z) - \alpha,
\]
and, by Pick’s theorem
\[
L_2(z) = A_2(z) + \frac{3}{2}\alpha + 1,
\]
\( A_i(z) \) the area of \( A_i(z) \).
Let \( \tilde{A}_1(z) \supset A_1(z) \) be the closed bounded region in the first quadrant between the line \( y = x \) and the curve \( y = \sqrt{t - x^2} + z \). The line and curve intersect at the point with
\[
x = y = \alpha_0 = \frac{z}{2} + \sqrt{\frac{t}{2} - \frac{z^2}{4}}.
\]
Since
\[
\tilde{A}_1(z) - A_1(z) \subset [\alpha, \alpha_0] \times [\alpha, \sqrt{t - \alpha^2} + z],
\]
it follows that for \( z \) not to large (e.g. \( z \leq t^{\eta} \)), with the obvious notation, \( \tilde{L}_1(z) = L_1(z) + O(1 + z^2) \) and \( \tilde{A}_1(z) - A_1(z) \ll 1 + z^2 \).
Let \( \tilde{A}_3(z) = \tilde{A}_3(z, \sqrt{t}) \) be the region in the plane consisting of points satisfying
\[
|y| \leq \sqrt{t-x^2} + z \quad , \quad |x| \leq |y|
|x| \leq \sqrt{t-y^2} + z \quad , \quad |x| \geq |y|.
\]

Clearly, for \( L_3(z) \) the number of lattice points in this slightly bumpy version of a circle,

\[
L_3(z) = 8L_1(z) - 4\sqrt{t} - 4\sqrt{t/2} + O(1 + z).
\]

The method for counting the number of lattice points in the circle set out in the beginning of this paper applies equally well to this region with only minor changes in the argument. In particular (compare (5.1) and (6) ), for any given \( \epsilon' > 0 \),

\[
L_3(z) = \sum_{p,q=-R}^{R} \left[ \int_{A_3(z)} e(px + qy) + \sum_{\mu \neq 0} \int_{A_3(z)} e((p + \mu Q)x + qy) + \sum_{\nu \neq 0} \int_{A_3(z)} e(px + (q + \nu Q)y) \right.
\]

\[
- \sum_{\mu,\nu \neq 0} \left( \frac{p}{2Q\mu} + \frac{q}{2Q\nu} \right) \int_{A_3(z)} e((p + \mu Q)x + (q + \nu Q)y) \left] \right.
\]

\[
+ O \left( \left( \frac{t^{1/2} + \epsilon'}{Q} + t^{1/4+\epsilon'} \right) \ln^2 Q \right).\]

In subsequent expressions, it will be understood that \( p \) and \( q \) range between \(-R\) and \( R = \frac{Q-1}{2} \), subject to whatever other restrictions are indicated.

Therefore

\[
\frac{1}{8}L_3(z) = \sum_{p,q} \left[ \int_{\tilde{A}_1(z)} e(px + qy) + \sum_{\mu \neq 0} \int_{\tilde{A}_1(z)} e((p + \mu Q)x + qy) + \sum_{\nu \neq 0} \int_{\tilde{A}_1(z)} e(qx + (q + \nu Q)y) \right.
\]

\[
- \sum_{\mu,\nu \neq 0} \left( \frac{p}{2Q\mu} + \frac{q}{2Q\nu} \right) \int_{\tilde{A}_1(z)} e((p + \mu Q)x + (q + \nu Q)y) \left] \right.
\]

\[
+ O \left( \left( \frac{t^{1/2} + \epsilon'}{Q} + t^{1/4+\epsilon'} \right) \ln^2 Q \right).\]

and combining these equations we end up with
\[
\sum_m \psi(\sqrt{t - m^2} + z) = - \sum_{(p,q) \neq (0,0)} \int_{\bar{A}_1(z)} \cos 2\pi(px + qy)
\]
\[\quad - \sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi((p + \mu Q)x + qy)
\]
\[\quad - \sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi(px + (q + \nu Q)y)
\]
\[\quad - \sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi((p + \mu Q)x + (q + \nu Q)y)
\]
\[\quad + O\left(\left(\frac{t^{1/2}+\epsilon'}{Q} + t^{1/4+\epsilon'}\ln^2 Q\right) + O(1 + z^2)\right).
\]

Therefore, up to an error of
\[O\left(\left(\frac{t^{1/2}+\epsilon'}{Q} + t^{1/4+\epsilon'}\ln^2 Q\right) + O(1 + \max|d_i|^2)(d_2 - d_1)\right),
\]
the term we are considering can be replaced by \(2\pi\) times
\[
\int_{d_1}^{d_2} \frac{\cos 2\pi Rz - \cos 2\pi z}{z} \left[\sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi(px + qy)
\right.
\]
\[\quad + \sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi((p + \mu Q)x + qy) + \sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi(px + (q + \nu Q)y)
\]
\[\quad - \sum_{p,q} \int_{\bar{A}_1(z)} \cos 2\pi((p + \mu Q)x + (q + \nu Q)y)\right] dz.
\]

Notice that if \(d_1 d_2 \geq 0\), the contribution of terms within the first integral with \(q = 0\) will cancel with the corresponding contribution for \(\Sigma(-d_2, -d_1)\), i.e. with the same expression with \(z\) replaced by \(-z\). In particular, these terms will vanish if the interval \((d_1, d_2)\) is symmetric around zero.

By explicit integration, the following holds for all \(p, q\) in the range of summation
and all integers \( \mu \) and \( \nu \), \( q \neq 0 \) if \( \nu = 0 \) :

\[
\int_{A(z)} \cos 2\pi((p + \mu Q)x + (q + \nu Q)y) \, dx \, dy
\]

\[
= \frac{1}{2\pi(q + \nu Q)} \int_0^{\sqrt{\pi}} \sin 2\pi((p + \mu Q)x + (q + \nu Q)(\sqrt{t - x^2} + z)) \, dx
\]

\[
- \frac{1}{2\pi(q + \nu Q)} \int_0^{\sqrt{\pi}} \sin 2\pi((p + \mu Q)x + (q + \nu Q)x) \, dx
\]

\[
= \frac{1}{2\pi(q + \nu Q)} \int_0^{\sqrt{\pi}} \sin 2\pi((p + \mu Q)x + (q + \nu Q)\sqrt{t - x^2}) \cos 2\pi(q + \nu Q)z \, dx
\]

\[
+ \frac{1}{2\pi(q + \nu Q)} \int_0^{\sqrt{\pi}} \cos 2\pi((p + \mu Q)x + (q + \nu Q)\sqrt{t - x^2}) \sin 2\pi(q + \nu Q)z \, dx
\]

\[
- \frac{1}{2\pi(q + \nu Q)} \int_0^{\sqrt{\pi}} \sin 2\pi((p + \mu Q)x + (q + \nu Q)x) \, dx.
\]

Again, assuming \( d_1 d_2 \geq 0 \), the contribution of first and third integral on the right to the term we are considering will cancel with the corresponding contribution for \( \Sigma(-d_2, -d_1) \). Hence, as such cancellations will come into play as we apply this, we will be left with (ignoring a factor of \( 2\pi \))

\[
\mathcal{B}(d_1, d_2) = \\
\sum_{p, q \neq 0} \int_0^{\sqrt{\pi}} \cos 2\pi(px + q \sqrt{t - x^2}) \, dx \cdot \int_{d_1}^{d_2} \frac{2\pi q z}{q z} \left[ \cos 2\pi R z - \cos 2\pi z \right] \, dz
\]

\[
+ \sum_{p, q \neq 0} \sum_{\mu \neq 0} \int_0^{\sqrt{\pi}} \cos 2\pi((p + \mu Q)x + q \sqrt{t - x^2}) \, dx.
\]

\[
\cdot \int_{d_1}^{d_2} \frac{2\pi q z}{q z} \left[ \cos 2\pi R z - \cos 2\pi z \right] \, dz
\]

\[
+ \sum_{p, q} \sum_{\nu \neq 0} \int_0^{\sqrt{\pi}} \cos 2\pi(px + (q + \nu Q)\sqrt{t - x^2}) \, dx.
\]

\[
\cdot \int_{d_1}^{d_2} \frac{2\pi q (q + \nu Q) z}{(q + \nu Q) z} \left[ \cos 2\pi R z - \cos 2\pi z \right] \, dz
\]

\[
- \sum_{p, q} \sum_{\mu, \nu \neq 0} \left( \frac{p}{2Q \mu} + \frac{q}{2Q \nu} \right) \int_0^{\sqrt{\pi}} \cos 2\pi((p + \mu Q)x + (q + \nu Q)\sqrt{t - x^2}) \, dx.
\]

\[
\cdot \int_{d_1}^{d_2} \frac{2\pi q (q + \nu Q) z}{(q + \nu Q) z} \left[ \cos 2\pi R z - \cos 2\pi z \right] \, dz.
\]
Note that $B(d_1, d_2) = B(-d_2, -d_1)$. Therefore, from above estimates and cancellations across $z = 0$,
$$|B(0, d)| \ll t^{\frac{1}{2} + \eta} d$$
for $d > 0$. Further, for example, for $d_i \geq 0$,
$$B(d_1, d_2) = B(0, d_2) - B(0, d_1).$$
Therefore, we may just consider $B(0, \gamma)$. In general,
\[
\int_0^\gamma \frac{\sin 2\pi \lambda z}{z} \cos 2\pi \mu z \, dz = \frac{1}{2} \int_0^\gamma \frac{\sin 2\pi z(\lambda + \mu)}{z} \, dz + \frac{1}{2} \int_0^\gamma \frac{\sin 2\pi z(\lambda - \mu)}{z} \, dz
\]
\[= \frac{1}{2} \left( \text{Si}(2\pi(\lambda + \mu)\gamma) + \text{Si}(2\pi(\lambda - \mu)\gamma) \right).\]

Hence,
$$B(0, \gamma) = \frac{1}{2} (B_+(0, \gamma) + B_-(0, \gamma)),$$
with
\[
B_\pm(0, \gamma) = \sum_{p,q \neq 0} \int_0^{\sqrt{t}} \frac{\cos 2\pi(px + q\sqrt{t - x^2})}{q} dx \cdot \left[ \text{Si}(2\pi(q \pm R)\gamma) - \text{Si}(2\pi(q \pm 1)\gamma) \right]
\]
\[+ \sum_{p,q \neq 0} \sum_{\mu \neq 0} \int_0^{\sqrt{t}} \frac{\cos 2\pi((p + \mu Q)x + qy)\sqrt{t - x^2})}{q} dx \cdot \left[ \text{Si}(2\pi(q \pm R)\gamma) - \text{Si}(2\pi(q \pm 1)\gamma) \right],
\]
\[+ \sum_{p,q \neq 0} \sum_{\nu \neq 0} \int_0^{\sqrt{t}} \frac{\cos 2\pi(px + (q + \nu Q)\sqrt{t - x^2})}{q + \nu Q} dx \cdot \left[ \text{Si}(2\pi(q + \nu Q \pm R)\gamma) - \text{Si}(2\pi(q + \nu Q \pm 1)\gamma) \right]
\]
\[+ \sum_{p,q} \sum_{\mu, \nu \neq 0} \left( \frac{p}{2Q\mu} + \frac{q}{2Q\nu} \right) \int_0^{\sqrt{t}} \frac{\cos 2\pi((p + \mu Q)x + (q + \nu Q)\sqrt{t - x^2})}{q + \nu Q} dx \cdot \left[ \text{Si}(2\pi(q + \nu Q \pm R)\gamma) - \text{Si}(2\pi(q + \nu Q \pm 1)\gamma) \right].
\]

Corresponding to the four terms, let
\( B_{\pm}(0, \gamma) = B_{\pm,1}(0, \gamma) + B_{\pm,2}(0, \gamma) + B_{\pm,3}(0, \gamma) + B_{\pm,4}(0, \gamma) \),

and

\( B(0, \gamma) = B_1(0, \gamma) + B_2(0, \gamma) + B_3(0, \gamma) + B_4(0, \gamma) \).

\( \text{Si}(x) \) is odd, bounded, and, for positive \( x \),

\[ \text{Si}(x) = \frac{\pi}{2} + O(\max\{1, |x|^{-1}\}) . \]

It therefore follows that

\[ B_i(0, \gamma) = \hat{B}_i(0, \gamma) + \tilde{B}_i \]

with

\[ L(t, R) = \tilde{B}_1 + \tilde{B}_2 = -\pi \sum_{p, q \neq 0} \int_0^{\sqrt{t}} \frac{\cos 2\pi(px + q\sqrt{t - x^2})}{|q|} dx \]

\[ -\pi \sum_{p, q \neq 0} \sum_{\mu \neq 0} \int_0^{\sqrt{t}} \frac{\cos 2\pi((p + \mu Q)x + qy)\sqrt{t - x^2})}{|q|} dx , \]

\[ \tilde{B}_3 = \tilde{B}_4 , \]

and, for example, assuming \( \gamma \geq 1 \),

\[ |\hat{B}_1(0, \gamma)| \ll \sum_{p, q \neq 0} \frac{1}{q^2} \left| \int_0^{\sqrt{t}} \cos 2\pi(px + q\sqrt{t - x^2}) dx \right| \]

\[ + \sum_{p, q \neq 0, R} \frac{1}{(R - q)|q|} \left| \int_0^{\sqrt{t}} \cos 2\pi(px + q\sqrt{t - x^2}) dx \right| \]

\[ + \sum_{p, q \neq 0, -R} \frac{1}{(R + q)|q|} \left| \int_0^{\sqrt{t}} \cos 2\pi(px + q\sqrt{t - x^2}) dx \right| , \]

and similarly for \( \hat{B}_2(0, \gamma) , \hat{B}_3(0, \gamma) , \) and \( \hat{B}_3(0, \gamma) . \) (The actual estimate has a factor of \( \gamma^{-1} \).

It is clear that \( L(t, R) \) is indeed independent of \( \eta \).

For general interest, here is a derivation of the description given in the first note after (2.5): First of all, it is not difficult to see that
\( \mathcal{L}(t, R) = -2\pi \sum_{p=1}^{\infty} \sum_{q=1}^{R} \int_{0}^{\sqrt{t/2}} \frac{\cos 2\pi px \cos 2\pi q \sqrt{t-x^2}}{q} dx \)

\[
= 4\pi^2 \sum_{p=1}^{R} \sum_{q=1}^{\infty} \int_{0}^{\sqrt{t}} \int_{0}^{\sqrt{t-x^2}} \cos 2\pi px \sin 2\pi qy \, dy \, dx \\
- 2\pi \sum_{p=1}^{\infty} \sum_{q=1}^{R} \int_{0}^{\sqrt{t/2}} \cos 2\pi px \sin 2\pi q \sqrt{t/2} \, dx \\
= 2\pi \sum_{p=1}^{\infty} \sum_{q=1}^{R} \left\{ \int_{0}^{\sqrt{t/2}} \sin 2\pi p \sqrt{t-y^2} \sin 2\pi qy \, dy - \frac{\sin 2\pi p \sqrt{t/2} \sin 2\pi q \sqrt{t/2}}{2\pi p} \right\} \\
= 2\pi \sum_{p=1}^{\infty} \sum_{q=1}^{R} \left\{ \int_{0}^{\sqrt{t}} \frac{w \sin 2\pi pw \sin q \sqrt{t-w^2}}{p \sqrt{t-w^2}} \, dw - \frac{\sin 2\pi p \sqrt{t/2} \sin 2\pi q \sqrt{t/2}}{2\pi p} \right\} \\
= 2\pi^2 \sum_{q=1}^{R} \left\{ \int_{0}^{\sqrt{t/2}} \psi(w) \sin 2\pi q \sqrt{t-w^2} \, dw \right\} \\
= \pi \sum_{q=1}^{R} \sum_{m=1}^{\lfloor \sqrt{t/2} \rfloor} q^{-1} \cos 2\pi q \sqrt{t-m^2} \\
- \pi \sum_{q=1}^{R} q^{-1} \int_{0}^{\sqrt{t/2}} \cos 2\pi q \sqrt{t-w^2} \, dw + O(R). \\
\]

The last equality is integration by parts, i.e., the one dimensional Euler-MacLaurin formula. We have

\[
\int_{0}^{\sqrt{t/2}} \cos 2\pi q \sqrt{t-w^2} \, dw = \int_{0}^{t^{1/4}} \cos 2\pi q \sqrt{t-w^2} \, dw + \int_{t^{1/4}}^{\sqrt{t/2}} \cos 2\pi q \sqrt{t-w^2} \, dw. \\
\]

On the domain of the second integral, the derivative of the phase function is at least \( t^{-\frac{1}{4}} \). It follows that

\[
\left| \sum_{q=1}^{R} q^{-1} \int_{0}^{\sqrt{t/2}} \cos 2\pi q \sqrt{t-w^2} \, dw \right| \ll t^{\frac{1}{4}} \ln R. \\
\]

This provides the stated description.

We now discuss an estimate for \( \hat{B}_1(0, \gamma) \). If \( p > q \geq 1 \), the derivative of the phase function satisfies

\[
|2\pi(p-q \frac{x}{\sqrt{t-x^2}})| \gg p-q. 
\]
Therefore,
\[ \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll \frac{1}{(p - q)}. \]

Hence,
\[ \sum_{p > q \geq 1} \frac{1}{q^2} \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll \sum_{p > q} \frac{1}{q^2(p - q)} \ll \ln R, \]

\[ \sum_{p > q \geq 1, q \neq R} \frac{1}{(R - q)|q|} \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll \sum_{p > q \geq 1, q \neq R} \frac{1}{(R - q)|q|(p - q)} \ll \frac{\ln^2 R}{R}, \]

and, similarly,
\[ \sum_{p > q \geq 1} \frac{1}{(R + q)|q|} \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll \frac{\ln^2 R}{R}. \]

A similar argument yields the same estimates for \( p < q \leq -1 \) or for \( p \) and \( q \) having opposite signs. On the other hand, if \( p \) and \( q \) have the same sign and \( |p| \leq |q| \), the absolute value of the second derivative of the phase function will be
\[ \frac{|q|t}{(t - x^2)^{\frac{3}{2}}} \gg \frac{q}{\sqrt{t}}, \]

and hence
\[ \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll \frac{t^{\frac{1}{2}}}{q^{\frac{1}{2}}}. \]

Hence,
\[ \sum_{pq \geq 0, |p| \leq |q| \neq 0} \frac{1}{q^2} \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll \sum_{pq \geq 0, |p| \leq |q| \neq 0} \frac{t^{\frac{1}{2}}}{q^{\frac{1}{2}}} \ll t^{\frac{1}{2}}, \]

Similarly,
\[ \sum_{pq \geq 0, |p| \leq |q| \neq 0} \frac{1}{(R - q)|q|} \left| \int_0^{\sqrt{\frac{t}{\pi}}} \cos 2\pi(px + q\sqrt{t - x^2})dx \right| \ll t^{\frac{1}{2}} \ln R \]

and
\[
\sum_{\substack{q \geq 0, |p| \leq |q| \neq 0 \\ q \neq -R}} \frac{1}{(R+q)|q|} \left| \int_0^{\sqrt{t}} \cos 2\pi (px + q \sqrt{t-x^2}) \, dx \right| \ll t^{\frac{1}{4}} \ln R
\]

Therefore, for \( \gamma \geq 1 \),

\[|\hat{B}_1(0, \gamma)| \ll t^{\frac{1}{2}} \ln R.\]

(The true estimate has a factor of \( \gamma^{-1} \).) Similar methods apply to \( B_i(0, \gamma) \), leading to the conclusion

\[|\hat{B}_i(0, \gamma)| \ll t^{\frac{1}{4}+\epsilon},\]

for \( i = 2, 3, 4 \). We leave the details to the reader.

What we have so far can be summed up as follows (recall \( Q = \lceil t^{\frac{1}{4}+\epsilon} \rceil \)), absorbing some logarithmic terms into \( t^{\epsilon'} \):

\[
\Sigma(d_1, d_2) + \Sigma(-d_2, -d_1) = \mathcal{L}(t, R) + 2 \int_{L(d_1, d_2)}^{R} \int_{1}^{\sqrt{t}} \cos 2\pi r(F(x,y)) \frac{dr}{r} \, dxdy \\
+ \sum \psi(y) \int_{1}^{\sqrt{t}} \cos(2\pi r F(m, y)) \frac{dr}{r} \sqrt{t-m^2+d_1} \\
+ \sum \psi(y) \int_{1}^{\sqrt{t}} \cos(2\pi r F(m, y)) \frac{dr}{r} \sqrt{t-m^2-d_2} \\
+ \sum \psi(y) \int_{1}^{\sqrt{t}} \cos(2\pi r F(m, y)) \frac{dr}{r} \sqrt{t-m^2-d_1} \\
+ O(t^{\frac{1}{4}+\eta+\epsilon'} + t^{\frac{1}{4}+\epsilon'})(d_2 - d_1) + t^{\frac{1}{4}+\epsilon'} d_1).
\]

The second and third terms cancel. Further, as also observed above,

\[|\Sigma(-d_1, d_1)| \ll d_1 t^{\frac{1}{4}+\epsilon'} .\]

and

\[
\left| \int_{L(-d_1, d_1)}^{R} \cos(r F(x,y)) \frac{dr}{r} \, dxdy \right| \ll d_1 t^{\frac{1}{4}+\epsilon'} .
\]

Hence, under the assumption that \( d_2 \) is an integer,

\[
\Sigma(-d_2, d_2) = \mathcal{L}(t, R) + 2 \int_{L(-d_2, d_2)}^{R} \int_{1}^{\sqrt{t}} \cos 2\pi r(F(x,y)) \frac{dr}{r} \, dxdy \\
+ O(t^{\frac{1}{4}+\eta+\epsilon'} d_2 + t^{\frac{1}{4}+\epsilon'} d_1)
\]

also.
With an error of at most $O(R\eta)$, the domain of the double integral can be replaced with the region $\theta_0 \leq \theta \leq \pi$ and $\sqrt{t} - \tau \leq \rho \leq \sqrt{t} + \tau$, in polar co-ordinates. Hence, with $d_2 = \tau$, this integral may be replaced by

$$
\int_0^R \int_{\theta_0}^{\pi} \int_{\sqrt{t} - \tau}^{\sqrt{t} + \tau} \cos 2\pi r (\sqrt{t} - \rho) \csc \theta \rho \, d\rho \, d\theta \, dr
$$

$$
= \sqrt{t} \int_0^R \int_{\theta_0}^{\pi} \int_{-\tau}^{\pi} \cos 2\pi rz \csc \theta \, dz \, d\theta \, dr + O(R\eta)
$$

$$
= \sqrt{t} \int_0^R \int_{\theta_0}^{\pi} (\sin 2\pi (r\tau) - (\sin 2\pi (-r\tau)) \sin \theta \, d\theta \, dr + O(R\eta)
$$

$$
= \frac{\alpha}{\pi} \int_0^R \sin 2\pi r\tau \, dr + O(R\eta).
$$

This completes the proof of (2.5).

There is the expansion, due to integration by parts, with all non-zero coefficients,

$$
\int_1^U \sin 2\pi w \frac{dr}{r^2} = \sum_{n=0}^{N-1} a_{2n+1} w^{-2n+1} + O(U^{-2}) + O(w^{-2N-1}).
$$

Using this and the preceding, we may restate the previous result as follows (absorbing some constants into the $a_i$):

**Proposition.** Let $0 < \eta < \epsilon'$. Let $\alpha$ be an integer with $\sqrt{t/2} - t^{1/4} - 1 \leq \alpha \leq \sqrt{t/2}$. Let $L = L(t, \alpha)$ denote the be the region in the plane with $0 \leq x \leq \alpha$ and

$$
|y - \sqrt{t - x^2}| \leq \lfloor t' \rfloor = \tau.
$$

Then

$$
\sum_{(m,n) \in L} \frac{\sqrt{t - m^2}}{t^{1/4} n} \int_1^R \cos 2\pi r (\sqrt{t - m^2} - n) \frac{dr}{r} =
$$

$$
t^{-1/4} L(t, R) + t^{1/4} \sum_{i=0}^{N(\eta)} a_{2i+1} \tau^{-2i-1} + O(t^{\eta+\epsilon'}). \quad (2.6)
$$

It is not difficult to use the same techniques (see the discussion of $\Sigma(d_1, d_2)$) to prove the following variant, in which the term that is independent of $\eta$ cancels:

**Proposition.** Let $0 < \eta < \epsilon'$. Let $\alpha$ be an integer with $\sqrt{t/2} - t^{1/4} - 1 \leq \alpha \leq \sqrt{t/2}$. Let $L_+ = L_+(t, \alpha)$ denote the be the region in the plane with $0 \leq x \leq \alpha$ and

$$
\lfloor t' \rfloor \leq y - \sqrt{t - x^2} \leq \lfloor t' \rfloor + 1
$$

and let $L_-$ be the region.

$$
\int_1^U \sin 2\pi w \frac{dr}{r^2} = \sum_{n=0}^{N(\eta)} a_{2n+1} w^{-2n+1} + O(U^{-2}) + O(w^{-2N-1}).
$$
\[ -[t^n] - 1 \leq y - \sqrt{t - x^2} \leq -[t^n]. \]

Then
\[
\sum_{(m,n) \in L_+ \cup L_-} \frac{\sqrt{t - m^2}}{t \frac{1}{4} n} \int_1^R \cos 2\pi r (\sqrt{t - m^2} - n) \frac{dr}{r} = t^{\frac{1}{4}} \sum_{i=2}^{N(\varepsilon)} b_i \tau^{-i} + O(t^{\eta+\varepsilon}).
\]

**Note:** In (2.5)-(2.7), it is obvious from the trivial estimate that the result remains the same if the range of \(x\) in the various regions of summation is replaced by \(\beta \leq x \leq \alpha\), for any \(0 < \beta \leq t^{\frac{1}{4}}\).

To summarize so far, let
\[ J = J_1 \cup J_2, \]
with \((m,n) \in J_1\) if \((m,n) \in J\) and
\[ |\sqrt{t - m^2} - n| \leq \tau = [t^n]\]
and \((m,n) \in J_2\) otherwise. Let
\[ E(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) \]
be the corresponding decomposition in (10). Then, from (13) and (2.6)
\[
\mathcal{E}_1(t) = t^{-\frac{1}{4}} \mathcal{L}(t,R) + t^{\frac{1}{4}} \sum_{i=0}^{N(\varepsilon)} a_{2i+1} \tau^{-2i-1} + O(t^{\eta+\varepsilon}). \tag{14}
\]

To study \(\mathcal{E}_2\), we use (12). It is easy to see that the remainder can be ignored. The second order terms in the expansion contributes at most, by integration by parts,
\[
\sum_{J_2} \frac{t}{mn(\sqrt{t - m^2} - n)} \cdot \frac{m\sqrt{t - m^2}}{t^{\frac{1}{4}}}
\]
\[
= \sum_{J_2} \frac{\sqrt{t - m^2}}{t^{\frac{1}{4}} n(\sqrt{t - m^2} - n)}
\]
\[
\ll \frac{1}{t^{\frac{1}{4}}} \sum_{J_2} \left( \frac{1}{n} + \frac{1}{\sqrt{t - m^2} - n} \right) \ll t^{-\frac{1}{4}} \ln t.
\]

Notice that this estimate remains valid when the summation is restricted to any subset of \(J_2\).
Finally, we deal with the contribution over $J_2$ of the leading term in (11). Split $J_2$ into $J^+_2$ and $J^-_2$, consisting of those $(m, n) \in J_2$ inside and outside the circle of radius $\sqrt{t}$, respectively. We will use complex notation, specifically replacing $\cos 2\pi(\cdot)$ with $e(\cdot)$ on $J^+_2$ and with $e(-\cdot)$ on $J^-_2$, for the term being considered. By integration by parts this leading term then leads to contributions of exactly

$$\sum_{J^\pm_2} \frac{\sqrt{t-m^2}}{n \sqrt{t}} \int_1^R e(\pm(\sqrt{t-m^2}-n)) \frac{dr}{r} =$$

$$\sum_{J^\pm_2} \left[ \sum_{i=1}^N (i-1)! \frac{\sqrt{t-m^2}}{n \sqrt{t} r^i \sqrt{t-m^2-n}^i} e(\pm r \sqrt{t-m^2}) \right]_1^R$$

$$+ N! \int_1^R \frac{\sqrt{t-m^2}}{n \sqrt{t} r^{N+1} \sqrt{t-m^2-n}^N} e(\pm r(\sqrt{t-m^2}-n)) dr \right] .$$

For the term involving the integral on the right, the trivial estimate gives

$$\left| \sum_{J_2} \int_1^R \cdots \right| \leq \sum_{J_2} \frac{t^{1/4-N\epsilon}}{n} \leq t^{3/4-N\eta} \ln t ,$$

which will be at most $O(t^{-1/2})$ for

$$N = N(\eta) > \frac{5}{4\eta} .$$

We can rewrite the sum over $J^+_1$ and over $J^-_1$ of the $i^{th}$ term as

$$\frac{(i-1)!}{r^i \sqrt{t}} \sum_{m_0}^{m_0} \left( \sum_{n_0}^{n_0(m)} \frac{\sqrt{t-m^2}}{n(\sqrt{t-m^2-n})^i} \right) e(r \sqrt{t-m^2}) \right]_1^R$$

and

$$\frac{(i-1)!}{r^i \sqrt{t}} \sum_{m_1}^{m_0} \left( \sum_{n_1}^{n_1(m)} \frac{\sqrt{t-m^2}}{n(\sqrt{t-m^2-n})^i} \right) e(-r \sqrt{t-m^2}) \right]_1^R ,$$

respectively, where $m_0 = \lfloor \sqrt{t/2} \rfloor$, $n_0(m) = \lfloor \sqrt{t-m^2} - \tau \rfloor$, and $n_1(m) = \lfloor \sqrt{t-m^2} + \tau \rfloor$.

If the inner sums were monotone functions of $m$, Abel summation and Lemma 3.5 of [GK] or even just Kusmin-Landau would apply as above to give the desired estimate. This type of argument does work if the inner sums are replaced by integrals.

To wit, consider

$$\frac{(i-1)!}{r^i \sqrt{t}} \sum_{m_1}^{m_0} \left( \int_m^{\sqrt{t-m^2-\tau}} \frac{\sqrt{t-m^2} du}{u(\sqrt{t-m^2-u})^i} \right) e(r \sqrt{t-m^2}) \right]_1^R$$
and
\[
\frac{(i - 1)!}{r^i \sqrt{t}} \sum_{m=1}^{m_0} \left( \int_{\sqrt{t} - m^2 + \tau}^{\sqrt{t} - m^2} \frac{\sqrt{t - m^2} \, du}{u(u - \sqrt{t - m^2})^i} \right) e(-r \sqrt{t - m^2}) \bigg|_1^R.
\]

Then it is not hard to show that the integrals are monotone functions of \( m \). By Kusmin-Landau or [GK, 3.5],
\[
\sum_{m=1}^{m_0} e(\sqrt{t - m^2}) \ll t^\frac{1}{4},
\]
and, more generally,
\[
\sum_{m=1}^{m_0} e(r \sqrt{t - m^2}) \ll t^\frac{1}{4} \sqrt{r}.
\]

It follows by Abel summation that the first expression will in modulus be at most a constant times
\[
\int_0^{n_0(1)} \frac{\sqrt{t - 1} \, du}{u(\sqrt{t - 1} - u)} \leq \int_1^{n_0(1)} \left( \frac{1}{u} + \frac{1}{\sqrt{t - 1} - u} \right) \, du \ll \ln t,
\]
and similarly, the second will be at most \( O(\ln t) \) as well.

To study the error introduced by replacing the sum by the integrals in the first expression, first note that by trapezoidal approximation
\[
E_i(m) = \int_{m}^{\sqrt{t} - m^2 - \tau} \frac{\sqrt{t - m^2} \, du}{u(\sqrt{t - m^2 - u})^i} - \sum_{n=m}^{n_0(m)} \frac{\sqrt{t - m^2}}{n(\sqrt{t - m^2 - n})^i}
\]
\[
= \int_{n_0(m)}^{\sqrt{t} - m^2 - \tau} \frac{\sqrt{t - m^2} \, du}{u(\sqrt{t - m^2 - u})^i}
\]
\[
- \frac{1}{2} \left( \frac{\sqrt{t - m^2}}{m(\sqrt{t - m^2 - m})^i} + \frac{\sqrt{t - m^2}}{n_0(m)(\sqrt{t - m^2 - n_0(m)})^i} \right) + O(m^{-2}).
\]

Our error in the \( i^{th} \) term will be
\[
\frac{(i - 1)!}{r^i \sqrt{t}} \sum_{m=1}^{m_0} E_i(m) e(r \sqrt{t - m^2}) \bigg|_1^R,
\]
which by the trivial estimate will be up to \( O(1) \) (actually even less) the same thing as
\[
\frac{(i - 1)!}{r^i \sqrt{t}} \sum_{m=1}^{m_0} E_i(m) e(r \sqrt{t - m^2}) \bigg|_1^R,
\]
\[ \bar{E}_i(m) = \int_{n_0(m)}^{\sqrt{t-m^2}-\tau} \frac{\sqrt{t-m^2} \, du}{u(\sqrt{t-m^2}-u)^i} - \frac{1}{2} n_0(m)(\sqrt{t-m^2} - n_0(m))^i, \]

We apply Taylor’s formula to expand the integral. Since

\[ \frac{\sqrt{t-m^2}}{u(\sqrt{t-m^2}-u)^i} = \left( \frac{1}{u} + \frac{1}{\sqrt{t-m^2}-u} \right) \frac{1}{(\sqrt{t-m^2}-u)^{i-1}}, \]

we have, up to an error of \( O(t^{-\frac{i}{2}}) \),

\[ \frac{d^j}{du^j} \bigg|_{n_0(m)} \frac{\sqrt{t-m^2}}{u(\sqrt{t-m^2}-u)^i} \approx n_0(m)^{-1} (\sqrt{t-m^2} - n_0(m))^{-i} (-1)^{j-i} i(i+1) \cdots (i+j-1) \frac{1}{(\tau + \sqrt{t-m^2} - [\sqrt{t-m^2}])^j}. \]

and

\[ \left| \frac{d^j}{du^j} \sqrt{t-m^2} \right|_{n_0(m)} \frac{u(\sqrt{t-m^2}-u)^i} \ll t^{-(i+j)\epsilon}, \]

for \( u \) in the range of integration. Moreover,

\[ \frac{1}{\tau + \sqrt{t-m^2} - [\sqrt{t-m^2}]} = \sum_{k=0}^{M} (-1)^k \frac{(\sqrt{t-m^2} - [\sqrt{t-m^2}])^k}{\tau^k + 1} + O(t^{-M\epsilon}). \]

Finally, it is obvious that

\[ \frac{1}{n_0(m)} = \frac{1}{\sqrt{t-m^2}} + O(t^{-\frac{1}{2}}) \]

and that

\[ \sqrt{t-m^2} - [\sqrt{t-m^2}] = \psi(\sqrt{t-m^2}) + \frac{1}{2}. \]

Hence, for \( M\epsilon \geq \frac{1}{4} \), we get from Taylor’s formula and some elementary manipulations an expression of the form

\[ \bar{E}_i(m) = \sum_{j=0}^{M-i} b_{i,j} \psi(\sqrt{t-m^2})^{j+1} \tau^{-i-j} \]

\[ + \sum_{j=1}^{M} \frac{\sqrt{t-m^2}}{n_0(m)(\sqrt{t-m^2} - n_0(m))^i} \sum_{k=0}^{M-j} a_{i,j,k} \tau^{-k} + O(t^{-\frac{1}{2}}). \]
The coefficients, specifically $b_{0,0}$, will be non-zero.

The contribution of the error term in this expression to the term $\mathcal{E}_2(t)$ we are studying will be at most $O(1)$, by the trivial estimate. The contribution of the first term, $j = 0$, for example, of the first sum will be

$$\frac{b_{i,0}(i - 1)!}{(\tau r)^{it^{\frac{1}{4}}}} \sum_{m=1}^{m_0} \psi(\sqrt{t - m^2}) e(r \sqrt{t - m^2}) R_1.$$

For the value $r = R$, this is clearly at most $O(1)$, so we need only consider the value $r = 1$. Plugging in the Fourier expansion of $\psi_1$ for $\psi$ (as the number of $m$ for which the Fourier expansions does not give the correct value, i.e. when $\sqrt{t - m^2}$ is an integer, will be at most $t^\epsilon$ and so can be ignored), this will be (up to the factor $b_{i,0}(i - 1)!$)

$$-\frac{1}{2\pi i t^{\frac{1}{4}}} \sum_{m=1}^{m_0} \left[ \sum_{p=1}^{\infty} \frac{1}{p} e((1 + p) \sqrt{t - m^2}) - \sum_{p=1}^{\infty} \frac{1}{p} e((1 - p) \sqrt{t - m^2}) \right]$$

$$= -\frac{1}{2\pi i t^{\frac{1}{4}}} \sum_{m=1}^{m_0} \left[ \sum_{q=2}^{\infty} \left( \frac{e(q \sqrt{t - m^2})}{q - 1} - \frac{e(-q \sqrt{t - m^2})}{q + 1} \right) - e(-\sqrt{t - m^2}) - 1 \right]$$

$$= \frac{1}{\tau t^{\frac{1}{4}}} \sum_{m=1}^{m_0} \psi(\sqrt{t - m^2}) + \frac{1}{2\pi i t^{\frac{1}{4}}} \sum_{m=1}^{m_0} \left[ e(\sqrt{t - m^2}) \right.$$

$$\left. + \sum_{q=2}^{\infty} e(q \sqrt{t - m^2}) \right] \frac{1}{2\pi i t^{\frac{1}{4}}}.$$

As above (3.5) of [GK] will yield the estimate

$$\left| \sum_{1}^{m_0} e(q \sqrt{t - m^2}) \right| \ll t^{\frac{1}{4}} \sqrt{q},$$

which can be applied to all the summations but the first to get at most $O(t^{\frac{1}{2}})$.

For the first summation, to begin with,

$$\sum_{1}^{m_0} \psi(\sqrt{t - m^2}) = \sum_{1}^{b} \psi(\sqrt{t - m^2}) + O(t^{\frac{1}{4}}),$$

with $b$ the greatest integer in $\sqrt{t/2}$. We repeat a special case of an argument used in (2.6). By trapezoidal approximation using integer points from 0 to $b$,

$$\sum_{1}^{b} \psi(\sqrt{t - m^2}) = A - L + \frac{\sqrt{t}}{2} + O(1),$$

where $A$ is the area under the circle of radius $y = \sqrt{t - x^2}$ for $0 \leq x \leq b$ and $L$ the number of lattice points in this regions, counting the top and sides but not the bottom.
Combining this with "Pick’s theorem" for the number of lattice points in the triangle with vertices the origin, \((b,0)\), and \((b,b)\), it follows that

\[
\sum_{1}^{b} \psi(\sqrt{t-m^2}) = \frac{\pi t}{8} - L + \frac{\sqrt{t}}{2} - \frac{1}{2} \sqrt{t} + O(1),
\]

\(L\) the number of lattice points contained in the sector of the circle of radius \(\sqrt{t}\) with angle \(\frac{\pi}{4} < \theta \leq \frac{\pi}{2}\). By symmetry with respect to interchange of co-ordinates, twice this sum will be the difference between \(\frac{\pi t}{4}\) and the number of lattice points in the part of the circle in the first quadrant, excluding the \(x\)-axis, plus \(O(1)\). Hence

\[
2 \sum_{1}^{m_0} \psi(\sqrt{t-m^2}) = \frac{\pi t - P(t)}{4} + O(t^{\frac{1}{4}}).
\]

Thus the contribution of the first term in the first sum may be written as

\[
\frac{1}{2\sqrt{2}} t^{\frac{1}{4}} \tau^{-i} + O(t^{-\frac{1}{4} - i|\pi t - P(t)|}) + O(1).
\]

More precisely, the first error term can be replaced with an expansion, i.e.

\[
\frac{1}{2\sqrt{2}} t^{\frac{1}{4}} \tau^{-i} + t^{-\frac{1}{4}} (\pi t - P(t)) \sum_{i=1}^{M(\eta)} c_i \tau^{-i} + O(1),
\]

in which in particular the leading coefficient is non-zero.

The contribution of the \(j^{th}\) term in the first sum of the expression for \(\bar{E}_i(m)\) will be a constant multiple of

\[
\frac{1}{\tau + j r t^{\frac{1}{2}}} \sum_{1}^{m_0} \psi(\sqrt{t-m^2})^{j+1} e(r\sqrt{t-m^2}) \bigg|_1^R.
\]

As above, the case \(r = R\) is \(O(1)\), so we may assume \(r = 1\). Suppose first that \(j \geq 1\) is odd. Then the Fourier expansion of \(\psi(j)\) has the form

\[
\psi(x)^{j+1} = \sum_{p=1}^{\infty} \sum_{k=2}^{j+1} \frac{c_{j,k}}{p^k} (e(px) + (-1)^k e(-px)) + \frac{1}{(j+2)2^{j+1}}.
\]

From another application of (3.5) of [GK], just as used above, it follows that we have an error of

\[
\frac{t^{\frac{1}{4}} \tau^{-i-j}}{(j+2)2^{j+1}} + O(1).
\]

If \(j\) is even, then \(\psi(x)^{j+1} - 2^j \psi(x)\) has a similar type of Fourier expansion:

\[
\psi(x)^{j+1} - 2^j \psi(x) = \sum_{p=1}^{\infty} \sum_{k=2}^{j+1} \frac{c_{j,k}}{p^k} (e(px) + (-1)^k e(-px)).
\]
So, combining both arguments, we will get a contribution to the total error of

\[
\frac{2^{j-1}}{\sqrt{2}} t^{\frac{j}{2}} \tau^{-j} + O(t^{-\frac{j}{2}-(i+j)\eta} |\pi t - P(t)|) + O(1)
\]

Again, we could replace the first error term with an expansion, but this will not be needed.

For the second sum in the expression for \( \bar{E}_i(m) \), we combine them for the different \( i \) and reverse the integration by parts used before (see (15):

\[
\sum_{i=1}^{N} (i-1)! \frac{\sqrt{t-m^2}}{\sqrt{t-r} n_0(m) (\sqrt{t-m^2} - n_0(m))^i} e(r \sqrt{t-m^2}) \bigg|_1^R =
\]

\[
= \frac{\sqrt{t-m^2}}{n_0(m) \sqrt{t}} \int_1^R e(r \sqrt{t-m^2} - n_0(m)) \frac{dr}{t} + O\left(\frac{1}{\sqrt{t}}\right)
\]

for \( N > 5/4\eta \).

The term on the right is the leading term in (12) or (13), and we have already dealt with the contribution of the higher term and remainder over any subset of \( J_2 \). Hence we get back to the left side of (11), summed over the points \((m,n_0(m))\) with \( 1 \leq m \leq m_0 \). This is the same as the set of points in Prop. (2.7), except when \( \sqrt{t-m^2} \) is an integer, in which case \((m,n_0(m)-1)\) is also included in the summation in (2.7). There at most \( O(t^\epsilon) \) such points, and hence the conclusion of (2.7) applies to this the real part of this sum together with the real part of the corresponding sum for \( J_1^- \).

Thus we can conclude that over \( J_2 = J_1^+ \cup J_1^- \) there is a total (real) contribution of the form

\[
\mathcal{E}_2(t) = t^{\frac{j}{2}} \sum_{i=1}^{M} d_i \tau^{-i} + \mathcal{L}(t,R) + c_1 t^{-\frac{i}{2} - \eta} (\pi t - P(t)) + O(t^{\eta + \epsilon'}) \quad (16)
\]

Therefore,

\[
\mathcal{E}(t) = t^{\frac{j}{2}} \sum_{i=1}^{M} d_i \tau^{-i} + \mathcal{L}(t,R) + c_1 t^{-\frac{i}{2} - \eta} (\pi t - P(t))
\]

\[
+ O(t^{-\frac{j}{2} - 2\eta |\pi t - P(t)|}) + O(t^{\eta + \epsilon'}) \quad (17)
\]

Now subtract from this the same result with \( \eta \) replaced by \( \eta/2 \) and then absorb \( c_1^{-1} \) into the coefficients \( d_i \); the result is the expression.

\[
t^{-\frac{i}{2}} (\pi t - P(t)) = t^{\frac{i}{2} + \frac{\eta}{2}} \sum_{i=1}^{M} d_i ([t^n]^{-i} - [t^n]^{-i}) + O(t^{-\frac{j}{2} - \eta} |\pi t - P(t)|) + O(t^{\eta + \epsilon'}) \quad (18)
\]

One could now go painstakingly through the above analysis to see that the coefficients \( d_i \) vanish. However, this is not necessary to complete our argument. Suppose we know that \( |\pi t - P(t)| \leq t^\alpha \). Let us take \( \eta = \epsilon' \) in the preceding expression.
that $\alpha - \frac{1}{4} > 2\epsilon$. It is well known that the left side (as well as exponential sums in general) changes sign infinitely often ([L, VIII, 6 or 7]; see also [T, 12.6] and [H, 1916a]). It therefore follows that $d_i = 0$ at least for

$$\frac{1}{4} + \epsilon \frac{1 - i}{2} \geq \alpha - \frac{1}{4},$$

and hence

$$t^{-\frac{1}{4}}|\pi t - P(t)| \ll t^{\alpha - \frac{1}{4} - \frac{1}{2} + \epsilon^2}.$$

Iterating this enough times (depending on $\epsilon$) we obtain

$$|P(t) - \pi t| \ll t^{\frac{1}{4} + 2\epsilon}$$

for any $\epsilon > 0$.

References


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