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\(\mathcal{F}K\)-convex functions on metric spaces

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Abstract. By an \(\mathcal{F}K\)-convex function on a length metric space, we mean one that satisfies \(f'' \geq -Kf\) on all unit speed geodesics. We show that natural \(\mathcal{F}K\)-convex (-concave) functions occur in abundance on metric spaces of curvature bounded above (below) by \(K\) in the sense of Alexandrov. We prove Lipschitz extension and approximation theorems for \(\mathcal{F}K\)-convex functions on CAT(\(K\)) spaces.

1. Introduction

This note deals with \(\mathcal{F}K\)-convex functions on metric spaces with curvature bounded above (CBA) by \(K\) in the sense of Alexandrov, and the dual concept, \(\mathcal{F}K\)-concave functions on finite-dimensional metric spaces with curvature bounded below (CBB) by \(K\). By an \(\mathcal{F}K\)-convex function on a length metric space, we mean a continuous real-valued function whose restriction \(f\) to every unit speed geodesic satisfies the differential inequality \(f'' \geq -Kf\) in the barrier sense. Letting \(\mathcal{F}K\) denote the family of solutions of the differential equation \(f'' + Kf = 0\), this means that \(f \leq g\) if \(g \in \mathcal{F}K\) coincides with \(f\) at the endpoints of a sufficiently short subsegment. Thus \(\mathcal{F}0\)-convexity is usual convexity. For \(\mathcal{F}K\)-concave functions the inequalities are reversed.

In this note, we first point out that natural nonnegative \(\mathcal{F}K\)-convex and \(\mathcal{F}K\)-concave functions occur in abundance on spaces with curvature bounded above and below by \(K\) respectively. Basic examples are collected in Theorem 1.1. Secondly, we use these constructions to show that non-Lipschitz \(\mathcal{F}K\)-convex functions, which occur naturally on CAT(\(K\)) spaces but are ill adapted to analytic arguments, can be approximated by Lipschitz ones.

The importance of convex and concave functions in nonpositively and non-negatively curved spaces is well documented; e.g., it is well known that Busemann functions are convex in CAT(0) spaces, and concave in spaces of CBB by 0. Perelman’s theorem ([Pe]; see [BBI, p. 400]) on concavity of distance from the boundary
of a space of CBB by 0 is (3B) below. Perelman also proved that if $X$ has CBB by 1, then distance from $\partial X$ is strictly concave. Statement (5B) makes this precise.

**Theorem 1.1. (Construction of $F_K$-convex and $F_K$-concave functions)**

Except in (1A), (2A) and (5A), assume $X$ is complete.

**CBA**

(1A) If $X$ is CAT($-1$), then $\cosh(\text{distance to a point})$ is $F(-1)$-convex.

(2A) If $X$ is CAT($-1$), then $\exp(\text{Busemann function})$ is $F(-1)$-convex.

(3A) If $X$ has CBA by 0, then distance to a TCS is convex.

(4A) If $X$ has CBA by $-1$, then $\sinh(\text{distance to a TCS})$ is $F(-1)$-convex.

(5A) If $X$ is CAT(1), then $\sin(\text{distance to a } \pi\text{-TCS})$ is $F1$-concave on the points whose distance from the $\pi\text{-TCS}$ is at most $\pi/2$.

**CBB**

(1B) If $X$ has CBB by $-1$, then $\cosh(\text{distance to a point})$ is $F(-1)$-concave.

(2B) If $X$ has CBB by $-1$, then $\exp(\text{Busemann function})$ is $F(-1)$-concave.

(3B) [Pe] If $X$ has CBB by 0, then distance to $\partial X$ is concave.

(4B) If $X$ has CBB by $-1$, then $\sinh(\text{distance to } \partial X)$ is $F(-1)$-concave.

(5B) If $X$ has CBB by 1, then $\sin(\text{distance to } \partial X)$ is $F1$-concave.

Here, a subset $T$ is $\ell$-totally convex (an $\ell$-TCS) if $T$ contains all geodesics of length less than $\ell$ joining pairs of points in $T$. A CAT($K$) space is a metric space in which any two points at distance less than $\pi/\sqrt{K}$ ($=\infty$ if $K \leq 0$) are joined by a distance-realizing geodesic, and any triangle $\Delta$ of perimeter less than $2\pi/\sqrt{K}$ is thinner than its model triangle in the simply connected, 2-dimensional space form $S_K$ of curvature $K$. That is, the distance between any two points of $\Delta$ is no greater than the distance between the corresponding points on a triangle with the same sidelengths in $S_K$. A length space has CBA by $K$ if it is locally CAT($K$). For CBB by $K$, the inequality is reversed. There is no distinction between the local and global triangle comparison properties in complete CBB spaces ([BGP]), or in complete, simply connected spaces with CBA by $K \leq 0$ ([G], [AB1]; see also [Ba], [BH]). (In the case of CBB by $K > 0$, circles of length $2\pi/\sqrt{K}$ and intervals of length $\pi/\sqrt{K}$ are understood to be excluded.)

Excellent texts and monographs on Alexandrov spaces have recently appeared ([Ba], [BN], [BH], [BBI], [Bu], [J], [Pl]). Here we take [BBI] and [BH] as references.

If a warped product space has CBA or CBB by $K$, the warping function must be, respectively, an $F_K$-convex or $F_K$-concave function that satisfies an additional first-order differential inequality [AB3]. All the functions in Theorem 1.1 belong, after scaling, to this class, and hence lead to new constructions of spaces with curvature bounds via the operation of warped product.

The example of warping functions underscores the fact that interesting $F_K$-convex functions need not be locally Lipschitz. (Indeed, there may be no point having a neighborhood on which the function is Lipschitz; see Example 5.1.) This is problematical; for instance, for an $F_K$-convex function $f$ on a CAT($K$)-space, the differential $Df$ is welldefined as a function on the direction space at $p$ when $f$ is Lipschitz [K], but not otherwise (see Example 5.2). The extension and
approximation theorems below circumvent the problems, and are an essential tool in [AB3]. By $(Df)_p \geq -A$ (resp. $| (Df)_p | \leq A$), we mean $(f \circ \gamma)'(0) \geq -A$ (resp. $| (f \circ \gamma)'(0) | \leq A$) for all geodesics $\gamma$ from $p$. We shall always take geodesics to be unitspeed.

**Theorem 1.2. (Convex extensions).** If $f$ is a convex function on a complete CAT$(0)$ space $X$, then for every sufficiently large $A$ there is a convex function $f^A$ on $X$ that agrees with $f$ on $X^A = \{ p \in X : (Df)_p \geq -A \}$ and satisfies $|Df^A| \leq A$ everywhere.

The correct analogue of Theorem 1.2 to nonzero curvature bounds is not at first apparent, and is given in the next theorem.

**Theorem 1.3. ($FK$-convex extensions).** Suppose $f$ is $FK$-convex on a complete CAT$(K)$ space $X$. If $K < 0$ and $f$ is bounded below, then for every $A > 0$ there exists an $FK$-convex function $f^A$ that agrees with $f$ on $X^A = \{ p \in X : (Df)_p \geq -\sqrt{A^2 - Kf(p)^2} \}$, and satisfies $(Df^A)_p \leq A^2 - Kf^A(p)^2$ everywhere. If $K > 0$ and $f(p) + f(q) \geq 0$ whenever $d(p, q) \geq \pi/\sqrt{K}$, then for sufficiently large $A$ there exists an $FK$-convex function $f^A$ that agrees with $f$ on $X^A = \{ p \in X : A^2 - Kf(p)^2 \geq 0, (Df)_p \geq -\sqrt{A^2 - Kf(p)^2} \}$, and satisfies $(Df^A)_p \leq A^2 - Kf^A(p)^2$ everywhere.

Theorems 1.2 and 1.3 yield the following Lipschitz approximation theorem:

**Theorem 1.4. (Lipschitz approximations).** A convex function on a complete CAT$(0)$ space is the limit of Lipschitz convex functions. For $K < 0$, an $FK$-convex function on a complete CAT$(K)$ space $X$ is the limit of $FK$-convex functions, each of which is Lipschitz on the sets in a convex exhaustion of $X$. For $K > 0$, an $FK$-convex function $f$, that satisfies $f(p) + f(q) \geq 0$ whenever $d(p, q) \geq \pi/\sqrt{K}$, on a complete CAT$(K)$ space is the limit of Lipschitz $FK$-convex functions.

We have not encountered settings that require similar approximations on CBB spaces. In particular, if a warped product has CBB by $K$, the warping function is necessarily Lipschitz [AB3].

Note that $FK$-convexity ($FK$-concavity) is not the same as the $\lambda$-convexity ($\lambda$-concavity) property in [BBI], [K], [Pl], [PP], namely, a generalized constant lower (upper) bound on $f''$. However, the first property implies the second locally. Lipschitz $\lambda$-concave functions give rise to an important construction of $F1$-concave functions, which was already used in [PP]. Recall that for a CBB (CBA) space, the direction space $\Sigma_p$ at a point $p$ has CBB by 1 ([BGP]) (respectively, CBA by 1 ([N])). If $f$ is Lipschitz and $\lambda$-concave ($\lambda$-convex), then the differential $Df$ is well-defined on the direction space $\Sigma_p$, and its extension to a homogeneous function of degree 1 on the tangent cone $C(\Sigma_p)$ is concave (convex) ([K]). It is straightforward to deduce that $Df$ is $F1$-concave ($F1$-convex) on $\Sigma_p$ (see Lemma 3.5).

We assume $FK$-convex and $FK$-concave functions to be continuous. For an investigation of discontinuous convex functions on 2-dimensional CBB spaces, see [M].
2. \(FK\)-affine functions on \(S_K\)

By a \(FK\)-affine function on a metric space, we mean one whose restriction to every geodesic lies in \(FK\). Thus the \(F0\)-affine functions are the affine ones. The following two lemmas are the starting points for all our theorems.

**Lemma 2.1.** If \(x\) is signed distance to the image of a complete geodesic in the model space \(S_K\), then \(x\) is affine if \(K = 0\); \(\sinh \sqrt{-K}x\) is \(FK\)-affine if \(K < 0\); and \(\sin \sqrt{K} x\) is \(FK\)-affine if \(K > 0\).

**Proof.** This is immediate in the Euclidean case. In the remaining cases, it follows from the fact that the warping function of a warped product must be \(FK\)-convex or \(FK\)-concave if the warped product has \(K\) as an upper or lower curvature bound, respectively. Specifically, we write \(H^3(K)\) – (line) as:

\[
\left( R > 0 \times \sinh \sqrt{-K} x \times \sinh \sqrt{-K} x^{-1} \right) \times \cosh \sqrt{-K} x \times \sinh \sqrt{-K} x^{-1},
\]

where \(S^1\) denotes a circle of length \(2\pi\). The righthand formula expresses \(H^3(K)\) – (line) as a warped product with base the hyperbolic halfplane \(x > 0\) and warping function \(\sinh \sqrt{-K} x\). Similarly in the spherical case, \(S^3(K)\) – (great circle) may be written:

\[
\left( 0, \pi \right) \times \cos \sqrt{K} x \times \sin \sqrt{K} x^{-1},
\]

where \(S^1\) denotes a circle of length \(2\pi\).

**Lemma 2.2.** If \(d_p\) is distance to a point \(p\) in \(S_K\), then \(\cosh \sqrt{-K} d_p\) is \(FK\)-affine if \(K < 0\); and \(\cos \sqrt{K} d_p\) is \(FK\)-affine if \(K > 0\).

**Proof.** For \(K > 0\), the claim follows from Lemma 2.1. Indeed, if \(x\) is signed distance to the image of a complete geodesic and \(p\) is the pole of this geodesic on its positive side, then \(\cos \sqrt{K} d_p = \sin \sqrt{K} x\).

Suppose \(K < 0\). Consider a geodesic segment \(pq_0q_2\) in \(S_K\) with length \(2m\) and midpoint \(q_1\). Let \(d(p, q_0) = a\) and \(d(p, q_2) = b\). The hyperbolic law of cosines applied to the triangles \(pq_0q\) and \(pq_2q\) implies

\[
\cosh \sqrt{-K} d(p, q_1) = (\cosh \sqrt{-K} a + \cosh \sqrt{-K} b) / 2 \cosh \sqrt{-K} m.
\]

Since the righthand side is the value at \(s = 0\) of the function \(Ae^{\sqrt{-K} s} + Be^{-\sqrt{-K} s}\) in \(FK\) whose values at \(s = -m\) and \(s = m\) respectively are \(\cosh \sqrt{-K} a\) and \(\cosh \sqrt{-K} b\), the claim follows.

**Remark 2.3.** The corresponding statement for \(K = 0\) is that the restriction of \(d_p^2/2\) to every geodesic satisfies \(f'' = 1\). In contrast to \(d_p\), the distance to a geodesic satisfies analogous \(FK\)-affine conditions for all \(K\), as given in Lemma 2.1. This analogy suggests constructions (4A), (4B) and (5A), (5B) in Theorem 1.1.

**Remark 2.4.** Lemmas 2.1 and 2.2 may be proved alternatively by showing that these functions are the restrictions to the quadric surface models of \(S_K\), of linear functions on \(R^3\) or \(R^3_1\).
More generally, for a simply-connected \( n \)-dimensional Riemannian manifold of constant curvature \( K \) there is a natural way of realizing the \( F_K \)-affine functions. There is an isometric imbedding as a quadric hypersurface in Euclidean space (for \( K = 0 \) the quadratic equation is degenerate, \((x_{n+1}-1)^2 = 0\)). Then the usual linear functions of the ambient space restrict to become the space of all the \( F_K \)-affine functions on the Riemannian manifold, a linear space of dimension \( n + 1 \).

**Corollary 2.5.** If \( d_p \) is distance to a point \( p \) in \( S_K \) and \( b \in \mathbb{R} \), then \( d_p + b \) is convex if \( K = 0 \); \( \sinh \sqrt{-K}(d_p + b) \) is \( F_K \)-convex if \( K < 0 \); and \( \sin \sqrt{K}(d_p + b) \) is \( F_K \)-convex if \( K > 0 \) and \(-\pi/2\sqrt{K} \leq b \leq \pi/2\sqrt{K}\).

**Proof.** If \( K = 0 \), then \( d_p \) is convex by a simple argument in Euclidean geometry [BH, p. 176]. A straightforward calculation using Lemma 2.2 and the identity relating \( \sinh \sqrt{-K}d_p \) and \( \cosh \sqrt{-K}d_p \) (resp. \( \sin \sqrt{K}d_p \) and \( \cos \sqrt{K}d_p \)) shows that the remaining claims when \( b = 0 \) follow from the fact \( |D(dp)| \leq 1 \). But the general claims follow from the cases when \( b = 0 \), by virtue of the sum formulas for sine and hyperbolic sine, together with Lemma 2.2. For example, if \( K = 1 \), then \( \sin(d_p + b) = \sin d_p \cos b + \cos d_p \sin b \), where the latter term is \( F_1 \)-affine by Lemma 2.2 and, as we have just seen, the former is \( F_1 \)-convex if \( \cos b > 0 \).

**Corollary 2.6.** If \( d_\sigma \) is distance to a geodesic segment \( \sigma \) in \( S_K \), then \( d_\sigma \) is convex if \( K = 0 \); \( \sinh \sqrt{-K}d_\sigma \) is \( F_K \)-convex for \( K < 0 \); and \( \sin \sqrt{K}d_\sigma \) is \( F_K \)-convex for \( K > 0 \).

**Proof.** The restriction of \( d_\sigma \) to a sufficiently short geodesic coincides either with distance to the image of a complete geodesic, or distance to an endpoint of \( \sigma \), or a join of these two. The first two have the appropriate \( F_K \)-convexity, by Lemma 2.1 and Corollary 2.5, respectively. This suffices, since the derivative at the join is two-sided, by the first variation formula.

3. **Construction of \( F_K \)-convex and concave functions**

**Distance to a point**

For (1A) and (1B) of Theorem 1.1, it suffices to note that by Lemma 2.2, saying that \( \cosh \sqrt{-K}d_p \) is \( F_K \)-concave (resp. \( F_K \)-convex) is equivalent to saying that CBB triangle comparisons (resp. CBA triangle comparisons) hold globally:

**Proposition 3.1.** \( X \) is complete with CBB by \( K < 0 \) if and only if \( \cosh \sqrt{-K}d_p \) is \( F_K \)-concave for all \( p \in X \); and is CAT(\( K \)) for \( K < 0 \) if and only if \( \cosh \sqrt{-K}d_p \) is \( F_K \)-convex for all \( p \in X \).

**Remark 3.2.** A unifying formulation of the characterizing differential inequalities for curvature bounds is given in [PP]. Specifically, \( X \) has CBB by \( K \) if and only if the restriction to every geodesic of the following function satisfies \( f'' \leq 1 - K f \) (in the barrier sense): \( (1/K)(1 - \cosh \sqrt{-K}d_p) \), if \( K < 0 \); \( (1/K)(1 - \cos \sqrt{K}d_p) \), if \( K > 0 \); \( d_p^2/2 \), if \( K = 0 \).
Busemann functions

(2A) and (2B) follow easily from (1A) and (1B), as the following proposition shows.

We remark that the concavity of Busemann functions for spaces of CBB by 0 may be deduced from (2B) by letting $K \to 0^-$.

**Proposition 3.3.** If $X$ is complete with CBB by $K < 0$ (resp., is CAT($K$) for $K < 0$) and $h$ is a Busemann function on $X$, then $\exp \sqrt{-K} h$ is $\mathcal{F}K$-concave (resp. $\mathcal{F}K$-convex).

**Proof.** By the definition of $h$ in terms of a ray $\gamma$ in $X$,

$$\exp \sqrt{-K} h = \lim_{t \to \infty} \exp \sqrt{-K}(d_{\gamma(t)} - t)$$

$$= \lim_{t \to \infty} \left( \exp \sqrt{-K}(d_{\gamma(t)} - t) + \exp \sqrt{-K}(-d_{\gamma(t)} - t) \right)$$

$$= \lim_{t \to \infty} 2 \cosh \sqrt{-K} d_{\gamma(t)} \exp \sqrt{-K}(-t).$$

Since $\mathcal{F}K$-convexity and $\mathcal{F}K$-concavity are preserved in the limit, (2A) and (2B) follow from (1A) and (1B). $\square$

Distance to a totally convex set

Now we verify (3A), (4A) and (5A). The case of (3A) for which $X$ is CAT(0) and $T$ is a point is well known.

**Proposition 3.4.** Let $T$ be a $\pi/\sqrt{K}$-totally convex set in $X$, and $d_T$ be distance to $T$. If $X$ is complete with CBA by $K \leq 0$, then $d_T$ is convex if $K = 0$ and $\sinh \sqrt{-K} d_T$ is $\mathcal{F}K$-convex if $K < 0$. If $X$ is CAT($K$) for $K > 0$, then $\sin \sqrt{K} d_T$ is $\mathcal{F}K$-convex on the set of points whose distance from $T$ is at most $\pi/2\sqrt{K}$.

Since the distances to $T$ and its closure agree, we assume $T$ is closed in $X$.

**Proof of Proposition 3.4 for $K \leq 0$.** Let $\pi : \hat{X} \to X$ be the universal covering map, and set $\hat{T} := \pi^{-1}(T)$. Then $\hat{X}$ is a CAT(0) space and $\hat{T}$ is a convex subset of $\hat{X}$. Therefore minimizers to $\hat{T}$ exist, are unique and vary continuously with their initial point ([BH, II.2]). The same statement holds for minimizers in $X$ to $T$, since minimizers to $T$ lift to minimizers to $\hat{T}$, and the latter project to the former.

We finish with what is essentially Alexandrov’s patchwork construction; however, quadrilaterals are more convenient than triangles as patches. Let $\tau$ be a geodesic in $X$, and $\gamma_t$ be the minimizer from $\tau(t)$ to $T$. Cover $\gamma_t$ by finitely many CAT($K$) neighborhoods whose union contains $\gamma_t$. $a - \epsilon \leq t \leq a + \epsilon$. Thus we may subdivide each of $\gamma_{a-\epsilon}$ and $\gamma_{a+\epsilon}$ into $k$ successive segments so that each corresponding pair of segments lies in a CAT($K$) neighborhood. Now apply Reshetnyak majorization [R] to each quadrilateral composed of a pair of corresponding segments of $\gamma_{a-\epsilon}$ and $\gamma_{a+\epsilon}$, and the minimizers (crossbars) joining their corresponding endpoints. Regard the comparison convex quadrilaterals in $\mathcal{S}_K$ as being glued along successive crossbars to form a region $R$. By angle comparisons, at any point where a crossbar meets a side, the interior angle of $R$ is at least $\pi$. For $\epsilon$ sufficiently small,
the length of the boundary of $R$ is less than $2\pi/\sqrt{K}$. Therefore $R$ lies in an open hemisphere of $S_K$ if $K > 0$, and in all cases the boundary of $R$ is a quadrilateral in the intrinsic metric of $R$. Now we majorize the boundary of $R$, to obtain a convex quadrilateral in $S_K$, one of whose sides, say $\tilde{\tau} : [a - \epsilon, a + \epsilon] \to S_K$, corresponds to $\tau = [\tilde{\tau}]$. By majorization, the distance in $S_K$ from $\tilde{\tau}(t)$ to the opposite side is at least equal to the distance in $X$ from $\tau(t)$ to a point on the crossbar joining the footpoints of $\gamma_{a-\epsilon}$ and $\gamma_{a+\epsilon}$. This crossbar lies in $T$. Therefore the distance in $X$ from $\tau(t)$ to $T$ is at most the distance in $S_K$ from $\tilde{\tau}(t)$ to a fixed geodesic segment $\sigma$ in $S_K$. By construction, these distances agree at $t = a \pm \epsilon$. Since the latter distance satisfies the appropriate $\mathcal{F}K$-convexity condition by Corollary 2.6, so does the former. This completes the proof if $K \leq 0$. \hfill \Box

We analyze the case $K > 0$ using the following lemma:

**Lemma 3.5.** Let $f$ be a real-valued function on a geodesic metric space $X$, and $C(f)$ be its linear homogeneous extension to the linear cone $C(X)$. Then $C(f)$ is convex on $C(X)$ if and only if $f$ is $\mathcal{F}1$-convex on $X$ and $f(x) + f(y) \geq 0$ whenever $d(x, y) \geq \pi$.

**Proof.** Recall that $C(X)$ is defined as $[0, \infty) \times X$ with the points of $[0] \times X$ identified. We denote the class of $[0] \times X$ by $0$. A geodesic in $C(X)$ either does not pass through 0 and projects to a reparametrized geodesic in $X$ of length less than $\pi$, or its image is the union of two segments from the vertex 0.

The cone over a geodesic $\sigma$ in $X$ of length $< \pi$ with arclength parameter $\theta$ is a flat totally geodesic subspace of $C(X)$ isometric to a sector of $E^2$ with polar coordinates $(r, \theta)$. On this sector, $C(f)$ has the form $rf(\theta)$. If $-m \leq \theta \leq m$, the midpoint $\mathcal{F}1$-convexity inequality for $f$ on $\sigma$ is $f(0) \leq (f(m) + f(-m))/2 \cos m$. The convexity inequality for $C(f)$ on a Euclidean segment from $(r_1, -m)$ to $(r_2, m)$, evaluated at the point that divides this segment in the ratio $r_1 : r_2$, is the same, namely:

$$2r_1r_2(r_1 + r_2)^{-1} \cos mf(0) \leq r_1(r_1 + r_2)^{-1} f(m) + r_2(r_1 + r_2)^{-1} r_1 f(-m).$$

Sequential application of this process yields the lemma. \hfill \Box

**Proof of Proposition 3.4 for $K > 0$.** By scaling, let $X$ be CAT(1) and $T$ be a $\pi$-totally convex subset of $X$. Let $S$ consist of points of $X$ at distance $\leq \pi/2$ from $T$. We claim $d_{C(T)}$ is $\mathcal{F}1$-convex on $S$, that is, on geodesics of $X$ that lie in $S$.

The linear cone $C(X)$ is CAT(0) ([Be]; see also [BH, p. 188]). Since $T$ is $\pi$-totally convex in $X$ and geodesics of $C(X)$ not passing through 0 have projections to $X$ of length less than $\pi$, it follows that $C(T)$ is convex in $C(X)$. The distance $d_{C(T)}$ to $C(T)$ in $C(X)$ coincides with the extension of $\sin(\min(d_T, \pi/2))$ to a homogeneous function of degree 1 on $C(X)$. That is, $d_{C(T)}$ agrees on $C(S)$ with the homogeneous extension of $\sin d_T$ to $C(S)$. We have already seen that $d_{C(T)}$ is convex, so the claim follows from Lemma 3.5. \hfill \Box

**Remark 3.6.** An interesting class of spaces for which the above coning argument fails consists of complete $X$ that have CBA by $K > 0$ but are not CAT($K$). Such
examples include thin cylinders; and barbell surfaces obtained by joining two spheres along a small negatively curved waist, where $T$ is a short closed geodesic in the waist. However, if $T$ is compact and $X$ is locally compact, one can still show that $\sin \sqrt{K} d_T$ is $\mathcal{F}K$-convex on the set $S^o$ of points at distance $< \pi/2 \sqrt{K}$ from $T$. The argument is as in the proof of Proposition 3.4 for $K \leq 0$, but requires more care in proving the uniqueness of minimizers to $T$ from points $q \in S^o$. Specifically, we show the uniqueness of geodesics of length less than $\pi/\sqrt{K}$ joining points $q$ to their opposite points $q'$ in the space obtained by gluing two copies of $X$ along $T$. The tool needed to do this is the following theorem [AB1]: *In a space of CBA by $K$, for any sufficiently small neighborhood $W$ of any geodesic of length less than $\pi/\sqrt{K}$, there are neighborhoods $U$ and $V$ of the endpoints such that every pair of points $p \in U, q \in V$ possesses a unique geodesic in $W$ from $p$ to $q$, varying continuously with $p$ and $q$.*

**Distance to the boundary**

Finally we verify (4B) and (5B). Propositions 3.4 and 3.7 both depend on comparisons with the distance to a geodesic in $S_X$, and hence on Lemma 2.1. This explains the duality between totally convex sets in CBA spaces and boundaries in CBB spaces, seen in (3A), (4A), (5A) and (3B), (4B), (5B), respectively. The following proof is Perelman’s proof of (3B) [Pe], with its last step modified slightly to show how Lemma 2.1 may be used. One could also derive (5B) from (3B) by a coning argument, but this approach is not available for (4B). For background on CBB spaces, see [BBI, Ch. 10].

**Proposition 3.7.** Suppose $X$ is complete with CBB by $K$. Then $\sin \sqrt{K} d_{\partial X}$ is $\mathcal{F}K$-concave if $K > 0$; and $\sinh \sqrt{-K} d_{\partial X}$ is $\mathcal{F}K$-concave if $K < 0$.

**Proof.** If $K > 0$, we may assume $\text{diam}(X) < \pi/\sqrt{K}$, since otherwise the double of $X$ is a spherical suspension, and the claim is immediate from Lemma 2.1.

Along geodesics realizing distance to $\partial X$, or lying in $\partial X$, the claim is trivial, so consider a geodesic $\tau : (-\epsilon, \epsilon) \to X$ which is neither. Set $h = d_{\partial X} \circ \tau, q_0 = \tau(0)$, and let $\gamma$ be a minimizer from $q_0$ to $\partial X$, say with footpoint $p_0 \in \partial X$, where $\gamma$ is the limit of minimizers from points $q_i = \tau(t_i), t_i > 0$. Then $dh/dt(0^+) = -\cos \theta_0$, where $\theta_0$ is the minimum angle between $\tau|[0, \epsilon)$ and minimizers from $q_0$ to $\partial X$, and also the angle between $\tau|[0, \epsilon)$ and $\gamma$ ([BBI, p. 126]). In particular, $(dh/dt)(0^+) + (dh/dt)(0^-) \leq 0$, that is, $h$ is infinitesimally concave.

In $S_X$, consider the model triangle $\tilde{p}_0 \tilde{q}_0 \tilde{q}_1$. Then $\angle \tilde{p}_0 \tilde{q}_0 \tilde{q}_1 \uparrow \theta_0$. Extend the model triangle to a quadrilateral $\tilde{p}_0 \tilde{q}_0 \tilde{q}_1 \tilde{r}_1$ with angles $\pi/2$ at both $\tilde{p}_0$ and $\tilde{r}_1$. Thus the points $\tilde{r}_1$ lie on a geodesic $\tilde{\sigma}$ from $\tilde{p}_0$ orthogonal to the segment $\tilde{p}_0 \tilde{q}_0$. Let $\tilde{r}$ be the geodesic from $\tilde{q}_0$ making angle $\theta_0$ with the segment $\tilde{q}_0 \tilde{p}_0$, and lying on the same side as $\tilde{\sigma}$. Set $\tilde{h}(t) = d(\tilde{r}(t), \tilde{\sigma})$. Since $\angle \tilde{p}_0 \tilde{q}_0 \tilde{q}_1 \leq \theta_0$,

$$d(\tilde{q}_0, \tilde{r}_1) \leq \tilde{h}(t_1). \quad (1)$$

This is because opening the hinge $\tilde{p}_0 \tilde{q}_0 \tilde{q}_1$, leaving $\tilde{p}_0 \tilde{q}_0$ fixed, moves $\tilde{q}_0$ transversely to the equidistant curves of $\tilde{\sigma}$. 


Throughout this section, \( f \) is an \( \mathcal{F}K \)-convex function on a complete CAT(\( K \)) space \( X \), where \( K \leq 0 \) is fixed. Our key idea for constructing a Lipschitz approximation to \( f \) is to obtain the epigraph of an approximating function as the closure of a union of solid \((\mathcal{F}K, A)\)-cones. In what follows, we use the simpler term \( A \)-cone since \( K \) is fixed.

Specifically, the \( A \)-cone with vertex \((p, a)\), where we always assume \( A > 0 \), is defined as follows: If \( K = 0 \), it is the epigraph in \( X \times \mathbb{R} \) of the function \( A(d_p + b) \), where \( a = Ab \). If \( K < 0 \), it is the epigraph in \( X \times \mathbb{R} \) of \((A/\sqrt{-K}) \sinh \sqrt{-K}(d_p + b) \), where \( a = (A/\sqrt{-K}) \sinh \sqrt{-K}b \) (see Figure 1). Thus the restriction of the defining function of a \( A \)-cone with vertex \((p, a)\) to a geodesic from \( p \) is an increasing function belonging to \( \mathcal{F}K \). \( A \)-cones may be defined similarly for \( K > 0 \), but their domain is restricted by the requirement that \( \sin \sqrt{K}(d_p + b) \) be increasing on geodesics from \( p \). Rather, we shall study the case \( K > 0 \) by referring it back to the case \( K = 0 \).

By an \( \mathcal{F}K \)-segment in \( X \times \mathbb{R} \), we mean the graph of an element of \( \mathcal{F}K \) over a geodesic segment parametrized by arclength. We say a subset of \( X \times \mathbb{R} \) is \( \mathcal{F}K \)-convex if it contains an \( \mathcal{F}K \)-segment whenever it contains the ends of that segment. When \( K = 0 \) this is the usual convexity of sets in \( X \times \mathbb{R} \) with the product metric. For a continuous function \( f : X \to \mathbb{R} \), \( \mathcal{F}K \)-convexity of the epigraph is equivalent to \( \mathcal{F}K \)-convexity of \( f \).

The following lemma, which we prove below, summarizes the properties of \( A \)-cones:
Lemma 4.1. For fixed $K \leq 0$ and $A > 0$:

1. $A$-cones are nested according to the position of their vertices; i.e., if the vertex of one $A$-cone is contained in another, then the first $A$-cone is contained in the second.

2. A union of $A$-cones is $FK$-convex if its vertex set is $FK$-convex.

For $g \in FK$, the expression $g^2 + Kg^2$ is constant. This invariant is positive if $K \geq 0$. If $K < 0$, the invariant is positive for $g(s) = B \sinh \sqrt{-K}(s + b)$, zero for $g(s) = B \exp \sqrt{-K}(s + b)$, and negative for $g(s) = B \cosh \sqrt{-K}(s + b)$. When it is positive, we refer to its square root as the steepness of $g$. In particular, the restriction of the defining function of a $A$-cone with vertex $(p,a)$ to a geodesic from $p$ is an element of $FK$ with steepness $A$.

Define

$$X^A = X^A(f) = \{ p \in X : (Df)_p \geq -\sqrt{A^2 - Kf(p)^2} \},$$

$$U^A = U^A(f) = \bigcup \{ A\text{-cone with vertex } (p, f(p)) : p \in X^A \}.$$

Note that $X^A$ is welldefined since we are assuming $K \leq 0$, and that $A_1 < A_2$ implies $X^{A_1} \subset X^{A_2}$.

Our approximation theorems will follow from the claim that $U^A$ is $FK$-convex, even though the epigraph of $f|X^A$ typically is not (since $X^A$ typically is not a convex subset of $X$):

Lemma 4.2. Fix $K \leq 0$, and suppose $f$ is bounded below if $K < 0$. If $X^A \neq \emptyset$:

1. The epigraph of $f$ is in $U^A$.

2. $U^A$ is $FK$-convex.

The intuition underlying Lemma 4.2(1) is that from any point of $X$, one can run down a gradient curve of $f$ to reach $X^A$.

Let us show how Lemma 4.2(2) follows from Lemmas 4.1 and 4.2(1):
Proof of Lemma 4.2(2). By definition, any two points of $U^A$ lie in $A$-cones with vertices on the graph of $f|X^A$. Since the epigraph of $f$ is \( \mathcal{F}K \)-convex, Lemma 4.2(1) implies that the \( \mathcal{F}K \)-segment joining the vertices lies in $U^A$. By Lemma 4.1(1), the union of $A$-cones with vertices on this segment lies in $U^A$. By Lemma 4.1(2), the \( \mathcal{F}K \)-segment joining the two original points lies in $U^A$. □

Now we prove our remaining claims:

Proof of Lemma 4.1(1): A-cones are nested by their vertex position.

The case $K = 0$ is an easy consequence of the triangle inequality. Suppose $K = -1$. It is to be proved that if the vertex $(p_1, a_1)$ of an $A$-cone $A_1$ is contained in the $A$-cone $A_2$ with vertex $(p_2, a_2)$, then $A_1 \subset A_2$. Let $(q, t) \in A_1$, that is, $t \geq A \sinh(d(p_1, q) + b_1)$, where $b_1 = \sinh^{-1}(a_1/A)$. We are given that $a_1 \geq A \sinh(d(p_1, p_2) + b_2)$, where $b_2 = \sinh^{-1}(a_2/A)$. Then

\[
b_1 \geq \sinh^{-1}\left((A \sinh(d(p_2, p_1) + b_2))/A\right) = d(p_1, p_2) + b_2,
\]

and

\[
t \geq A \sinh(d(p_1, q) + b_1) \geq A \sinh(d(p_1, q) + d(p_1, p_2) + b_2) \geq A \sinh(d(p_2, q) + b_2).
\]

Thus $(q, t)$ is in the epigraph of $A \sinh(d(p_2, q) + b_2)$, as desired. □

Proof of Lemma 4.1(2): A union of $A$-cones is \( \mathcal{F}K \)-convex if its vertex set is.

First observe that a single $A$-cone is \( \mathcal{F}K \)-convex, or equivalently, its defining function is \( \mathcal{F}K \)-convex on $X$. Indeed, this claim follows directly from Corollary 2.5, via CAT(\( K \)) triangle comparisons and the fact that hyperbolic sine is increasing, and sine is increasing on \((-\pi/2, \pi/2)\).

It suffices for the lemma to prove:

(2') A union of $A$-cones is \( \mathcal{F}K \)-convex if its vertex set is an \( \mathcal{F}K \)-segment.

Now we show it suffices to prove (2') when the base space is the model plane $S_K$. Indeed, in $X \times \mathbb{R}$ suppose we have the ends of an \( \mathcal{F}K \)-segment $(p_1, a_1), (p_2, a_2)$ and another pair of points $(\psi_1, c_1), (\psi_2, c_2)$ in the $A$-cones with vertices at the first pair. By Reshetnyak majorization, there is a convex quadrilateral $p_1'p_2'q_2'q_1'$ in $S_K$ and a distance nonincreasing map $\psi : p_1'p_2'q_2'q_1' \to X$ carrying the sides isometrically to the sides of the geodesic quadrilateral $p_1p_2q_2q_1$. Due to these isometries, the \( \mathcal{F}K \)-segments built on the sides of the $S_K$-quadrilateral with the same end heights $a_1, a_2, c_1, c_2$ have the same heights at all corresponding points as the \( \mathcal{F}K \)-segments in $X \times \mathbb{R}$. In particular, $(\psi_1', c_1), (\psi_2', c_2)$ are in the $A$-cones with vertices $(p_1', a_1), (p_2', a_2)$. Now if (2') holds for $S_K$, then every point $(\psi', c)$ of the \( \mathcal{F}K \)-segment connecting $(\psi_1', c_1), (\psi_2', c_2)$ is in an $A$-cone with vertex at some $(p', a)$ in the \( \mathcal{F}K \)-segment connecting $(p_1', a_1), (p_2', a_2)$. Since $q = \psi(\psi')$ is an arbitrary point of the geodesic connecting $q_1$ and $q_2$, and $d(p, q) \leq d(p', q')$ for $p = \psi(p')$, it follows that $(q, c)$ is in the $A$-cone with vertex $(p, a)$. That is, (2') also holds for $X$.

Finally, we describe directly an \( \mathcal{F}K \)-convex function $F$ on $S_K$ whose epigraph is the union of $A$-cones with vertices on a given \( \mathcal{F}K \)-segment $\sigma = (\gamma, h)$, $h \in \mathcal{F}K$. See Figure 2. It may occur that $h$ has steepness $\geq A$, in which case all of the $A$-cones along $\sigma$ are contained in the lowest one by Lemma 4.1(1). Then the union is
simply that lowest one, which is $\mathcal{F}K$-convex, and there is nothing more to prove. Henceforth we assume that $h$ has steepness less than $A$.

The construction of $F$ is based on Lemma 2.1, which we can restate, with a scale factor $A$, to say that for a given baseline in $S_K$ there is an $\mathcal{F}K$-affine function which is 0 on the baseline and has steepness $A$ on every perpendicular line directed by a chosen positive side of the baseline. For every such $\mathcal{F}K$-affine function, the gradient is perpendicular to the baseline, from which we conclude that the epigraph is the union of all $A$-cones whose vertices lie on the graph.

We shall realize $F$ as the supremum of a family of $\mathcal{F}K$-affine functions on $S_K$ chosen from this class. The family will have a uniform Lipschitz constant in a neighborhood of each point, and hence have continuous supremum. The $\mathcal{F}K$-convexity of $F$ follows, since the supremum of a set of $\mathcal{F}K$-convex functions satisfies the $\mathcal{F}K$-convexity inequality along geodesics, and hence is $\mathcal{F}K$-convex if it is continuous. The baselines for the family will be the tangent geodesics to a curve $\tau$ which we now describe.

Let $a_1, a_2$ be the values of $h$ at the ends of $\gamma$. For a function $g \in \mathcal{F}K$ having steepness $A$ and value 0 at 0 let $r_1, r_2$ be nonnegative numbers such that $g(r_i) = |a_i|, \ i = 1, 2$ (take the least such $r_i$ if $K > 0$). If $a_1$ and $a_2$ have the same sign, the fact that the steepness of $h$ is at least $A$ implies $|r_1 - r_2| < |\gamma|$. Then the two circles in $S_K$ centered at the ends of $\gamma$, with radii equal to the corresponding $r_i$, have two common external tangents, which we select. If $a_1$ and $a_2$ have opposite sign, then $r_1 + r_2 \leq |\gamma|$ and the two circles have in addition two common internal tangents, which we select. Let $\tau$ be the closed curve consisting of the segments of the two selected tangents between the points of tangency, joined to arcs of the circles as in Figure 3. The signs of $a_1$ and $a_2$ determine not only which arcs are selected,
but also the directions from the baselines in which the corresponding $\mathcal{FK}$-affine functions are positive. In Figure 3, the positive direction is indicated at one point, and moves continuously as $\tau$ is traced.

Thus the collection of all tangent lines to $\tau$ form the baselines for our family of $\mathcal{FK}$-affine functions, and the positive direction is chosen so that $\sigma$ is in the epigraph of them all. For any baseline tangent to an open circular arc in $\tau$, the graph of the corresponding $\mathcal{FK}$-affine function contains the point of $\sigma$ that projects to the corresponding endpoint of $\gamma$. For baselines extending the two selected segments of $\tau$, the graph of of the corresponding $\mathcal{FK}$-affine function contains all of $\sigma$. Thus, the intersection of all the epigraphs of the family, which is the epigraph of $F$, has $\sigma$ in its boundary.

It remains to show that the epigraph of $F$ coincides with the union $U$ of the $A$-cones having vertices on $\sigma$. We have arranged that the epigraph of $F$ contains $U$. On the other hand, the value at any point $q$ of an $\mathcal{FK}$-affine function in our family is determined by the signed distance from $q$ to the baseline of that function, and is an increasing function of that distance. Thus, $F(q)$ takes its value from the $\mathcal{FK}$-affine function whose baseline has the greatest signed distance from $q$. A given member of the family coincides with $F$ precisely on a ray starting at a point $\gamma(s)$ and running perpendicular to the corresponding baseline in the positive direction. The graph of these coincident values is a generator of the $A$-cone with vertex $(\gamma(s), h(s))$. Thus the whole graph of $F$ must lie in $U$.  

**Proof of Lemma 4.2(1):** The epigraph of $f$ is in $U^A$.

It suffices, for any fixed $p_0 \notin X^A$, to prove the existence of $p \in X^A$ such that $(p, f(p))$ and $(p_0, f(p_0))$ are joined by an $\mathcal{FK}$-segment of steepness at least $A$. The proof proceeds by applying Zorn’s lemma to a family $Z$ of curves in $X$ ending at $p_0$. Each curve of $Z$ will be parametrized by the values of $f$ along the curve; in addition, for every geodesic secant to the curve the $\mathcal{FK}$-segment supported by that geodesic segment and having its ends on the graph of $f$ is required to have steepness at least $A$. The partial ordering of $Z$ is by restriction: $\gamma \leq \tilde{\gamma}$ means that $\gamma$ is obtained from $\tilde{\gamma}$ by restriction to the domain of $\gamma$. The domain of each curve in $Z$ is an interval $(a, f(p_0))$ or $[a, f(p_0)]$.

First suppose $f$ is bounded below. Then there is a uniform positive lower bound $1/B \leq \sqrt{A^2 - Kf(p)^2}$ for all points $p$ lying on any curve in $Z$. For $t < t'$ in the domain of $\gamma \in Z$, if $\sigma$ is the geodesic secant in $X$ connecting $\gamma(t)$ and $\gamma(t')$
and \( g \in \mathcal{F}K \) is defined on the domain of \( \sigma \) and has end values \( t, t' \), we have
\[
g' = \sqrt{A_g^2 - Kg^2},
\]
where the steepness \( A_g \) is at least \( A \). Hence, we obtain the Lipschitz inequality
\[
l' - t \geq \text{length}(\sigma)/B = d(\gamma(t), \gamma(t'))/B.
\]
That is, \( B \) is a uniform Lipschitz constant for all elements of \( Z \).

We apply Zorn’s Lemma in the form: there is a maximal linearly ordered subset. In this case, that means that there is a curve \( \gamma_m \in Z \), namely, the union curve of a maximal linearly ordered subset, which has no proper extension to another element of \( Z \). By the Lipschitz continuity of \( \gamma_m \), the assumption that \( f \) is bounded below, and the completeness of \( X \), it follows that the domain of \( \gamma_m \) is a closed interval \([a_m, f(p_0)]\).

To conclude we show that \( p = \gamma_m(a_m) \in X^A \). Otherwise, there would be a geodesic segment from \( p \) along which \( f' < -\sqrt{A^2 - K a_m^2} \). But then \( \gamma_m \) could be extended by that geodesic segment, and it is a consequence of Lemma 4.1(1) that the extension is in \( Z \). This contradicts the maximality of \( \gamma \).

We have shown that if \( K \leq 0 \) and \( f \) is bounded below, then every \( X^A \) is non-empty and the epigraph of \( f \) is in \( U^A \). This completes the proof of Lemma 4.2 in the case \( K < 0 \).

Now suppose \( K = 0 \), \( X^A_{-\epsilon} \neq \emptyset \). In this case the hypothesis that \( f \) is bounded below is not required to show that the elements of \( Z \) are uniformly Lipschitz, since the lower bound \( 1/B \) can be taken to be \( A \). Our proof above would fail only if \( \gamma_m \) were defined on \((-\infty, f(p_0)]\), and \( \lim_{t \to -\infty} f \circ \gamma(t) = -\infty \). But then for any given \( q \in X \), the limit sup of the slopes of the geodesics in \( X \times \mathbb{R} \) joining \((q, f(q))\) and \((\gamma_m(t), t)\) is at least \( A \). Since \( f \) is convex, this means that if \( \gamma \) is the unit speed geodesic in \( X \) from \( q \) to \( \gamma_m(t) \), then \( \liminf_{t \to -\infty} (f \circ \gamma)'(0) \leq -A \). Choosing \( q \) to lie in \( X^A_{-\epsilon} \) yields a contradiction. Therefore the epigraph of \( f \) is in \( U^A \).

Finally, suppose \( K = 0 \), \( X^A \neq \emptyset \). Since by definition \( U^A = \cap_{\epsilon > 0} U^{A+\epsilon} \), and the preceding paragraph shows that the epigraph of \( f \) is in \( U^{A+\epsilon} \), then the epigraph of \( f \) is in \( U^A \).

\section{5. Lipschitz approximations to \( \mathcal{F}K \)-convex functions}

In this section, \( f \) is an \( \mathcal{F}K \)-convex function on a complete CAT(\( K \)) space \( X \).

\textbf{Proof of Theorems 1.2 and 1.3: Construction of \( \mathcal{F}K \)-convex extensions.}

If \( K < 0 \) and \( f \) is bounded below, we have just seen that \( X^A \neq \emptyset \) for all \( A > 0 \). By Lemma 4.2(2), \( U^A \) is \( \mathcal{F}K \)-convex. Since \( \mathcal{F}K \)-segments in \( X \times \mathbb{R} \) vary continuously with their endpoints, the closure \( \text{cl}U^A \) is \( \mathcal{F}K \)-convex. Moreover, it follows from the nesting property of \( A \)-cones (Lemma 4.1(1)) that the function \( f^A \) of which \( \text{cl}U^A \) is the epigraph is continuous. Therefore \( f^A \) is \( \mathcal{F}K \)-convex, for all \( A > 0 \). If \( K = 0 \), \( X^A \) is nonempty for \( A \) sufficiently large by [AB2, Lemma 7.1]. Just as before, \( \text{cl}U^A \) is the epigraph of an \( \mathcal{F}K \)-convex function \( f^A \) for \( A \) sufficiently large.
In either of these cases, we claim that \( f^A \) agrees with \( f \) on \( X^A \). Otherwise, there would be points \( p \) and \( q \) in \( X^A \) such that \( f^A(q) < f(q) \), and \( (q, f(q)) \) lies in the interior of the \( A \)-cone with vertex \( (p, f(p)) \). That is, the \( \mathcal{F}K \)-segment \((\gamma, h)\) from \((p, f(p))\) to \((q, f(q))\) would have steepness greater than \( A \). But \( h \circ \gamma \geq f \circ \gamma \) by the \( \mathcal{F}K \)-convexity of \( f \). It follows that \((f \circ \gamma)'(L^-) < -\sqrt{A^2 - Kf(q)^2}\), where \( \gamma(L) = q \), contradicting the definition of \( f^A \). This completes the proof of Theorem 1.2 and the case of Theorem 1.3 in which \( K < 0 \).

For the remaining case of Theorem 1.3, we scale so that \( K = 1 \). Consider the linear cone \( C(X) \), and denote \((r, x) \in C(X)\) by \( r x \). We are assuming \( f(x) + f(y) \geq 0 \) for points \( x \), \( y \in X \) satisfying \( d(x, y) \geq \pi \). By Lemma 3.5, the linear homogeneous extension \( C(f) \) of \( f \) to \( C(X) \) is a convex function. Since \( C(X) \) is a CAT(0) space, we have just proved that for every sufficiently large \( A \) there is a convex function \( C(f)^A \) on \( X^A \) that agrees with \( C(f) \) on

\[
C(X)^A = \{ rx \in C(X) : DC(f)^A_{rx} \geq -A \}
\]

and satisfies \(|DC(f)^A| \leq A \) everywhere.

We claim that \( C(X)^A \) is the cone over

\[
X^A = \{ x \in X : A^2 - f(x)^2 \geq 0, \ (Df)_x \geq -\sqrt{A^2 - f(x)^2} \}.
\]

Indeed, the restriction of \( C(f) \) to the Euclidean sector obtained by coning over a short geodesic segment of \( X \) with arclength parameter \( \theta \) has the form \( rf(\theta) \). A short geodesic segment \( \gamma \) in \( C(X) \) is represented by a straight line segment in such a sector. If \( \gamma \) has arclength parameter \( t \), and makes angle \( \phi \) with the circle \( r = \text{constant} \) at \( \gamma(0) \), then

\[
d(rf(\theta))/dt|_{t=0} = f(\theta) \sin \phi + f'(\theta) \cos \phi.
\]

Since \( r \) does not appear, this expression takes its minimum (resp. maximum) in \([\pi/2, \pi/2]\) at the same angle \( \phi \) for every point on a given ray \( \theta = \text{constant} \). Specifically, the minimum is:

\[
-|f(\theta)|, \text{ if } f'(\theta) \geq 0; \quad -\sqrt{f(\theta)^2 + f'(\theta)^2}, \text{ if } f'(\theta) \leq 0. \tag{4}
\]

The maximum is:

\[
|f(\theta)|, \text{ if } f'(\theta) \leq 0; \quad \sqrt{f(\theta)^2 + f'(\theta)^2}, \text{ if } f'(\theta) \geq 0. \tag{5}
\]

It follows from (4) that \( C(X)^A \) is the cone over the points \( x \) of \( X \) that satisfy:

\[
-A \leq -f(x)^2 + \inf Df_x, \text{ if } \inf Df_x \leq 0; \quad A^2 - f(x)^2 \geq 0, \text{ if } Df_x \geq 0.
\]
It is straightforward to check that these conditions are equivalent to the defining conditions of $X^A$.

By definition, $C(f)^A$ is the function on $C(X)$ whose epigraph is the union of $(F_0, A)$-cones with vertices on the graph of $C(f)C(X)^A$. We claim $C(f)^A$ is linear homogeneous. Indeed, for any $r_1, x_1 \in C(X)$, there is $r_0, x_0 \in C(X)^A$ such that

$$C(f)^A(r_1, x_1) = r_0 f(x_0) + Ad(r_0 x_0, r_1 x_1).$$

(6)

Since $C(X)^A = C(X^A)$ contains the entire ray through $x_0$, it follows that

$$C(f)^A(\lambda r_1, x_1) \leq \lambda C(f)(r_1, x_1).$$

If this were a strict inequality, then the same argument starting at $\lambda r_0 x_0$ and using a factor of $\lambda^{-1}$ would violate (6). This proves the linear homogeneity of $C(f)^A$.

Set $f^A = C(f)^A|X$. By linear homogeneity, $C(f)^A = C(f^A)$. By Lemma 3.5, $f^A$ is $F^1$-convex on $X$. It only remains to prove that $(Df^A)^2 \leq A^2 - f^A(x)^2$, given that $|D(C(f)^A)_{x_0}| \leq A$. Applying the analysis of (4) and (5) to $f^A$ shows that $(\inf(Df^A)_{x_0})^2 \leq A^2 - f^A(x)^2$ if $\inf(Df^A)_{x_0} < 0$, and $(\sup(Df^A)_{x_0})^2 \leq A^2 - f^A(x)^2$, as required.

Proof of Theorem 1.4: Construction of Lipschitz approximations.

Let $A_n > 0$ be a sequence that increases without bound. If $K \geq 0$, set $f_n = f^{A_n}$, where $f^{A_n}$ is the $F^K$-convex function constructed in the preceding proof. If $K < 0$, let $f_n$ be the $F^K$-convex function that results from applying the same construction to $\max\{f, -\sqrt{A_n}\}$ rather than to $f$. We may do so because the negative constant functions are $F^K$-convex for $K < 0$, and hence the base function $\max\{f, -\sqrt{A_n}\}$ is $F^K$-convex and bounded below.

Lemmas 4.1(1) and 4.2(1), respectively, imply that for any $p \in X$, the values $f_n(p)$ are increasing for $n$ sufficiently large, and are bounded above by $f(p)$. Therefore $f_n$ has a pointwise limit function $\bar{f} \leq f$, where both $f$ and $\bar{f}$ satisfy the $F^K$-convexity inequality along geodesics. We claim $\bar{f} = f$.

Suppose, to the contrary, that $\bar{f}(p) < f(p)$.

First suppose $K = 0$. By Theorems 1.2 and 1.3, setting $X_n = X^{A_n}(f)$. We have $f_n|X_n = f|X_n$, so $p \notin X_n$. Therefore $(p, \bar{f}(p))$ lies, for every $n$, in a $A_n$-cone with vertex $(q_n, f(q_n))$ for some $q_n \in X_n$. That is, the $F^K$-segment from $(q_n, f(q_n))$ to $(p, \bar{f}(p))$ has slope at least $A_n$. If the $f(q_n)$ are bounded below, it follows that $q_n \to p$. Since $f(q_n) \leq \bar{f}(p)$ and $f$ is continuous, we conclude that $f(p) \leq \bar{f}(p)$, a contradiction. If the $f(q_n)$ are not bounded below, then for any fixed $q \in X$, the slope of the geodesic in $X \times R$ joining $(q, f(q))$ to $(q_n, f(q_n))$ diverges to $-\infty$. Choosing $q \in X_n$ yields a contradiction.
Suppose \( K < 0 \). Set \( X_n = X^{A_n}(\max\{f, -\sqrt{A_n}\}) \). For \( n \) sufficiently large, \( f(p) > -\sqrt{A_n} \). Since \( p \notin X_n \) for \( n \) sufficiently large, then there exists \( q_n \in X_n \) such that the \( FK \)-segment \((\gamma_n, h_n)\) from \((q_n, f(q_n))\) to \((p, f(p))\) has steepness at least \( A_n \). That is, \( h_n^2 + K h_n^2 > A_n^2 \). Therefore \( h_n > A_n \). Since the absolute difference in endpoint values is at most \( \|f\| + \sqrt{A_n} \), we obtain \( q_n \to p \), a contradiction as in the preceding paragraph.

If \( K > 0 \), recall that the construction of \( f_n = f^{A_n} \) proceeds by normalizing so that \( K = 1 \), then reducing to the case \( K = 0 \) by coning. Thus the construction for \( K = 0 \) is applied to the function \( C(f) \) on \( C(X) \). Then \( f_n = C(f)^{A_n}|X \). Since we have just seen that \( C(f)^{A_n} \) converges to \( C(f) \) on \( C(X) \), it follows that \( f_n \) converges to \( f \) on \( X \).

Thus in every case, \( f_n \to f \). Each \( f_n \) is an \( FK \)-convex function satisfying \( |(Df_n)(p)|^2 \leq A_n^2 - K f_n(p)^2 \). If \( K \geq 0 \), then \( f_n \) is Lipschitz and the theorem is proved. If \( K < 0 \), then \( f_n \) is Lipschitz if it is bounded above, and in any case is Lipschitz on its sublevel sets. It only remains to prove that these sublevel sets, \( \{p : f_n(p) > A\} \), are convex if \( A > 0 \). But since \( K < 0 \), the restriction of an \( FK \)-convex function to a geodesic is strictly convex where it is positive, and so cannot take a positive internal maximum. \( \square \)

Example 5.1. It is easy to give examples of highly non-Lipschitz convex functions on CAT(0) spaces. Following a remark of Kleiner ([K, p.411]), let \( X_0 \) be a basepoint and construct a metric tree \( X_n \) recursively by attaching \( 2^n \) edges of length \( 2^{-n} \) to each end vertex of \( X^{n-1} \). Let \( X = \text{the completion of } \bigcup X_n \), and \( T = X - \bigcup X_n \). Then \( T \) has infinite Hausdorff dimension. (To check this, let an \( n \)-fan consist of all \( T \) with a common footpoint on \( X_n \). An \( n \)-fan has diameter \( 2^{-(n-1)} \), and \( n \)-fans with distinct footpoints on \( X_n \) are disjoint. For an arbitrary finite cover, each element may be replaced by the fan of the same diameter containing it, to obtain a cover by disjoint \( x \) for which the sum of \( r \)-th powers of the diameters is no more than for the original cover. For a fan of smallest diameter in the new cover, say an \( n \)-fan with footpoint \( p \) in \( X_{n-1} \), all the \( n \)-fans with footpoint \( p \) in \( X_{n-1} \) also lie in the cover. These \( 2^n \) fans contribute \( 2^n 2^{-(n-1)r} \) to the \( r \)-th power sum. Thus if \( n \geq r \), these \( 2^n \) fans may be replaced by the single fan with footpoint \( p \) and diameter \( 2^{-(n-2)} \), without increasing the \( r \)-th power sum. Thus we may assume the cover has constant diameter equal to the largest in the original cover, say \( 2^{-(m-1)} \). If \( m \geq r \), the corresponding \( r \)-th power sum is \( 2^{m(m+1)/2} 2^{-(m-1)r} \geq 2^{-r^2/2} \).

Now modify the above definition of \( X_n \), \( n \geq 2 \), by in addition putting a vertex at the midpoint of each edge of \( X_{n-1} \) with an attached edge of length \( 2^{-n} \). Again set \( X = \text{the completion of } \bigcup X_n \), and \( T = X - \bigcup X_n \). Starting with value 0 at the basepoint \( X_0 \), recursively define a continuous convex function \( f \) on \( X \), to be linear with slope \( n \) on each edge of length \( 2^{-n} \) that is newly attached to \( X_{n-1} \) in forming \( X_n \). Then

\[
T = \{ p : (f \circ \gamma)'(0) = -\infty \text{ for all geodesics } \gamma \text{ from } p \}.
\]

This set \( T \) is dense in \( X \), which is compact. \( T \) has infinite Hausdorff dimension while its complement in \( X \) has Hausdorff dimension 1.
Example 5.2. Let $X$ be the closed domain in the Euclidean plane bounded by the curves $y = x^2, 0 \leq x \leq 1, x = 1, 0 \leq y \leq 1,$ and $y = 0, 0 \leq x \leq 1$; see Figure 4. Then $X$ is a CAT(0) space in which the geodesics are line segments, arcs of the parabolic boundary, or smooth concatenations of those two types. It is easily calculated from the second derivative along these geodesics that the function $f(x, y) = 1 - \frac{1}{\sqrt{y}}$ is a convex function on $X$. However, $Df$ is not well-defined at $(0, 0)$ on the single direction starting at that point because the derivative at $x = 0$ of the restriction of $f$ to the $x$-axis is 0, and to the parabola $y = x^2$ is negatively infinite.

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References

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