A concept of convergence in geodesic spaces

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Abstract

A CAT(0) space is a geodesic space for which each geodesic triangle is at least as ‘thin’ as its comparison triangle in the Euclidean plane. A notion of convergence introduced independently several years ago by Lim and Kuczumow is shown in CAT(0) spaces to be very similar to the usual weak convergence in Banach spaces. In particular many Banach space results involving weak convergence have precise analogues in this setting. At the same time, many questions remain open.

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1. Introduction

In 1976 Lim [16] introduced a concept of convergence in a general metric space setting which he called strong \(\Delta\)-convergence. We show here that CAT(0) spaces provide a natural framework for Lim’s concept, and that in such a setting \(\Delta\)-convergence shares many properties of the usual notion of weak convergence in Banach spaces. As a consequence we are able to show that many Banach space concepts and results which involve weak convergence can be extended to a CAT(0) setting. On the other hand, the weak convergence analogy is by no means complete. We list several open questions at the end of the paper. (We should mention that in [15] Kuczumow introduced an identical notion of convergence in Banach spaces, which he calls ‘almost convergence’.)

A metric space is a CAT(0) space (the term is due to Gromov — see, e.g., [1], p. 159) if it is geodesically connected, and if every geodesic triangle in \(X\) is at least as ‘thin’ as its comparison triangle in the Euclidean plane. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [1] or Burago et al. [2]. We note in particular that the complex Hilbert ball with the hyperbolic metric (see [9]; also inequality (4.3) of [19] and subsequent comments) is a CAT(0) space.

2. Preliminary remarks

Let \((X, d)\) be a metric space. A geodesic path joining \(x \in X\) to \(y \in X\) (or, more briefly, a geodesic from \(x\) to \(y\))
is a map $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic is denoted $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\Delta$) and a geodesic segment between each pair of vertices (the edges of $\Delta$). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\Delta$ be a geodesic triangle in $X$ and let $\overline{\Delta}$ be a comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

Let $X$ be a complete CAT(0) space, let $(x_n)$ be a bounded sequence in a complete $X$ and for $x \in X$ set

$$r((x_n)) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ of $(x_n)$ is given by

$$r((x_n)) = \inf \{ r(x, (x_n)) : x \in X \},$$

and the asymptotic center $A((x_n))$ of $(x_n)$ is the set

$$A((x_n)) = \{ x \in X : r(x, (x_n)) = r((x_n)) \}.$$

It is known (see, e.g., [5], Proposition 7) that in a CAT(0) space, $A((x_n))$ consists of exactly one point.

### 3. Basic properties of $\Delta$-convergence

We restrict our study of Lim’s concept to CAT(0) spaces, and throughout this section, $X$ denotes a complete CAT(0) space.

**Definition 3.1.** A sequence $(x_n)$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $(u_n)$ for every subsequence $(u_n)$ of $(x_n)$. In this case we write $\Delta$-lim$_n x_n = x$ and call $x$ the $\Delta$-limit of $(x_n)$.

Next recall that a bounded sequence $(x_n)$ in $X$ is said to be regular if $r((x_n)) = r((u_n))$ for every subsequence $(u_n)$ of $(x_n)$. It is known that every bounded sequence in a Banach space has a regular subsequence (see, e.g., [8], p. 166). The proof is metric in nature and carries over to the present setting without change. Since every regular sequence $\Delta$-converges, we see immediately that every bounded sequence in $X$ has a $\Delta$-convergent subsequence.

Notice that given $(x_n) \subset X$ such that $(x_n)$ $\Delta$-converges to $x$ and given $y \in X$ with $y \neq x$,

$$\limsup_{n} d(x_n, x) < \limsup_{n} d(x_n, y).$$

Thus $X$ satisfies a condition which is known in Banach space theory as the Opial property.

**Remark.** Every bounded closed convex subset $K$ of $X$ is $\Delta$-closed in the sense that it contains the $\Delta$-limits of all of its $\Delta$-convergent sequences (see [4], Proposition 2.1). The following fact is a consequence of this.

**Proposition 3.2.** If a sequence $(x_n)$ in $X$ $\Delta$-converges to $x \in X$, then

$$x \in \bigcap_{k=1}^{\infty} \text{conv} \{ x_k, x_{k+1}, \ldots \},$$

where $\text{conv}(A) = \bigcap \{ B : B \supseteq A \text{ and } B \text{ is closed and convex} \}$. 
The preceding ideas readily extend to nets. We define the asymptotic radius and asymptotic center for nets analogous to the way they are defined for sequences.

**Definition 3.3.** A net \( (x_\alpha) \) in \( X \) is said to \( \Delta\)-converge to \( x \in X \) if \( x \) is the unique asymptotic center of \( (u_\xi) \) for every subnet \( (u_\xi) \) of \( (x_\alpha) \).

The following is Proposition 4 of [13] (in different terminology).

**Proposition 3.4.** A bounded ultranet is \( \Delta\)-convergent.

Since every net has a subnet which is an ultranet, we immediately have the following.

**Proposition 3.5.** Every bounded net in \( X \) has \( \Delta\)-convergent subnet.

The preceding fact can be reformulated as follows. (Cf., Theorem 3 of [16].)

**Proposition 3.6.** Every bounded closed convex set in \( X \) space is \( \Delta\)-compact.

**Proposition 3.7.** Let \( K \) be a closed convex subset of \( X \), and let \( f : K \rightarrow X \) be a nonexpansive mapping. Then the conditions \( (x_\alpha) \Delta\)-converges to \( x \) and \( d(x_\alpha, f(x_\alpha)) \rightarrow 0 \), imply \( x \in K \) and \( f(x) = x \).

**Proof.** Since
\[
\limsup d(f(x), x_\alpha) \leq \limsup \left[ d(f(x), f(x_\alpha)) + d(x_\alpha, f(x_\alpha)) \right] = r(x, (x_\alpha)),
\]
it must be the case that \( f(x) = x \) by uniqueness of asymptotic centers.

**Corollary 3.8 (Theorem 21 of [12]).** Let \( K \) be a bounded closed convex subset of a complete CAT(0) space \( X \). Suppose \( f : K \rightarrow X \) is a nonexpansive mapping for which
\[
\inf \{ d(x, f(x)) : x \in K \} = 0.
\]
Then \( f \) has a fixed point in \( K \).

We have seen that CAT(0) spaces satisfy the Opial property. We now show that they also satisfy what is known in Banach space theory as the Kadec–Klee property. For a bounded sequence \( (x_n) \) in a metric space we denote
\[
\text{sep}(x_n) := \inf \{ d(x_n, x_m) : n \neq m \}.
\]

**Theorem 3.9 (Kadec–Klee Property).** Let \( X \) be a complete CAT(0) space, let \( p \in X \), and let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that \( d(p, x) \leq 1 - \delta \) for every sequence \( (x_n) \subset X \) such that \( d(p, x_n) \leq 1, \text{sep}(x_n) > \varepsilon \) and \( \Delta - \lim_{n} x_n = x \).

**Proof.** For convenience, and without loss of generality, we assume \( d(p, x_n) = 1 \). By passing to a subsequence if necessary we may suppose \( d(x_n, x) \geq \frac{\varepsilon}{2} \) for all \( n \). Let \( \Delta(p, \bar{x}, \bar{x}_n) \) be a comparison triangle for \( \Delta(p, x, x_n) \) in \( \mathbb{E}^2 \).

Then \( x \) is the asymptotic center of \( (x_n) \) relative to the segment \([p, x] \), and \( \limsup d(x_n, x) = r((x_n)) \). For each \( n \), let \( \bar{u}_n \) be the point of the segment \([p, \bar{x}] \) which is nearest to \( \bar{x}_n \), and let \( u_n \) be the point of the segment \([p, x] \) for which \( d(p, u_n) = d(p, \bar{u}_n) \) and \( d(u_n, x) = d(\bar{u}_n, \bar{x}) \). Let \( \theta_n = \angle_{\bar{p}}(\bar{x}, \bar{x}_n) \). By passing to subsequences again we may suppose \( (\bar{u}_n) \) converges to \( \bar{u} \in [\bar{p}, \bar{x}] \), \( (u_n) \) converges to \( u \in [p, x] \), and \( \theta_n \rightarrow \theta \). Since \( d(\bar{x}_n, \bar{x}) = d(x_n, x) \geq \frac{\varepsilon}{2} > 0 \) it must be the case that \( \theta > 0 \). If \( d(p, x) = d(p, \bar{x}) \leq \cos \theta \), take \( \delta = 1 - \cos \theta \). Otherwise \( d(p, x) > \cos \theta \) from which \( \angle_{\bar{u}_n}(\bar{p}, \bar{x}_n) = \frac{\theta}{2} \) and \( d(p, \bar{u}_n) = \cos \theta_n \). This implies \( d(p, \bar{u}) = \cos \theta \) and \( \cos \theta = \lim \cos \theta_n \) can be estimated in terms of \( \varepsilon \). In this case, we have (using the CAT(0) inequality)
\[
r((x_n)) = \limsup d(x_n, x) \\
= \limsup d(\bar{x}, \bar{x}_n) \\
\geq \limsup d(\bar{u}_n, \bar{x}_n) \\
= \limsup d(\bar{u}, \bar{x}_n) \\
\geq \limsup d(u, x_n).
\]
Thus \( r(u, (x_n)) \leq r((x_n)) \). This implies that \( u = x \) by uniqueness of the asymptotic center. Hence \( \bar{u} = \bar{x} \). But \( d(p, u) = d(\bar{p}, \bar{u}) \leq \cos \theta < 1 \). We thus conclude that in either case \( d(p, u) \leq 1 - \delta \), where \( \delta \) is positive and depends on \( \varepsilon \).  

We now turn to a wider class of mappings. We show first that nonexpansive mappings in a CAT(0) space are mappings of type \( \Gamma' \) in the sense of Bruck [3], that mappings of type \( \Gamma' \) are \( \alpha \)-almost convex in the sense of García-Falset et al. [7], and that \( \alpha \)-almost convex mappings are of convex type in the sense of Khamsi [10]. Finally we show that Corollary 3.8 holds for continuous mappings of convex type. Several examples of \( \alpha \)-almost convex mappings that are not nonexpansive are given in [7], so this result is a proper extension of Corollary 3.8.

Crucial to this aspect of our development is the so-called CN inequality of Bruhat and Tits (see [1, p. 163]). Let \( X \) be a CAT(0) space, let \( p, q, r \in X \), and let \( m \) be the mid-point of the segment \([q, r]\). Then

\[
d(p, q)^2 + d(p, r)^2 \geq 2d(m, p)^2 + \frac{1}{2}d(q, r)^2.
\]

We now list the relevant definitions. (We state these definitions using only mid-points since this is sufficient for our purpose.)

**Definition 3.10** ([3]). A mapping \( T: K \rightarrow X \) is said to be of type \( \Gamma' \) if there exists a continuous strictly increasing convex function \( \gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \gamma(0) = 0 \) such that, if \( x, y \in K \) and if \( m \) is the mid-point of the segment \([x, y]\), and if \( m' \) is the mid-point of the segment \([T(x), T(y)]\), then

\[
\gamma \left( d(m', T(m)) \right) \leq |d(x, y) - d(T(x), T(y))|.
\]

**Definition 3.11** ([7]). A mapping \( T: K \rightarrow X \) is called \( \alpha \)-almost convex for \( \alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) continuous, strictly increasing, and \( \alpha(0) = 0 \), if for \( x, y \in K \),

\[
J_T(m) \leq \alpha(\max \{J_T(x), J_T(y)\}),
\]

where \( m \) is the mid-point of the segment joining \( x \) and \( y \), and \( J_T \) is defined by

\[
J_T(x) := d(x, T(x)).
\]

**Definition 3.12** (Cf. [10,11]). Let \( K \) be a convex subset of a CAT(0) space \( X \). A mapping \( T: K \rightarrow X \) is said to be convex type on \( K \) if for \( \{x_n\}, \{y_n\} \in K \) and \( m_n \) the mid-point of the segment \([x_n, y_n]\),

\[
\lim_{n \to \infty} d(x_n, T(x_n)) = 0 \quad \lim_{n \to \infty} d(y_n, T(y_n)) = 0 \quad \Rightarrow \quad \lim_{n \to \infty} d(m_n, T(m_n)) = 0.
\]

**Proposition 3.13.** Let \( K \) be a nonempty bounded closed convex subset of a CAT(0) space \( X \) and let \( T: K \rightarrow X \). Then the following implications hold.

- If \( T \) is nonexpansive \( \Rightarrow \) \( T \) is of type \( \Gamma' \).
- If \( T \) is \( \alpha \)-almost convex \( \Rightarrow \) \( T \) is of convex type.

**Proof.** For the first implication, let \( m \) denote the mid-point of \([x, y]\) for \( x, y \in K \), and let \( m' \) denote the mid-point of the segment \([T(x), T(y)]\). It follows from the (CN) inequality that

\[
d(m', T(m))^2 \leq \frac{1}{2}d(T(x), T(m))^2 + \frac{1}{2}d(T(y), T(m))^2 - \frac{1}{4}d(Tx, Ty)^2
\]

\[
\leq \frac{1}{2}d(x, m)^2 + \frac{1}{2}d(y, m)^2 - \frac{1}{4}d(T(x), T(y))^2
\]

\[
\leq \frac{1}{4}(d(x, y)^2 - d(T(x), T(y))^2)
\]

\[
\leq \frac{1}{4}(d(x, y) + d(T(x), T(y))) (d(x, y) - d(T(x), T(y)))
\]

\[
\leq \frac{1}{2}D(d(x, y) - d(T(x), T(y))),
\]
where \( D = \text{diam}(K) \). This implies that
\[
\frac{2}{D} \cdot d (m’, T(m))^2 \leq |d(x, y) - d(T(x), T(y))|.
\]
Thus it suffices to take \( \gamma(t) = \frac{2t^2}{D} \).

The second implication can be proved by following the steps of the argument of Example 6 of [7], simply replacing \( \| \cdot \| \) with \( d(\cdot, \cdot) \), and the third implication is immediate. ■

**Theorem 3.14.** Let \( K \) be a bounded closed convex subset of a complete \( \text{CAT}(0) \) space \( X \), and let \( T : K \to X \) be continuous and of convex type. Suppose
\[
\inf \{ d(x, T(x)) : x \in K \} = 0.
\]
Then \( T \) has a fixed point in \( K \).

The proof closely follows the proof of Proposition 1 in [11] and Theorem 21 of [12]. For the convenience of the reader we include the details.

**Proof of Theorem 3.14.** Let \( x_0 \in X \) be fixed and define
\[
\rho_0 = \inf \{ \rho > 0 : \inf \{ d(x, f(x)) : x \in B(x_0, \rho) \cap K \} = 0 \}.
\]
Obviously \( \rho_0 < \infty \), and if \( \rho_0 = 0 \) then \( x_0 \in K \) and \( f(x_0) = x_0 \) by continuity of \( f \). So we suppose \( \rho_0 > 0 \).

Now choose \( \{ x_n \} \subset K \) so that \( d(x_n, f(x_n)) \to 0 \) and \( d(x_0, x_n) \to \rho_0 \). Since any convergent subsequence of \( \{ x_n \} \) would have a fixed point of \( f \) as its limit, we may suppose there exist \( \varepsilon > 0 \) and subsequences \( \{ u_k \} \) and \( \{ v_k \} \) of \( \{ x_n \} \) such that \( d(u_k, v_k) \geq \varepsilon \). Passing again to subsequences if necessary we may also suppose \( d(x_0, u_k) \leq \rho_0 + \frac{1}{k} \) and \( d(x_0, v_k) \leq \rho_0 + \frac{1}{k} \). Let \( m_k \) be the mid-point of the segment \( [u_k, v_k] \) and let \( \bar{m}_k \) be the point corresponding to \( m_k \) on the comparison triangle \( \Delta (\bar{x}_0, \bar{u}_k, \bar{v}_k) \) in \( \mathbb{E}^2 \). Then by the CAT(0) inequality
\[
d(x_0, m_k) \leq d(\bar{x}_0, \bar{m}_k) \leq \sqrt{\left( \rho_0 + \frac{1}{k} \right)^2 - \left( \frac{\varepsilon}{2} \right)^2}.
\]
Clearly \( d(x_0, m_k) \leq \rho^* \leq \rho_0 \) for \( k \) sufficiently large. On the other hand, by condition (1), \( d(m_k, f(m_k)) \to 0 \) as \( k \to \infty \). This contradicts the definition of \( \rho_0 \). ■

**4. A four point condition**

In this section \( X \) always denotes a complete \( \text{CAT}(0) \) space, and we assume that \( X \) satisfies the following seemingly mild geometric condition.

\( (Q_4) \) For points \( x, y, p, q \in X \),
\[
\begin{align*}
& d(x, p) < d(x, q) \\
& d(y, p) < (y, q) \\
\end{align*}
\]
for any point \( m \) on the segment \( [x, y] \).

It is easy to see that this condition holds in many \( \text{CAT}(0) \) spaces, including Hilbert spaces and \( \mathbb{R} \)-trees. It is not clear to us whether it holds in all \( \text{CAT}(0) \) spaces.

Now assume that \( K \) is a bounded closed convex subset of a complete \( \text{CAT}(0) \) space \( X \). Let \( \mathcal{U} \) be a nontrivial ultrafilter on the natural numbers \( \mathbb{N} \). Fix \( p \in X \), and let \( \check{X} \) denote the metric space ultrapower of \( X \) over \( \mathcal{U} \) relative to \( p \). Thus the element of \( \check{X} \) consist of equivalence classes \( \check{x} := \{(x_i)\}_{i \in \mathbb{N}} \) for which
\[
\lim_{\mathcal{U}} d(x_i, p) < \infty,
\]
with \( (u_i) \in \{(x_i)\} \) if and only if \( \lim_{\mathcal{U}} d(x_i, u_i) = 0 \). Note that \( \check{X} \) is also a \( \text{CAT}(0) \) space ([1], p. 187). We use \( \check{x} \) to denote the class \( \{(x_i)\} \) with \( x_i \equiv x \), and \( \check{X} \) denotes the canonical embedding of \( X \) in \( \check{X} \).

The following ultrapower characterization of \( \Delta \)-convergence can be found in [4] (Proposition 3.1).
Proposition 4.1. A regular sequence \((x_n) \subset X\) \(\Delta\)-converges to \(x \in X\) if and only if for any nontrivial ultrafilter \(U\) over \(\mathbb{N}\), \(\hat{x}\) is the unique point of \(\hat{X}\) which is nearest to \(\hat{x} := \{(x_n)\}\) in the ultrapower \(\hat{X}_U\).

Proposition 4.2. Suppose \(X\) satisfies (Q4), and suppose \((x_n)\) and \((y_n)\) both \(\Delta\)-converge to \(p \in X\). Suppose \(m_n \in [x_n, y_n]\) satisfies \(d(x_n, m_n) = \lambda d(x_n, y_n)\) for fixed \(\lambda \in (0, 1)\). Then \((m_n)\) also \(\Delta\)-converges to \(p\).

Proof. We pass to the ultrapower \(\hat{X}_U\) of Proposition 4.1. Thus \(\hat{p}\) is the unique point of \(\hat{X}\) which is nearest to both \(\hat{x}\) and \(\hat{y}\). Then some subsequence of \((m_n)\), which we again denote \((m_n)\), \(\Delta\)-converges to \(q\), and \(\hat{q}\) is the unique point of \(\hat{X}\) which is nearest to \(\hat{m}\). We pass to corresponding subsequences of \((x_n)\) and \((y_n)\) and relabel as at the outset. Assume \(\hat{q} \neq \hat{p}\). Then \(d_U(\hat{x}, \hat{p}) < d_U(\hat{x}, \hat{q})\) and \(d_U(\hat{y}, \hat{p}) < d_U(\hat{y}, \hat{q})\), while \(d_U(\hat{m}, \hat{q}) < d_U(\hat{m}, \hat{p})\). It follows that one can choose \(n\) so that \(d(x_n, p) < d(x_n, q)\), \(d(y_n, p) < d(y_n, q)\), and \(d(m_n, q) < d(m_n, p)\). This contradicts condition (Q4). Thus every subsequence of the original sequence \((m_n)\) \(\Delta\)-converges to \(p\), and so \((m_n)\) itself \(\Delta\)-converges to \(p\). ■

We will need Proposition 4.2 in the next section.

5. LANE mappings

While this may not be a significant observation, we mention in passing that the approach of Kirk–Sims [14] carries over to CAT(0) spaces which satisfy (Q4) if one makes certain minor adjustments. If \(K\) is a closed convex subset of a Banach space \(X\), a continuous mapping \(f : K \to X\) is said to be \(\text{locally almost nonexpansive (LANE)}\) if for each \(x \in K\) and \(\varepsilon > 0\) there exists a weak neighborhood \(N_x\) of \(x\) such that for \(u, v \in N_x\), \(\|f(u) - f(v)\| \leq \|u - v\| + \varepsilon\).

The concept is due to Nussbaum [17]. He proved that if \(X\) is uniformly convex and if \(f : K \to X\) is a LANE mapping, then \(I - f\) is demiclosed on \(K\), in the sense that the conditions \((x_n) \subset K\) \(\Delta\)-converges weakly to \(x \in X\) and \(\|(I - f)(x_n) - y\| \to 0 \Rightarrow x \in K\) and \(x - f(x) = y\).

It is not even possible to formulate the above result, as stated, in a CAT(0) setting. However it is possible to use the notion of \(\Delta\)-convergence to formulate a precise analogue.

Definition 5.1. Let \(K\) be a closed convex subset of a complete CAT(0) space. A continuous mapping \(f : K \to X\) is said to be LANE if for each \(x \in K\) and \(\varepsilon > 0\) the following condition holds: If \((u_n), (v_n)\) are two sequences in \(K\) which \(\Delta\)-converge to \(x\), then there exists \(N \in \mathbb{N}\) such that

\[
d(f(u_n), f(v_n)) \leq d(u_n, v_n) + \varepsilon \quad \text{whenever} \quad n \geq N.
\]

(2)

It is now possible to follow the approach of [14]. Assume \(f : K \to X\) is a LANE mapping. For \(\hat{x} = [(x_i)] \in \hat{K}\), define \(\hat{f} : \hat{K} \to X\) by setting

\[
\hat{f}([x_i]) = [(f(x_i))].
\]

For each \(x \in K\), let

\[
W_x = \left\{ \hat{x} = [(x_i)] \in \hat{K} : \Delta-lim_{U} x_i = x \right\}.
\]

If \(\Delta-lim x_n = x\) and \(\Delta-lim y_n = y\), \(\hat{x}\) and \(\hat{y}\) are the unique points of \(\hat{X}\) which are nearest to \(\hat{x}\) and \(\hat{y}\), respectively. Since the nearest point projection from \(\hat{X}\) onto \(\hat{X}\) is nonexpansive ([1, p. 176]), \(d(x, y) = d_U(\hat{x}, \hat{y}) \leq d_U(\hat{x}, \hat{y})\). It follows that the sets \(W_x\) are closed, and by Proposition 4.2 they are also convex. The remaining details follow as in [14] and lead to the following analogue of Proposition 3.7.

Theorem 5.2 (Cf. [20]). Let \(K\) be a closed convex subset of a complete CAT(0) space \(X\), suppose \(X\) satisfies (Q4), and let \(f : K \to X\) be a continuous LANE mapping. Then the conditions \((x_n)\Delta\)-converges to \(x\) and \(d(x_n, f(x_n)) \to 0\), imply \(x \in K\) and \(f(x) = x\).

Proof. Suppose \((x_n) \subset K\) satisfies \(d(x_n, f(x_n)) \to 0\) and \(\Delta-lim x_n = x\). Letting \(\hat{x} = [(x_n)]\), we have \(\hat{f}(\hat{x}) = \hat{x}\). By Proposition 4.1 \(\hat{x}\) is the unique point of \(\hat{X}\) which is nearest to \(\hat{x}\). Since \(\hat{f} : \hat{K} \to X\), and since the restriction of \(\hat{f}\) to \(W_x\) is nonexpansive, it must be the case that \(\hat{f}(\hat{x}) = \hat{x}\). Since \(f\) is continuous, this implies that \(f(x) = x\). ■
Remark. In a Banach space a LANE mapping is always a 1-set contraction [17]. Consequently, if $K$ is a bounded closed and convex subset of a uniformly convex Banach space, and if $f : K \rightarrow K$ is a continuous LANE mapping, then $f$ always has a fixed point. This is because it is possible to uniformly approximate $f$ with $k$-set contractions, $k < 1$, and then obtain a sequence $(x_n)$ in $K$ such that $\|x_n - f(x_n)\| \rightarrow 0$. In [18] Reich extends this fact to mappings of the form $f = g + h$, where $g$ is a LANE mapping and $h$ is strongly continuous (i.e., if $x_n \subset K$ converges weakly to $x$ then $h(x_n)$ converges strongly to $x$). No analogue of these facts seems to exist in CAT(0) spaces, a fact that probably limits interest in LANE mappings in such a context.

6. $J$-type mappings

Definition 6.1 ([6]). Let $K$ be a nonempty subset of a metric space $(X, d)$. We say that $y_0 \in X$ is a center for the mapping $T : K \rightarrow X$ if, for each $x \in K$,

$$d(y_0, T(x)) \leq d(y_0, x).$$

We say that $T : K \rightarrow X$ is a $J$-type mapping whenever it is continuous and it has some center $y_0 \in X$. We use $Z(T)$ to denote the set of all centers of the mapping $T$.

If a mapping $T : K \rightarrow X$ has a center $y_0 \in K$, then trivially $T(y_0) = y_0$. Thus, fixed point results for $J$-type mappings are only nontrivial provided they have a center $y_0 \not\in K$.

The following is an analogue of Proposition 13 of [6].

Proposition 6.2. Let $K$ be a bounded closed convex subset of a complete CAT(0) space $X$ and $T : K \rightarrow X$ be a $J$-type mapping. Suppose there exists $x_0 \in K$ such that, for every $n \geq 0$, $T^n(x_0) \in K$ and

$$w_\infty(x_0) \subset Z(T).$$

Then $T$ has a fixed point $x$ and the Picard iterates sequence $(T^n(x_0))$ is $\Delta$-convergent to $x$. Here $w_\infty(x_0) := \bigcup A((u_n))$ where the union is taken over all subsequences $(u_n)$ of $T^n(x_0)$.

Proof. By Proposition 3.5, the set $w_\infty(x_0)$ is nonempty and moreover it is contained in $K \cap Z(T)$. Therefore, each point in $w_\infty(x_0)$ is a fixed point of $T$.

We now let $A(T^n(x_0)) = \{x\}$ and let $(u_n)$ be a subsequence of $(T^n(x_0))$. Suppose $A((u_n)) = \{y\}$ with $y \neq x$. Since $y \in Z(T)$ we know that the sequence $(d(T^n(x_0), y))$ is decreasing, and hence it is convergent. Then by the uniqueness of asymptotic centers,

$$\lim \sup_n d(u_n, y) < \lim \sup_n d(u_n, x) \leq \lim \sup_n d(T^n(x_0), x) < \lim \sup_n d(T^n(x_0), y) = \lim \sup_n d(u_n, y)$$

a contradiction. Therefore $w_\infty(x_0) = \{x\}$, and then the conclusion follows.  

The following is an analogue of Proposition 10 of [6]. We observe that the result holds in any complete metric space.

Proposition 6.3. Let $K$ be a closed subset of a complete metric space $X$ and $T : K \rightarrow X$ be a $J$-type mapping. Suppose there exists $x_0 \in K$ such that, for every $n \geq 0$, $T^n(x_0) \in K$. If

$$\lim_{n \rightarrow \infty} d(T^n(x_0), Z(T)) = 0,$$

then $T$ has a fixed point $x$ and the sequence of Picard iterates $(T^n(x_0))$ converges to $x$.

Finally, we include with another result which is known as the $J$-fixed point property.

Proposition 6.4. Let $K$ be a closed convex subset of a complete CAT(0) space $X$ and $T : K \rightarrow K$ be a $J$-type mapping. Then $T$ has a fixed point.
Proof. Let \( y_0 \in X \) be a center of \( T \). If \( y_0 \in K \), then it is a fixed point of \( T \) and nothing needs to be proved. If \( y_0 \not\in K \), we let \( x \) be the unique point of \( K \) nearest to \( y_0 \). It is followed from the definition of \( T \) that
\[
d(T(x), y_0) \leq d(x, y_0).
\]
Since \( T(x) \in K \), by the uniqueness of \( x \), we have \( x = T(x) \) and the proof is complete. \( \blacksquare \)

7. Concluding remarks

It would seem that the results of this paper along with those of [4] make a strong case for calling \( \Delta \)-convergence ‘weak’ convergence, at least in the CAT(0) context (and indeed this terminology was adopted in [4]). On the other hand, the analogy is far from complete. We now list a few of the many open questions that remain. Throughout, \( X \) denotes a complete CAT(0) space.

1. First and foremost, is there a topology \( \tau \) on \( X \) such that a sequence \((x_n)\) converges relative to \( \tau \) if and only if it \( \Delta \)-converges? This is true in Hilbert space and surely in some other CAT(0) spaces as well, but in general \( \Delta \)-convergence seems to provide only the quasi-topology that in [16] Lim calls the strong \( \Delta \)-topology.

2. If \((x_n) \subset X \) \( \Delta \)-converges to \( x \in X \), and if \( y_n \in \text{conv} \{x_n, x_{n+1}, \ldots\} \) for each \( n \in \mathbb{N} \), then does \((y_n) \) \( \Delta \)-converge to \( x \)?

3. (Cf. Proposition 3.2.) If a sequence \((x_n)\) in \( X \) \( \Delta \)-converges to \( x \in X \), then is the case that
\[
\{x\} = \bigcap_{k=1}^{\infty} \text{conv} \{x_k, x_{k+1}, \ldots\}?
\]

4. Is the condition \((Q_4)\) necessary in Proposition 4.2?

5. Does every CAT(0) space satisfy \((Q_4)\)? Obviously many CAT(0) spaces, including Hilbert spaces, have this property.

6. Sosov [21] has recently introduced two analogues of weak convergence which are meaningful in CAT(0) spaces as well. The relationship between his notions and \( \Delta \)-convergence is not immediately clear.

References