Convex functions on symmetric spaces and
geometric invariant theory for spaces of weighted
configurations on flag manifolds

Bernhard Leeb and John Millson

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1 Introduction

In 1912, Hermann Weyl studied a version of the following problem, see [Fu97] and [Fu00]:

**Eigenvalues of a sum Problem 1.1 ([We12])** Given three $n$-tuples of real numbers arranged in decreasing order, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\sum_i \alpha_i = \sum_i \beta_i = \sum_i \gamma_i = 0$, when are there traceless Hermitian matrices $A, B, C \in i \cdot su(n)$ such that the spectrum of $A$ is $\alpha$, the spectrum of $B$ is $\beta$, the spectrum of $C$ is $\gamma$ and $A + B = C$?

This can be regarded as a problem for triangles in $i \cdot su(n)$, equipped with the geometry, in the sense of Felix Klein, having as automorphisms the translations and the adjoint action of $K = SU(n)$. In this geometry, pairs of points are equivalent if and only if their difference matrices have equal spectra. The spectra of the matrices can hence be interpreted as side lengths, and Weyl’s question amounts to finding the triangle inequalities in this geometry.

Problem 1.1 has a natural generalization. Let $G$ be a connected semisimple real Lie group of noncompact type, $K$ a maximal compact subgroup and $X = G/K$ the associated symmetric space of noncompact type. Decompose the Lie algebra $\mathfrak{g}$ of $G$ according to $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ relative to the Killing form. $\mathfrak{p}$ is canonically identified with the tangent space $T_0X$ to $X$ at the point fixed by $K$. Then $K$ acts on $\mathfrak{p}$ by the restriction of the Adjoint representation and the orbits are parametrized by the Euclidean model Weyl chamber $\Delta^e_{\text{model}}$. There is hence a natural notion of $\Delta^e_{\text{model}}$-valued side lengths for polygons in $\mathfrak{p}$, and Problem 1.1 generalizes to finding the $n$-gon inequalities in this geometry. Let us denote by $\mathcal{P}_n(\mathfrak{p})$ the set of $h = (h_1, \ldots, h_n) \in (\Delta^e_{\text{model}})^n$ such that there exists a closed $n$-gon with side lengths $h$.

**Problem 1.2** Prove that $\mathcal{P}_n(\mathfrak{p})$ is a finite-sided polyhedron and find a system of linear inequalities for $h$ which describes $\mathcal{P}_n(\mathfrak{p})$.

The $K$-action on open $n$-gons naturally extends to a $G$-action as follows. Let $e = (e_1, \ldots, e_n) \in \mathfrak{p}^n$ denote the $n$-gon with vertices $v_i = \sum_{j \leq i} e_j$, $0 \leq j \leq n$. To emphasize that we allow polygons to be open we will also call them linkages. The linkage $e$ can be interpreted as weighted configuration on the geometric boundary $\partial_\infty X$ at infinity of the symmetric space. ¹ Namely, using the radial projection $\phi : \mathfrak{p} \rightarrow \partial_\infty X$ to infinity, $e$ maps to the configuration $\xi = (\xi_i) \in \partial_\infty X^n$ where $\xi_i := \phi(e_i)$. If the polygon has side lengths $h$ then the directions $h_i/\|h_i\|$ determine on which $G$-orbits in $\partial_\infty X$ the point $\xi_i$ lies, indeed we have $\xi_i \in G\tau_i$ where $\tau_i := \phi(h_i/\|h_i\|)$. ² We attach the real-valued Euclidean lengths $r_i := \|h_i\|$ as weights to the ideal points $\xi_i$.

In constructing a moduli space of weighted configurations one faces problems due to the noncompactness of $G$. They can be dealt with as in Geometric Invariant

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¹$\partial_\infty X$ denotes the *geometric or ideal boundary* in the sense of the theory of spaces of nonpositive curvature.

²Here, the radial projection $\Delta^e_{\text{model}} \rightarrow \Delta^e_{\text{model}}$ is also denoted by $\phi$. 

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Theory by defining a suitable notion of stability and taking a Mumford type quotient, $\mathcal{M}_h$, of the semistable points by extended orbit equivalence. Namely, one can associate to a weighted configuration on $\partial_\infty X$ a natural convex function on $X$, its weighted Busemann function. It is well-defined up to an additive constant. We call the configuration semistable (respectively stable) if the weighted Busemann function is bounded below (respectively proper), see 2.4. We prove in 3.16 that there exist semistable configurations of type $h$ (i.e. the corresponding Mumford type moduli space is non-empty) if and only if there exist closed polygons with side lengths $h$.

In the special case $G$ is complex then $\mathfrak{p} = i \mathfrak{k}$. The $K$-orbits in $\mathfrak{p} - \{0\}$ carry a natural symplectic structure and the $G$-orbits in $\partial_\infty X$ a natural complex structure. Both fit together to give a Kähler structures on the orbits. The extension of the $K$-action on linkages to a $G$-action is the complexification. The closing condition for polygons in $\mathfrak{p}$ translates into the momentum zero condition from symplectic geometry, i.e. the moduli space of closed polygons with fixed side lengths is a symplectic quotient. The correspondence between the moduli spaces of closed polygons and semistable configurations is then an example for the identification of symplectic quotients with Mumford quotients.

This paper deals with the question for which values of the side length parameter $h \in (\Delta_{\text{model}})^n$ the moduli spaces are non-empty. We use techniques from the theory of symmetric spaces to give a solution of the general problem 1.2. Our main result is a description of $\mathcal{P}_n(\mathfrak{g})$ for all $\mathfrak{g}$ and all $n$.

In order to state our main theorem we need to explain a generalization of the Schubert calculus, $\partial_\infty X$ carries a natural structure as a spherical building induced from the Tits angle metric $\angle_{\text{Tits}}$. We choose an apartment $a_{\text{model}}$ containing $\Delta_{\text{model}}^{\text{sph}}$ in $\partial_{\text{Tits}} X$ and call it the standard apartment. Then the Weyl group $W$ acts on $a_{\text{model}}$. For each vertex $\zeta$ of the model spherical Weyl chamber $\Delta_{\text{model}}^{\text{sph}}$, we denote by $\text{Grass}_\zeta$ the corresponding $G$-orbit of maximally singular points in $\partial_\infty X$, a "generalized Grassmannian". Assume the orbit $W \zeta$ is given by $W \zeta = \{\eta_1, \ldots, \eta_N\}$. To relate our parametrization of the Schubert cells in $\text{Grass}_\zeta$ to the classical one from Lie theory we note that if $W \zeta$ is the isotropy of $\zeta$ then there is a natural bijection between $W \zeta$ and the relative Weyl set i.e. the coset space $W/W_\zeta$. It is customary to parametrize Schubert cells by this coset space - for us it is more convenient to use the vertices in the orbit $W \zeta$.

In order to define the Schubert cells in $\text{Grass}_\zeta$ we need to choose a spherical Weyl chamber $\sigma$. Such a choice is equivalent to a choice of Borel (i.e. minimal parabolic) subgroup $B$. The correspondence is that the fixed point set of $B$ is the chamber $\sigma$. We now define the Schubert cells in $\text{Grass}_\zeta$ to be the orbits of $B$. The Bruhat decomposition then states the images in $\text{Grass}_\zeta$ of the elements in the relative Weyl set $W/W_\zeta$ give orbit representatives for these orbits. These images are distinct and consequently the relative Weyl set parametrizes the Schubert cells.

It is immediate that the elements in the orbit $W \zeta$ also parametrize the orbits but it is important for us to see this as a direct consequence of the Tits geometry. Given a spherical Weyl chamber $\sigma$ and a vertex $\eta$ of $\partial_{\text{Tits}} X$ we define the relative position $(\sigma, \eta)$ to be the vertex of $a_{\text{model}}$ obtained as follows. Choose $g \in G$ such that $g \sigma$...
coincides with \( \Delta_{\text{sph \ model}} \). Now choose \( b \in B \) such that \( b \eta \gamma = \eta^* \) lies in the standard apartment \( a_{\text{model}} \). We then define the relative position \((\sigma, \eta)\) to be \( \eta^* \). We prove that the relative position is well-defined and is a complete invariant of \( G \)-orbits of pairs \((\sigma, \eta)\) in Section 4.

Now we can give the second definition of Schubert cells. Let \( \sigma \), and \( \zeta \) be as above. For \( \eta_i \in W \zeta \) we define the Schubert cell \( C_{\eta_i} \) to be the set of vertices \( \eta \) in \( \text{Grass} \zeta \) such that \( (\sigma, \eta) = \eta_i \). The Schubert cycles are then defined as the closures \( \overline{C_{\eta_i}} \) of the Schubert cells; they are unions of Schubert cells. Note that, as real algebraic varieties, the Schubert cycles represent homology classes \([C_{\eta}] \in H^*(\text{Grass} \zeta, \mathbb{Z}/2\mathbb{Z})\); in the complex case they even represent \textit{integral} homology classes.

**Non-emptiness of the moduli space.** We can now describe our inequalities for \( h \in (\Delta_{\text{arc \ model}})^n \) giving necessary and sufficient for the non-emptiness of \( \mathcal{M}_h \) or, equivalently, the existence of a solution to the closing condition

\[
\sum_{i=1}^n e_i = 0. \tag{1}
\]

The spherical model apartment \( a_{\text{model}} \) is a Coxeter complex, and we think of the spherical model Weyl chamber as being one of its top-dimensional simplices. \( \angle \) denotes the spherical distance.

The data \( h \) correspond to the \( n \)-tuple of points \( \tau_i = \phi(h_i/||h_i||) \in \Delta_{\text{sph \ model}} \) and real numbers \( r_i := ||h_i|| \). For all vertices \( \zeta \) of \( \Delta_{\text{sph \ model}} \) and all \( n \)-tuples of vertices \( \eta_1, \ldots, \eta_n \in W \zeta \) consider the inequality

\[
\sum_i r_i \cdot \cos \angle(\tau_i, \eta_i) \leq 0. \tag{2}
\]

We may rewrite the inequality as follows using standard terminology of Lie theory. The inverse image of the standard apartment \( a_{\text{model}} \) under \( \phi \) is the set of nonzero vectors in a Cartan subspace \( \mathfrak{a} \in \mathfrak{p} \) (so \( \mathfrak{a} \) is a maximal subalgebra of \( \mathfrak{p} \), necessarily abelian and reductive). By standard Lie theory, \( \mathfrak{a} \) comes equipped with a root system, the restricted roots, and any other theorems. Let \( \{\alpha_1, \ldots, \alpha_f\} \) be the simple restricted roots corresponding to the positive chamber \( \Delta_{\text{arc \ model}} \). The restricted simple roots are a basis for \( \mathfrak{a}^* \). The members of the basis for \( \mathfrak{a}^* \) to the basis of simple restricted roots will be called the fundamental (restricted) coweights. The elements of the lattice generated by the fundamental coweights (the dual of the lattice generated by the restricted roots) will be called coweights. The edges of \( \Delta_{\text{arc \ model}} \) are the positive multiples of the fundamental coweights. Let \( \lambda^\downarrow \in \Delta_{\text{arc \ model}} \) be the fundamental coweight contained in the edge with direction \( \zeta \), and let \( \lambda_i^\downarrow := w_i \lambda^\downarrow \) where \( [w_i] \in W/W \zeta \) such that \( w_i \zeta = \eta_i \). Then (2) becomes (up to a positive multiple)

\[
\sum_i \langle h_i, \lambda_i^\downarrow \rangle \leq 0. \tag{3}
\]

The following result describes, in terms of the Schubert calculus over the integers modulo 2, a subset of these inequalities which is equivalent to the moduli space being non-empty. This generalizes earlier results of Helmke-Rosenthal, Klyachko, Belkale, Berenstein-Sjamaar and O'Shea-Sjamaar.
Main Theorem 1.3 \( \mathcal{M}_h \neq \emptyset \) iff (2) (respectively (3)) holds whenever the intersection of the Schubert classes \([C_{\eta_1}], \ldots, [C_{\eta_n}]\) in \( H^*_c(\text{Gr} \mathcal{F}_\zeta, \mathbb{Z}/2\mathbb{Z})\) equals \([pt]\).

If \( G \) is complex, then \( \mathcal{M}_h \neq \emptyset \) iff (2) (respectively (3)) holds whenever the intersection of the integral Schubert classes \([C_{\eta_1}], \ldots, [C_{\eta_n}]\) in \( H^*_c(\text{Gr} \mathcal{F}_\zeta, \mathbb{Z})\) equals \([pt]\).

Note that, in the complex case, the set of necessary and sufficient inequalities obtained from the integral Schubert calculus is in general smaller.

In Section 5 of this paper we make the above inequalities explicit for the classical simple complex Lie groups \( G \). In this case our formulas are a sharper version of those of Berenstein and Sjamaar ([BeSi]) in that we need only consider those \( n \)-fold intersections of Schubert classes with intersection number one. If we further specialize to the case \( g = sl(m, \mathbb{C}) \) we find that \( p \) is the space of \( m \times m \) Hermitian matrices of trace zero. In this case we obtain the inequalities recently proved to be sufficient by Klyachko ([Kly98]) as refined by Belkale, cf. [Bel]. For the complex case our inequalities bear the same relation to those of Berenstein-Sjamaar as those of Belkale do to those of Klyachko.

The general case considered in Problem 1.2 was solved in [OSj99]. In their theory the real Lie algebra \( g \) is replaced by the dual compact form \( u := \mathfrak{t} + i\mathfrak{p} \) so their polygons lie in \( i\mathfrak{p} \) instead of \( \mathfrak{p} \). Of course multiplying by \( i \) gives an isomorphism of their moduli space to the one we consider and an isomorphism of their polyhedron to ours. However their inequalities are quite different from ours. They are associated to the integral Schubert calculus of the complexification \( g \otimes \mathbb{C} \) and are efficient for the case of split \( g \) but become less and less efficient as the real rank of \( g \) (i.e. the rank of the symmetric space \( X \)) decreases. For instance, for the case of real rank one they have a very large number of inequalities when the triangle inequalities alone will suffice (see below). This is recognized in [OSj99] and the problem is posed as to whether a formula of the type we found above in terms of the Schubert calculus modulo 2 would exist.

We illustrate our inequalities by describing them for the trivial case when \( X \) has rank one (i.e. \( \text{dim } a = 1 \)). In this case the \( K \)-orbits are spheres and 1.2 is the problem of constructing a closed \( n \)-gon in \( p \) with given real-valued side-lengths \( r_1, \ldots, r_n \). Of course, the system of triangle inequalities

\[
r_i \leq r_1 + \cdots + r_i + \cdots + r_n \quad (1 \leq i \leq n)
\]

gives necessary and sufficient conditions for its existence. Let us check that these inequalities coincide with the inequalities (3): Since \( X \) has rank one, the group \( G \) acts transitively on the sphere \( \partial_\infty X \). A minimal parabolic subgroup \( B \) fixes a unique point on \( \partial_\infty X \) and acts transitively on the complement. Thus there are only two mod 2 Schubert classes, the class of a point and the top class (the whole sphere). Moreover we have \( a = \mathbb{R} \lambda^\vee \) where \( \lambda^\vee \) is the fundamental coweight and \( W \cong \mathbb{Z}/2 \). The model spherical chamber \( \Delta^\text{sph}_{\text{model}} \) consists just of the vertex \( \zeta = \phi(\lambda^\vee) \) and the isotropy group \( W_\zeta \) is trivial. Thus the relative Weyl set \( W/W_\zeta \) is equal to \( W = \{1, w\} \). The Schubert class corresponding to 1 is the class of a point and the Schubert class corresponding to \( w \) is the top class. We note that the only \( n \)-fold transverse intersections of Schubert classes equal to a point are obtained by taking the top class in all but one place and the class of a point in the remaining place. If the point class is in the \( i \)-th place then
the inequality (3) becomes
\[ \sum_j \langle h_j, \lambda_j^\vee \rangle \leq 0 \]
with \( \lambda_j^\vee = -\lambda_j^\vee \) for \( j \neq i \) and \( \lambda_i^\vee = \lambda_i^\vee \). Substituting \( h_j = r_j \lambda_j^\vee \) we obtain the \( i \)-th triangle inequality (4).

We next give a brief sketch of the proof of our theorem. As explained above we replace the given data \( h_1, \cdots, h_n \) with the boundary data \( \tau_1, \cdots, \tau_n \) together with the lengths \( r_1, \cdots, r_n \). We code a boundary configuration \( \xi \) corresponding to these data into an atomic measure \( \nu \) on \( \partial_\infty X \) such that \( \nu \) has an atom of mass \( r_i \) at the point \( \xi_i \in \partial_\tau \), \( 1 \leq i \leq n \). We then integrate the Busemann function \( b_\xi \) over \( \partial_\infty X \) to obtain the weighted Busemann function \( b_\nu \), a convex Lipschitz function on \( X \), see section 2. The Busemann function comes into play naturally because the vanishing of its gradient turns out to be equivalent to the closing condition (1). Therefore \( \mathcal{P}_n(p) \neq \emptyset \) if and only if the convex function \( b_\nu \) has a minimum for at least one of the measures \( \nu \) in consideration (i.e. of type \( h \)). The existence of a minimum for \( b_\nu \) in turn can be read off from its asymptotic slope which is defined as
\[ \text{slope}_\nu(\xi) := \lim_{t \to \infty} \frac{b_\nu(r(t))}{t} \]
with \( r(t) \) an arbitrary unit speed geodesic ray asymptotic to \( \xi \in \partial_\infty X \). We prove that \( \mathcal{P}_n(p) \neq \emptyset \) if and only if \( b_\nu \) is bounded below if and only if \( \text{slope}_\nu \geq 0 \) for some measure \( \nu \).

We can now explain how the inequality (2) arises. Namely, one easily computes that
\[ \text{slope}_\nu = -\sum_i r_i \cos \angle_{\text{Tits}}(\xi_i, \cdot). \]
To have \( \text{slope}_\nu \geq 0 \) everywhere it suffices that \( \text{slope}_\nu(\eta) \geq 0 \) for all vertices \( \eta \). Thus the uncountably infinite collection of inequalities
\[ \text{slope}_\nu(\eta) \geq 0, \quad \text{for all vertices } \eta, \tag{5} \]
are necessary and sufficient in order that \( \nu \) be a semistable configuration. We will now explain why the infinite collection of inequalities (5) in fact contains only finitely many different inequalities and that these inequalities are parametrized by the intersections of Schubert cells.

We choose a vertex \( \zeta \) in the model spherical Weyl chamber \( \Delta_\text{model}^{\text{sph}} \). Of course this involves finitely many choices. The orbit of \( \zeta \) is the generalized Grassmannian \( \text{Grass}_\zeta \). We will consider the inequalities in the system (5) such the \( \eta \in \text{Grass}_\zeta \). The points \( \xi_i \) belong to spherical Weyl chambers \( \sigma_i \). The \( n \) relative positions \( \eta_i = (\sigma_i, \eta), 1 \leq i \leq n \), determine Schubert cells \( C_{\eta_i}, 1 \leq i \leq n \). By definition \( \eta \in \cap_{i=1}^n C_{\eta_i} \) and we have
\[ \text{slope}_\nu(\eta) = -\sum_i r_i \cos \angle_{\text{Tits}}(\xi_i, \eta) = -\sum_i r_i \cos \angle_{\text{Tits}}(\tau_i, \eta_i). \]
The second equality follows from 3.8 because
\[ (\sigma_i, \eta) = (\Delta_\text{model}^{\text{sph}}, \eta_i) \]
and \( \text{acc}(\xi_i) = \tau_i \) (see Section 2 for the definition of acc). Since there are only finitely many relative positions \((\sigma_i, \eta_i)\) we have proved that the infinite collection of inequalities (5) gives rise to only finitely many different inequalities which are of the form of those of (2).

Thus if there exists a semistable configuration \( \nu \) then the parameters \((r_1, \ldots, r_m, \tau_1, \ldots, \tau_n)\) must satisfy (2) whenever \( \cap_{i=1}^n C_{\eta_i} \neq \emptyset \). We deduce from the semicontinuity of Tits distance that (2) will continue to hold whenever \( \cap_{i=1}^n \overline{C}_{\eta_i} \neq \emptyset \). This proves the necessity of the inequalities in (1.3). To obtain the sufficiency of the inequalities described in (1.3) we use a version of Kleiman transversality (3.13) to deduce that it suffices to consider the above inequalities in the case that the Schubert cells \( C_{\eta_1}, \ldots, C_{\eta_n} \) intersect transversally. We then borrow a trick from Belkale’s thesis (based on our generalization of the “Harder-Narasimhan Lemma”) to prove that it suffices to consider only those intersections of Schubert cells that intersect in a single point. From these last two statements it is easy to deduce the homological formulation of (1.3).

We should point out that the weighted Busemann function was first considered (for the case of hyperbolic space) in [DE].

To conclude we mention some related results which are not included in this paper. We will later give a proof that the polyhedron \( \mathcal{P}_n(p) \) does not depend on the restricted root system but only on the Coxeter group \( W \) as a subgroup of \( O(n) \) i.e on the representation of the abstract Coxeter group.

In a separate paper, we intend to discuss the polyhedra \( \mathcal{P}_n(X) \) for the case in which \( X \) is a symmetric space. The noncompact case is solved, see 3.2.4 and also [AMW] and [Kly99]. Our own proof will appear in [KLM]. There are some partial results for the compact case, [AW] and [Ga].

2 Convex functions on symmetric spaces

2.1 Geometric preliminaries

In this section we will briefly review some basic facts about spaces of nonpositive curvature and especially Riemannian symmetric spaces of noncompact type. We will omit most of the proofs. For more details on spaces with upper curvature bound and in particular spaces with nonpositive curvature, we refer to [Ba, KLLe], for the theory of buildings from a geometric viewpoint, i.e. within the framework of spaces with curvature bounded above, we refer to [KLLe].

2.1.1 Metric spaces with curvature bounds

Curvature bounds for metric spaces. Consider a complete metric length space, that is, a complete metric space \( Y \) such that any two points \( y_1, y_2 \in Y \) can be joined by a rectifiable curve with length \( d(y_1, y_2) \); such curves are called geodesic segments. Although there is no smooth structure nor a Riemann curvature tensor around, one can still make sense of a sectional curvature bound in terms of distance comparison. Namely, we say that \( Y \) has (globally) curvature \( \leq k \) if all triangles in \( Y \) are thinner than corresponding triangles in the model plane (or sphere) \( M^2_k \) of constant curvature \( k \). Here, a geodesic triangle \( \Delta \) in \( Y \) is a one-dimensional object consisting of three points and geodesic segments joining them. A comparison triangle \( \tilde{\Delta} \) for \( \Delta \) in \( M^2_k \)
is a triangle with the same side lengths. To every point \( p \) on \( \Delta \) corresponds a point \( \hat{p} \) on \( \Delta \), and we say that \( \Delta \) is thinner than \( \hat{\Delta} \) if for any points \( p \) and \( q \) on \( \Delta \) we have \( d(p, q) \leq d(\hat{p}, \hat{q}) \). One can similarly define lower curvature bounds by requiring triangles to be thicker than their comparison triangles. We will only be concerned with upper curvature bounds in this paper.

Due to Toponogov's Theorem, a complete simply-connected manifold has curvature \( \leq k \) in the distance comparison sense if and only if it has sectional curvature \( \leq k \).

**Angles and spaces of directions.** The presence of a curvature bound allows to define *angles* between segments \( \sigma_i : [0, \varepsilon) \to Y \) initiating in the same point \( y = \sigma_1(0) = \sigma_2(0) \) and parametrized by unit speed. Let \( \hat{\alpha}(t) \) be the angle of a comparison triangle for \( \Delta(y, \sigma_1(t), \sigma_2(t)) \) in the appropriate model plane at the vertex corresponding to \( y \). If \( Y \) has an upper (resp. lower) curvature bound then the comparison angle \( \hat{\alpha}(t) \) is monotonically decreasing (resp. increasing) as \( t \searrow 0 \). It therefore converges, and we define the angle \( \angle_y(\sigma_1, \sigma_2) \) of the segments at \( y \) as the limit. In this way, one obtains a pseudo-metric on the space of segments emanating from a point \( y \in Y \). Identification of segments with angle zero and metric completion yields the *space of directions* \( \Sigma_y Y \). One can show that if \( Y \) has an upper (lower) curvature bound, then \( \Sigma_y Y \) has curvature \( \leq 1 \) \((\geq 1)\).

### 2.1.2 Spaces with nonpositive curvature

Assume from now on that \( Y \) is a *Hadamard space*, that is, a space of nonpositive curvature. Examples are Riemannian symmetric spaces of noncompact type and Euclidean buildings, which we will be mainly concerned with in this paper.

A basic consequence of the definition of nonpositive curvature is that the distance function \( Y \times Y \to \mathbb{R}^+_0 \) is *convex*. It follows that geodesic segments between any two points are unique and that \( Y \) is contractible.

**Structure at infinity.** A geodesic ray is an isometric embedding \( \rho : [0, \infty) \to Y \). Two rays \( \rho_1 \) and \( \rho_2 \) are called *asymptotic* if \( t \mapsto d(\rho_1(t), \rho_2(t)) \) stays bounded (and hence decreases by convexity). An equivalence class of asymptotic rays is called an *ideal point* or a point at infinity, and we define the *geometric boundary* \( \partial_\infty Y \) as the set of ideal points. The topology on \( Y \) can be canonically extended to the *cone topology* on \( \bar{Y} := Y \cup \partial_\infty Y \); if \( Y \) is locally compact, \( \bar{Y} \) is a compactification of \( Y \).

There is a natural metric on \( \partial_\infty Y \), the *Tits metric*: We define the Tits distance of ideal points \( \xi, \eta \in \partial_\infty Y \) as \( \angle_{\text{Tits}}(\xi, \eta) := \sup_{y \in Y} \angle_y(\xi, \eta) \). It is useful to know that one can compute \( \angle_{\text{Tits}}(\xi, \eta) \) by only looking at the angles along a ray \( \rho \) asymptotic to one of the ideal points \( \xi \) and \( \eta \); namely \( \angle_{\rho(t)}(\xi, \eta) \) is monotonically increasing and converges to \( \angle_{\text{Tits}}(\xi, \eta) \) as \( t \to \infty \). The resulting metric space \( \partial_{\text{Tits}} Y = (\partial_\infty Y, \angle_{\text{Tits}}) \) is called the *Tits boundary*. The Tits distance is lower semicontinuous with respect to the cone topology and induces in general a finer topology on \( \partial_\infty Y \). It turns out that \( \partial_{\text{Tits}} Y \) is a complete metric length space with curvature \( \leq 1 \). The Tits boundary of a product space is the spherical join of the Tits boundaries of the factors.

One can refine the structure at infinity by blowing up ideal points. This is reminiscent of a construction due to Karpelevich for symmetric spaces. For an ideal point \( \xi \in \partial_\infty Y \) consider all rays representing it. The usual distance between the rays as sets induces a pseudo-metric, and we call two rays *strongly asymptotic* if they have
distance zero. Metric completion of the space of strong asymptote classes yields a metric space \( \bar{Y} \) which again is Hadamard.

**Convex functions and Busemann functions.** The convexity of the distance \( d(\cdot, \cdot) \) provides natural convex functions on a Hadamard space \( Y \). First of all, the distance \( d(y, \cdot) \) from a point is a convex function. Other important examples are the Busemann functions which can be regarded as measuring the relative distance from an ideal point \( \xi \in \partial_{\infty} Y \): Take a ray \( \rho \) asymptotic to \( \xi \) and define the Busemann function \( b_{\xi} \) as the pointwise limit of the normalized distance functions \( d(\rho(t), \cdot) - t \). It is easy to see that, up to an additive constant, \( b_{\xi} \) is independent of the ray \( \rho \) representing \( \xi \). \( b_{\xi} \) is Lipschitz continuous with Lipschitz constant 1. The level and sublevel sets of \( b_{\xi} \) are called horospheres and horoballs centered at \( \xi \). We denote the horosphere passing through \( x \) by \( H_{s}(\xi, y) \) and the horoball which it bounds by \( Hb(\xi, y) \). The horoballs are convex subsets and their ideal boundaries are convex subsets of \( \partial_{\text{Tit}} Y \), namely balls of radius \( \pi/2 \) around the centers of the horoballs: \( \partial_{\infty} Hb(\xi, y) = \{ \angle_{\text{Tit}}(\xi, \cdot) \leq \pi/2 \} \).

Consider a Lipschitz continuous convex function \( f : Y \rightarrow \mathbb{R} \). It is asymptotically linear along any ray, and we define the asymptotic slope of \( f \) at \( \xi \in \partial_{\infty} X \) as follows: Pick any geodesic ray \( \rho \) asymptotic to \( \xi \) and set

\[
slope{f}(\xi) := \lim_{t \to \infty} \frac{f(\rho(t))}{t}.
\]

The right-hand side is independent of the ray \( \rho \) representing \( \xi \). \( \slope{f} \) is Lipschitz continuous with the same Lipschitz constant as \( f \). Note that the sublevel sets \( \{ f \leq a \} \) are convex, and their ideal boundaries are given by \( \partial_{\infty} \{ f \leq a \} = \{ \slope{f} \leq 0 \} \). We collect a few basic properties:

**Proposition 2.1** (i) The set \( \{ \slope{f} \leq 0 \} \subset \partial_{\infty} Y \) of asymptotic decrease is convex with respect to the Tits metric.\(^4\) The function \( \slope{f} \) is convex on \( \{ \slope{f} \leq 0 \} \) and strictly convex on \( \{ \slope{f} < 0 \} \).

(ii) If \( Y \) is locally compact, then \( f \) is proper and bounded below iff \( \slope{f} > 0 \) everywhere on \( \partial_{\infty} Y \).

(iii) If \( Y \) is locally compact and if \( \{ \slope{f} < 0 \} \neq \emptyset \), then \( \{ \slope{f} < 0 \} \) contains no points with distance \( \pi \) and \( \slope{f} \) has a unique minimum.

(iv) Suppose that \( Y \) is locally compact and that \( \partial_{\text{Tit}} Y \) is connected. If \( \{ \slope{f} < 0 \} \neq \emptyset \) then \( \{ \slope{f} \leq 0 \} \) is the closure of \( \{ \slope{f} < 0 \} \). If \( \{ \slope{f} = 0 \} \) is open, then \( \slope{f} \geq 0 \) everywhere.

**Sketch of proof.** (i) If \( \angle_{\text{Tit}}(\xi_1, \xi_2) < \pi \) and \( \rho_i \) are rays asymptotic to \( \xi_i \), then the midpoints \( m(t) \) of the segments \( \rho_1(t)\rho_2(t) \) converge to the midpoint \( \mu \) of \( \xi_1\xi_2 \) in \( \partial_{\text{Tit}} Y \), which therefore belongs to \( \{ \slope{f} \leq 0 \} \subset \partial_{\infty} Y \) as well. This implies convexity for the set of asymptotic decay. Regarding the slope function, one has

\[
slope{f}(\mu) \leq \frac{\slope{f}(\xi_1) + \slope{f}(\xi_2)}{2 \cos(\angle_{\text{Tit}}(\xi_1, \xi_2)/2)}.
\]

\(^4\) A subset \( C \) of a space with curvature \( \leq 1 \) is called convex if for any two points in \( C \) with distance \( < \pi \) the unique shortest segment joining them is contained in \( C \).
If $\text{slope}_f(\xi_i) \leq 0$, the right-hand side is $\leq (\text{slope}_f(\xi_1) + \text{slope}_f(\xi_2))/2$ and convexity follows.

(ii) If $f$ were not proper, sublevel sets would be noncompact and hence contain rays. It follows that there is an ideal point with asymptotic slope $\leq 0$. Conversely, if $f$ is proper then clearly $\text{slope}_f > 0$.

(iii) Assume that $\angle_{\text{Tits}}(\xi_1, \xi_2) = \pi$ and $\text{slope}_f(\xi_i) < 0$. As before, we pick rays $\rho_i$ asymptotic to $\xi_i$. The midpoints $m(t)$ of the segments $\rho_1(t)\rho_2(t)$ diverge to infinity because $f(m(t)) \to -\infty$. Hence they subconverge in the geometric compactification to an ideal point $\eta$ with $\angle_{\text{Tits}}(\eta, \xi_i) = \pi/2$, i.e. $\eta$ is the midpoint of a shortest geodesic segment $\xi_1\xi_2$.\(^5\) (6) now leads to a contradiction and shows the first assertion. (6) implies also that a sequence $\eta_n$ such that $\text{slope}_f(\eta_n) \to \inf \text{slope}_f < 0$ must be Cauchy. Hence there is one and only one minimum for $\text{slope}_f$.

(iv) Since $\partial\text{Tits}Y$ is connected, every point $\xi$ with $\text{slope}_f(\xi) \leq 0$ can be connected by a segment of length $< \pi$ to a point $\eta$ with $\text{slope}_f(\eta) < 0$. The convexity of $\text{slope}_f$ on $\{\text{slope}_f \leq 0\}$ implies the first assertion. The second is a direct consequence. \(\square\)

**Parallel sets.** A line $l \subset Y$ is an isometrically embedded copy of $\mathbb{R}$. Two lines are parallel if they have finite Hausdorff distance. It is easy to see that in this case they actually bound an embedded flat strip. A stronger statement holds: The parallel set $P(l)$ is defined as the union of lines parallel to $l$, and there is a canonical isometric splitting $P(l) \cong l \times CS(l)$ where the cross section $CS(l)$ is again Hadamard. The cross sections $pt \times CS(l)$ are level sets of the Busemann functions for the two ideal endpoints of $l$. If $l$ is asymptotic to the ideal point $\xi$, there is a canonical isometric embedding $CS(l) \hookrightarrow Y_\xi$.

### 2.1.3 Symmetric spaces of noncompact type

A complete simply connected Riemannian manifold $X$ is called a symmetric space if in every point $x \in X$ there is a reflection, that is, an isometry $\sigma_x$ fixing $x$ with $d\sigma_x = -id_x$. $X$ has noncompact type if it has nonpositive sectional curvature and no Euclidean de Rham factor. Then the identity component of its isometry group is a noncompact semisimple Lie group $G$. The point stabilizers $K_x$ in $G$ are maximal compact subgroups. We will assume from for the rest of this section that $X$ is a symmetric space of noncompact type.\(^6\)

**Flats.** A flat in $X$ is a convex subset isometric to a Euclidean space. $G$ operates transitively on the set of all maximal flats, and their dimension is called the rank of the symmetric space.

The rich structure of the pattern of flats in $X$ induces the additional structure of an affine Coxeter complex on maximal flats: A non-maximal flat of positive dimension is called singular if it arises as the intersection of maximal flats. Every partial flag of singular flats can be extended to a complete flag, i.e. the dimensions of successive flats differ by one. Each maximal flat $F$ contains finitely many families of parallel codimension-one singular flats. The reflections at these hyperplanes generate the affine Weyl group $W_{aff}(F)$ acting on $F$ and make $F$ into the affine Coxeter complex $(F, W_{aff}(F))$. For a point $x \in F$, there are finitely many singular hyperplanes $f \subset F$

\(^5\)In general, points with Tits distance $\pi$ cannot be connected by a geodesic!

\(^6\)Most of the results discussed in the present section will be valid for Euclidean buildings, as well.
passing through \( x \). They divide \( F \) into cones whose closures are called a **Euclidean Weyl chamber** with **tip** \( x \). The maximal flats, as well as the Euclidean Weyl chambers can be canonically identified with each other by isometries in \( G \), and hence they can be simultaneously identified with a model Euclidean Coxeter complex \((F_{\text{model}}, W_{\text{aff}})\), respectively, a model Weyl chamber \( \Delta_{\text{model}}^{\text{eucl}} \).

**Structure at infinity.** With respect to the cone topology, \( \hat{X} \) is a standard ball and \( \partial_{X} X \) a standard sphere of dimension \( \text{rank}(X) - 1 \). Let us look at the additional structure given by the Tits metric. \( \angle_{\text{Tits}}(\cdot, \cdot) \) is just a discrete metric if \( \text{rank}(X) = 1 \), but it becomes non-trivial if \( \text{rank}(X) \geq 2 \). Namely, \( \partial_{\text{Tits}} X \) carries a natural structure as a spherical simplicial complex of dimension \( \text{rank}(X) - 1 \). We call the points in the 0-skeleton of \( \partial_{\text{Tits}} X \) **vertices**. The top-dimensional faces are ideal boundaries of Euclidean Weyl chambers, and are therefore called **spherical Weyl chambers**. The ideal boundaries of flats in \( X \) are isometrically embedded round spheres in \( \partial_{\text{Tits}} X \); every convex subset in \( \partial_{\text{Tits}} X \) isometric to a round sphere arises in this way. If \( f \) is a singular flat in \( X \) then \( \partial_{\text{Tits}} f \) is a subcomplex. **Apartments** \( a \subset \partial_{\text{Tits}} X \) are ideal boundaries of maximal flats, \( a = \partial_{\text{Tits}} F \). \( a \) is tessellated by chambers, and there is a natural finite reflection group, the Weyl group \( W(a) \), acting on them; \( W(a) \) arises as the rotational part of the affine Weyl group \( W_{\text{aff}}(a) \). The pair \( (a, W(a)) \) is a **spherical Coxeter complex**. \( G \) acts transitively on spherical Weyl chambers and apartments. They can again be simultaneously and compatibly identified with a spherical model Weyl chamber \( \Delta_{\text{model}}^{\text{sph}} \), respectively, a model spherical Coxeter complex \((a_{\text{model}}, W)\). There is a canonical projection map

\[
\text{acc} : \partial_{\text{Tits}} X \to \Delta_{\text{model}}^{\text{sph}};
\]

it is an isometry on each chamber and we call it the **accordion map** because of the way it folds the spherical building onto the model chamber. There is a canonical identification \( \Delta_{\text{model}}^{\text{sph}} \cong \partial_{\text{Tits}} \Delta_{\text{model}}^{\text{eucl}} \). The faces of the complex \( \partial_{\text{Tits}} X \) correspond to parabolic subgroups stabilizing them and, as a simplicial complex, \( \partial_{\text{Tits}} X \) is canonically isomorphic to the spherical Tits building associated to \( G \). We refer to [KlLe] for a study of spherical buildings as metric spaces with curvature \( \leq 1 \).

For an ideal point \( \xi \), the space of strong asymptote classes is metrically complete. In other words, every point in \( X_{\xi} \) is represented by a ray. \( X_{\xi} \) is a symmetric space of rank \( \text{rank}(X) - 1 \); it has no Euclidean de Rham factor iff \( \xi \) is a vertex in the Tits building \( \partial_{\text{Tits}} X \).

**Subspaces and subgroups.** A subset \( Z \subseteq X \) is called **totally geodesic** if, with any two distinct points, it contains the unique line passing through them. It is easy to see that such subsets are embedded submanifolds which are again symmetric spaces. They are called **totally geodesic subspaces**. We denote by \( G_{Z} \subseteq G \) the identity component of the subgroup generated by the reflections at all points of \( Z \).

**Parallel sets.** For a line \( l \), the parallel set \( P(l) \cong l \times CS(l) \) is a symmetric space of the same rank as \( X \). The cross section \( CS(l) \) has a Euclidean de Rham factor iff the line is a singular 1-flat. If \( l \) is asymptotic to \( \xi \), then the canonical isometric embedding \( CS(l) \hookrightarrow X_{\xi} \) is onto, i.e. an isometry. For any two ideal points \( \xi, \eta \in \partial_{\text{Tits}} X \) which are antipodal, i.e. \( \angle_{\text{Tits}}(\xi, \eta) = \pi \), there are lines with \( \xi \) and \( \eta \) as ideal endpoints. These lines are parallel and we denote their union by \( P(\xi, \eta) \).

More generally, one can define the parallel set \( P(Z) \) of an arbitrary totally-geodesic subspace \( Z \subseteq X \). It again splits isometrically, \( P(Z) \cong Z \times CS(Z) \), and the cross
section $P(Z)$ is a symmetric space of rank $\leq \text{rank}(X) - \text{rank}(Z)$.

**Busemann functions.** The Busemann functions $b_\xi$ are smooth and their differential is given by

$$
(d b_\xi)_x(v) = -\cos \angle_x(v, \xi),
$$

(7)
i.e. $-\text{grad } b_\xi$ is the unit vector field pointing towards $\xi$. The horospheres centered at $\xi$ are orthogonal to the foliation by geodesics asymptotic to $\xi$. The Busemann functions are convex, however, not strictly convex. One can precisely describe where the Hessian is degenerate:

**Lemma 2.2** Let $u, v \in T_xX$ be a non-zero tangent vectors, let $l$ be the geodesic with initial condition $v$, and suppose that $u$ points towards the ideal point $\xi$. Then the following are equivalent:

1. $D^2_{u,v} b_\xi = 0$.
2. $b_\xi$ is affine linear on $l$.
3. The 2-plane in $T_xX$ spanned by $u$ and $v$ has sectional curvature zero.
4. $u, v$ are tangent to a 2-flat.

### 2.2 Measures on the ideal boundary of symmetric spaces

$X = G/K$ denotes a symmetric space of noncompact type without Euclidean de Rham factor. We will define stability for finite measures on the ideal boundary and, more generally, for convex Lipschitz functions, and discuss the properties of measures with various degrees of stability.

#### 2.2.1 Definition of stability

Let $\text{Conv}(X)$ denote the space of Lipschitz continuous convex functions on $X$ equipped with the compact-open topology.

**Definition 2.3 (Stability of convex functions)** The convex function $f \in \text{Conv}(X)$ is called

- **semistable** if it is bounded below,
- **unstable** otherwise,
- **asymptotically semistable** if $\text{slope}_f \geq 0$ everywhere on $\partial_\infty X$.
- **nice semistable** if $f$ has a minimum, and
- **stable** if $f$ is proper and bounded below.
We deduce from 2.3 a concept of stability for measures on \( \partial_\infty X \). We use the cone topology on \( \partial_\infty X \) which makes it homeomorphic to a sphere. Let \( \mathcal{M}(\partial_\infty X) \) be the space of finite measures on \( \partial_\infty X \) equipped with the weak * topology. To a measure \( \mu \in \mathcal{M}(\partial_\infty X) \), we assign a convex function on \( X \), namely the weighted Busemann function given as the convex combination

\[
b_\mu := \int_{\partial_\infty X} b_\xi d\mu(\xi). \tag{8}
\]

\( b_\mu \) is well-defined up to additive constants, convex and continuous with Lipschitz constant \( \|\mu\| \). We abbreviate \( \text{slope}_\mu := \text{slope} b_\mu \).

**Definition 2.4 (Stability of measures at infinity)** The finite measure \( \mu \) on \( \partial_\infty X \) is called

- **semistable** if \( b_\mu \) is bounded below,
- **unstable** otherwise,
- **asymptotically semistable** if \( \text{slope}_\mu \geq 0 \) everywhere on \( \partial_\infty X \).
- **nice semistable** if \( b_\mu \) has a minimum, and
- **stable** if \( b_\mu \) is proper and bounded below.

**Remark 2.5** We will see later that semistability and asymptotic semistability are equivalent for measures with finite support, cf. 2.26. In general, however, asymptotic semistability does not imply semistability.\(^7\)

The natural map

\[
\mathcal{M}(\partial_\infty X) \longrightarrow \text{Conv}(X)/\mathbb{R}, \quad \mu \mapsto [b_\mu] \tag{9}
\]

given by (8) is continuous, however, not injective. There is a certain freedom in redistributing the measure without changing the weighted Busemann function. For instance, there is a canonical way of concentrating the mass on vertices of the Tits building without changing the Busemann function:

**Lemma 2.6** For each measure \( \mu \in \mathcal{M}(\partial_\infty X) \) there is a measure \( \mu_{\text{vert}} \) supported on vertices of the Tits building such that \( b_\mu = b_{\mu_{\text{vert}}} + \text{const.} \)

**Proof.** Consider first the case of a Busemann function \( b_\xi \). Let \( \sigma \) be the face of the Tits building spanned by \( \xi \), i.e. containing \( \xi \) as an interior point. For any flat \( f \subset X \) with \( \partial_\infty f \supset \sigma \ni \xi \), \( b_\xi \) is affine linear on \( f \); the gradient lines in \( f \) are the parallel lines asymptotic to \( \xi \). There is a unique way of expressing \( b_\xi \) as the weighted Busemann function for a measure supported on the vertices of the simplex \( \sigma \), i.e. of assigning to \( \xi \) a measure supported on vertices. This assignment is continuous, and the assertion for arbitrary \( \mu \) follows by integration over \( \partial_\infty X \).

Similarly, if \( X \) is reducible, one can concentrate the mass on the ideal boundaries of the factors.

\(^7\)Here is an example of a measure on the ideal boundary of hyperbolic plane \( \mathbb{H}^2 \) where \( b_\mu \) is not bounded below although \( \text{slope}_\mu \geq 0 \) everywhere: Let \( \xi_n \rightarrow \xi \) on \( \partial_\infty \mathbb{H}^2 \) and put mass 1 on \( \xi \) and mass \( 2^{-n} \) on \( \xi_n \) for \( n \geq 1 \). Using 2.24 one sees that \( b_\mu \) is not bounded below along rays asymptotic to \( \xi \) if \( \xi_n \rightarrow \xi \) fast enough.
Lemma 2.7 Suppose that $X$ decomposes as a product $X = X_1 \times \cdots \times X_m$. Then for every measure $\mu \in \mathcal{M}(\partial_{\infty}X)$ there is a canonical choice of measures $\mu_i \in \mathcal{M}(\partial_{\infty}X_i)$ such that $b_\mu = b_{\mu_1} + \cdots + b_{\mu_m} + \text{const}$. The $b_{\mu_i}$ are unique up to additive constants.

Proof. $\partial_{\text{Tits}} X$ is the spherical join of the $\partial_{\text{Tits}} X_i$. For an ideal point $\xi \in \partial_{\infty} X$ there is a minimal subproduct $X_{i_1} \times \cdots \times X_{i_r}$ containing $\xi$ in its ideal boundary. There are unique $\xi_{i_j} \in \partial_{\infty} X_{i_j}$ such that the right-angled simplex in $\partial_{\text{Tits}} X$ spanned by them contains $\xi$ as an interior point. There is a unique way of expressing $b_\xi$ as a linear combination of the $b_{\xi_{i_j}}$ and, as in the proof of 2.6, integration over $\partial_{\infty} X$ yields the desired decomposition of $b_\mu$. Uniqueness of the summands is clear.

Next we see how the asymptotic growth behavior of $b_\mu$ can be read off the measure $\mu$ and we express the stability conditions in terms of the Tits geometry. Using the formula (7) for the differential of Busemann functions, we see that the differential of the weighted Busemann function $b_\mu$ is given by

$$(db_\mu)_x \cdot v = - \int_{\partial_{\infty} X} \cos \angle_x(v, \xi) d\mu(\xi)$$

where $v \in T_x X$. To compute the asymptotic slope of $b_\mu$ at $\eta$ we pick a ray $\rho$ asymptotic to $\eta$ and recall that $\angle_{\rho(t)}(\xi, \eta) \nearrow \angle_{\text{Tits}}(\xi, \eta)$. It follows that

$$\text{slope}_\mu(\eta) = - \int_{\partial_{\infty} X} \cos \angle_{\text{Tits}}(\eta, \cdot) d\mu. \quad (10)$$

2.2.2 Stable measures

According to 2.1, stability is equivalent to positivity of asymptotic slopes because $X$ is locally compact. Using (10), we can express stability in terms of the Tits geometry:

Proposition 2.8 $\mu$ is stable iff

$$\int_{\partial_{\infty} X} \cos \angle_{\text{Tits}}(\eta, \cdot) d\mu < 0$$

for all $\eta \in \partial_{\infty} X$.

2.2.3 Unstable measures and directions of steepest descent

We know from 2.1 that for unstable measures $\mu$ there is a unique ideal point $\xi_{\text{min}}$ of steepest descent, i.e. where $\text{slope}_\mu$ attains its minimum. We will show that there exists a vertex of steepest descent among vertices which is almost unique. The following result is analogous to the Harder-Narasimhan Lemma, cf. Lemma 5 in [Bel].

Proposition 2.9 Let $\mu \in \mathcal{M}(\partial_{\infty} X)$ be unstable. There exist vertices with minimal $\mu$-slope (among vertices). These vertices are vertices of the simplex $\sigma_{\text{min}}$ containing the unique ideal point $\xi_{\text{min}}$ of steepest descent for $b_\mu$. In particular, there are only finitely many vertices of steepest descent, and every $G$-orbit of vertices contains at most one vertex of them.
Proof. Each ideal point lies in an apartment containing \( \sigma_{\min} \). It therefore suffices to prove the assertion for such apartments.

Let \( a \) be an apartment containing \( \sigma_{\min} \) and choose a base point \( o \). \( b_\mu \) has on each Euclidean Weyl chamber the same asymptotic slopes as a suitably chosen linear function, cf. (10). Thus there exists an, up to additive constants, unique convex functions \( \beta : a \to \mathbb{R} \) whose restriction to each Weyl chamber with tip in \( o \) is affine linear. \( \beta \) is not bounded below, because \( \text{slope}_\mu(\xi_{\min}) < 0 \). There is a unique ray \( \rho_{\text{steep}} \) emanating from \( o \) with steepest decay of \( \beta \), namely the ray asymptotic to \( \xi_{\min} \).

Among the finitely many maximally singular\(^8\) rays in \( a \) initiating in \( o \), let \( \rho_{\text{sing}} \) be one with steepest decay. The assertion reduces to showing that each closed Weyl chamber with tip \( o \) which contains \( \rho_{\text{steep}} \) also contains \( \rho_{\text{sing}} \).

This is clear if \( \beta \) is linear: Then \( \rho_{\text{steep}} \) is parallel to \(- \text{grad} \beta \) and we have \( \angle_o(\rho_{\text{steep}}, \rho_{\text{sing}}) \leq \angle_o(\rho_{\text{steep}}, \rho) \) for any other maximally singular ray \( \rho \) emanating from \( o \). Hence no wall through \( o \) can separate \( \rho_{\text{steep}} \) from \( \rho_{\text{sing}} \) because reflection at this wall would move \( \rho_{\text{sing}} \) to a maximally singular ray of smaller angle with \( \rho_{\text{steep}} \). Hence Weyl chambers containing \( \rho_{\text{steep}} \) have \( \rho_{\text{sing}} \) as an edge.

In the general case, if \( \beta \) is not linear, there is a unique linear minorant \( \lambda \leq \beta \) with same direction and slope of steepest decay. We have \( \lambda = \beta \) on the intersection \( V \) of all Weyl chambers containing \( \rho_{\text{steep}} \). Since rays of steepest \( \lambda \)-descent are edges of \( V \) they are rays of steepest \( \beta \)-descent as well and there are no other maximally singular rays of steepest \( \beta \)-descent. \( \square \)

**Remark 2.10** Proposition 2.9 holds more generally for convex functions which are asymptotically linear on Weyl chambers.

### 2.2.4 Nice semistable measures: The structure of minimum sets

Suppose that \( \mu \in \mathcal{M}(\partial_\infty X) \) is nice semistable, and denote by \( \text{MIN}(\mu) \) the non-empty set of minima for \( b_\mu \).

2.2 implies an analogous result for weighted Busemann functions:

**Lemma 2.11** Let \( \nu \in \mathcal{M}(\partial_\infty X) \), let \( v \) be a non-zero tangent vector and let \( l \) be the geodesic with initial condition \( v \). Then the following are equivalent:

1. \( D^2_{v,v}b_\nu = 0 \).
2. \( b_\nu \) is affine linear on \( l \).
3. \( \nu \) is supported on \( \partial_\infty P_l \).

**Proof.** Integration yields \( D^2_{v,v}b_\nu = \int_{\partial_\infty X} D^2_{v,v}b_\xi \, d\nu(\xi) \). Since \( D^2_{v,v}b_\xi \geq 0 \), we have \( D^2_{v,v}b_\nu = 0 \) iff \( D^2_{v,v}b_\xi = 0 \) for \( \nu \)-almost all \( \xi \). Hence (1.) \( \Rightarrow \) (3.). Clearly (3.) \( \Rightarrow \) (2.) \( \Rightarrow \) (1.). \( \square \)

**Corollary 2.12** \( \text{MIN}(\mu) \) is a complete totally geodesic subspace of \( X \). In particular, if \( \mu \) is stable, then \( b_\mu \) is strictly convex and has a unique minimum.

---

\(^8\)A ray is maximally singular if it is asymptotic to a vertex.
Proof. Lemma 2.11 implies that $MIN(\mu)$ contains with any two distinct points also the complete geodesic passing through these points. \hfill \Box

The next result describes the relation between the location of the measure and its minimum set. We say that a product subspace $Y_1 \times \cdots \times Y_k$ of $X$ has non-enlargable factors if for any product subspace $Y'_1 \times \cdots \times Y'_k$ with $Y'_i \supseteq Y_i$ holds $Y'_i = Y_i$.

**Theorem 2.13 (Structure of minimum sets)** There is a product subspace $Y \times Z \times F$ with non-enlargable factors such that the following holds:

- $Y$ and $Z$ have no Euclidean factor. $F$ is Euclidean.
- $MIN(\mu) = Y \times F$.
- $supp(\mu) \subseteq \partial_{\infty}(Z \times F)$.
- $b_\mu|_F$ is constant and $b_\mu|_Z$ is proper and bounded below.

**Proof.** The totally geodesic subspace $MIN(\mu)$ splits as $MIN(\mu) = Y \times F$ where $F$ is its Euclidean de Rham factor. The parallel set of $MIN(\mu)$ splits as $P_{MIN(\mu)} = Y \times F' \times Z$ where $F' \supseteq F$ is a flat and $Z$ has no Euclidean factor. Then $Cent(G_{MIN(\mu)})_{\partial} = G_{F' \times Z}$.\footnote{For a symmetric subspace $W \subseteq X$, $G_W$ denotes the subgroup of $Isom(W)$ generated by transvections along geodesics in $W$.} According to Lemma 2.11, $\mu$ is supported on the ideal boundary of $\cap_{t \in MIN(\mu)} P_t = F' \times Z$.

The Busemann function $b_\mu$ must be constant on $F'$ and proper bounded below on $Z$, so $F' = F$ and $\mu$ induces a stable measure on $\partial_{\infty}Z$. Since $P_{Y \times F} = Y \times Z \times F$, the factors $Z$ and $F$ are non-enlargable, and $MIN(\mu) = Y \times F$ implies that $Y$ (and again $F$) are non-enlargable. \hfill \Box

Immediate consequences of 2.13 are:

**Corollary 2.14** The fixator of $supp(\mu)$ contains $G_{MIN(\mu)}$. In particular, it acts transitively on $MIN(\mu)$.

**Corollary 2.15 (Strict convexity transverse to the minimum set)** There exist constants $A$ and $C > 0$ depending on $\mu$ such that

$$b_\mu > \min(b_\mu) + C \cdot d(MIN(\mu), \cdot) + A. \quad (11)$$

The sublevel sets of $b_\mu$ are within bounded Hausdorff distance of the minimum set $MIN(\mu)$.

**Proof.** By 2.11, $b_\mu$ is strictly convex along every ray emanating from $MIN(\mu)$ in normal direction. By compactness, there is an estimate of the form (11) for all such rays with a fixed starting point. The first claim then follows from the homogeneity of $MIN(\mu)$ (cf. 2.14). The second claim is a direct consequence. \hfill \Box
Corollary 2.16 Nice semistable measures have weak * closed $G$-orbits within the space of asymptotically semistable measures.

Proof. Assume that $g_n \mu$ converge to an asymptotically stable measure $\nu$, i.e., $b_{g_n \mu} \to b_\nu$ uniformly on compacta (after additive renormalization). By 2.15, the distance of $MIN(g_n \mu)$ from a base point must remain bounded because $\text{slope}_\nu \geq 0$. 2.14 allows to modify the $g_n$ so that they stay bounded and hence subconverge. \qed

2.2.5 Asymptotically semistable measures and making them nicer by folding

Lemma 2.17 (Structure of slope zero set at infinity) Let $\mu$ be asymptotically semistable, i.e. $\text{slope}_\mu \geq 0$. Then:

- $\{\text{slope}_\mu = 0\}$ is a subcomplex with respect to the simplicial structure of the Tits boundary.
- $\mu$ is nice semistable iff $\{\text{slope}_\mu = 0\}$ is empty or has dimension $d \geq 0$ and contains a $d$-dimensional (singular) sphere.\(^{10}\)

Proof. For the first claim we observe that if $\text{slope}_\mu$ is zero at an interior point of a face $\sigma$ of $\partial_{\text{Tits}} X$ then it vanishes on the whole face $\sigma$. We use here that $\text{slope}_\mu \geq 0$ everywhere.

For the second claim we consider the non-trivial case that $\mu$ is not stable. Then $\{\text{slope}_\mu = 0\}$ is a non-empty subcomplex $C$ of some dimension $d \geq 0$. Let us assume that $C$ contains a singular $d$-sphere $s$. Then $b_\mu$ is constant along any flat $F$ with $\partial_\infty F = s$. The induced convex function on the cross section of $P_F$ must be proper bounded below because otherwise $\dim \{\text{slope}_\mu = 0\} > d$. Hence $b_\mu$ assumes a minimum on $P_F$. Let now $x \in X$ be an arbitrary point. Consider a ray $\rho$ emanating from $x$ and asymptotic to an interior point of $s$. Then $\rho$ is strongly asymptotic to $P_F$. $b_\mu$ is non-increasing along $\rho$ because it is convex of asymptotic slope zero. Hence $b_\mu(x) \geq \min(b_\mu|_{P_F})$, i.e., the minima of $b_\mu$ on $P_F$ are minima on all of $X$. This means that $\mu$ is nice semistable.

Vice versa, assume that $b_\mu$ assumes a minimum in $x$. Pick a $d$-dimensional face $\sigma$ of $C$. Then $b_\mu$ is constant on the $(d+1)$-flat $F$ through $x$ asymptotic to $\sigma$, and hence $C$ contains the top-dimensional sphere $\partial_\infty F$. \qed

Folding. Let us now assume that $\mu$ is asymptotically semistable but not nice semistable. We describe a construction of nicer measures in the weak * closure of $G \cdot \mu$.

By assumption, there exists $\xi$ with $\text{slope}_\mu(\xi) = 0$. Pick a point $\hat{\xi} \in \partial_\infty X$ antipodal to $\xi$. Let $(T_t)$ be the 1-parameter group of translations whose translation axes are the geodesics asymptotic to $\xi$ and $\hat{\xi}$ and so that $T_t$ translates towards $\xi$ for $t > 0$. As $t \to \infty$, the induced boundary maps $T_{-t}$ on $\partial_\infty X$ converge pointwise to a map

$$\text{Fold} : \partial_\infty X \to \partial_\infty P(\xi, \hat{\xi})$$

\(^{10}\)Sphere here means a closed convex subset isometric to a unit sphere. Singular means that it is a subcomplex.
which can be described as follows:\textsuperscript{11} For $\eta \in \partial_\infty X$, $Fold(\eta)$ is the unique point in $\partial_\infty P(\xi, \hat{\xi})$ with $\angle_{Tits}(Fold(\eta), \xi) = \angle_{Tits}(\eta, \xi)$ and such that there are minimizing geodesics $\overline{\eta\xi}$ and $\overline{\xi Fold(\eta)}$ with same initial directions at $\xi$. (These geodesics are unique unless $\eta$ is antipodal to $\xi$.) $Fold$ is 1-Lipschitz for the Tits metric and isometric on polyhedral faces. Note that

$$(T_{-\epsilon})_*\mu \rightarrow Fold_*\mu$$

in the weak $\ast$ topology.

Lemma 2.18  $Fold_*\mu$ is again asymptotically semistable.

Proof. $slope_{(T_{-\epsilon})_*\mu} = slope_{\mu}$ on the Tits neighborhood $\text{star}(\xi)$ of $\xi$.\textsuperscript{12} Hence also $slope_{Fold_*\mu} = slope_{\mu}$ on a neighborhood of $\xi$. Since $\{slope_{Fold_*\mu} \geq 0\}$ has non-empty interior and $slope_{Fold_*\mu} = 0$ in $\xi$, 2.1 implies that $Fold_*\mu$ is asymptotically semistable, as claimed. \qed

Proposition 2.19  Let $\mu$ be asymptotically semistable. Then $G\mu$ contains a nice semistable measure.

Proof. Pick $\xi$ as a maximally regular point of slope zero. Then $\{slope_{Fold_*\mu} = 0\}$ contains a top-dimensional sphere and hence $Fold_*\mu$ is nice semistable by 2.17. \qed

Remark 2.20  2.19 shows that our definition of asymptotic semistability agrees with the definition of semistability used in geometric invariant theory. For measures with finite support, semistability in our sense and asymptotic semistability are equivalent, cf. 2.26.

As in geometric invariant theory, there is a uniqueness result:

Theorem 2.21 (Uniqueness of nice semistable degenerations)  Let the measure $\mu$ be asymptotically semistable. Then the orbit closure $G\mu$ contains a unique $G$-orbit of nice semistable measures.

Proof. If $\mu$ is nice semistable then $G\mu$ is closed within asymptotically semistable measures, cf. 2.16. We therefore assume that $\mu$ is not nice. In view of 2.6, we may assume without loss of generality that $\mu$ is supported on vertices.

Suppose that $\nu$ is a nice semistable degeneration of $\mu$, i.e.

$$g_n\mu \rightarrow \nu$$

in the weak $\ast$ topology with $g_n \in G$. Pick a point $\xi$ in the non-empty subcomplex $\{slope_{\mu} = 0\}$ and let $H = \text{Stab}_G(\xi)_\circ$. We may assume without loss of generality that $g_n \in H$. For otherwise, using that $K$ acts transitively on $G$-orbits in $\partial_\infty X$, we can replace $g_n$ by $k_ng_n \in H$ and $k_ng_n\mu$ subconverges to $k\nu$.

\textsuperscript{11}We denote by $P(\xi, \hat{\xi})$ the union of all geodesics asymptotic to $\xi$ and $\hat{\xi}$.

\textsuperscript{12}$\text{star}(\xi)$ is the union of all (closed) polyhedral faces containing $\xi$. 

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Lemma 2.22  For all $\eta \in \partial_{\infty}X$ exists $\epsilon = \epsilon(\eta) > 0$ such that

$$\angle_{\text{Tits}}(\xi, \cdot) \leq \angle_{\text{Tits}}(\xi, \eta) - \epsilon$$

on $\overline{H\eta} - H\eta$.

Proof. Consider a sequence of points $\eta_n := g_n\eta$ in $H\eta$ which converges to $\eta_{\infty} \in \overline{H\eta}$. Then $d(\eta_{\infty}, \xi) \leq d(\eta, \xi)$ by semi-continuity of the Tits metric. If $d(\eta_{\infty}, \xi) = d(\eta, \xi)$ then $\xi \in H\eta$. Otherwise $d(\eta_{\infty}, \xi) < d(\eta, \xi)$. Since $\eta_{\infty}$ has the same type as $\eta$ (i.e. $G\eta_{\infty} = G\eta$) only finitely many values are possible for $d(\eta_{\infty}, \xi)$, whence the claim. □

$H$ has only finitely many orbits on vertices, and we decompose $\mu$ into the finitely many restrictions $\mu|_{H\eta}$ to the vertex orbits $H\eta$. The measures $g_n \cdot \mu|_{H\eta}$ converge weakly to a measure $\nu|_{H\eta}$ supported on $\overline{H\eta}$. Due to 2.22 we have

$$\text{slope}_{\nu}(\xi) = 0$$

with strict inequality iff not all $\nu|_{H\eta}$ are already supported on $H\eta \subset \overline{H\eta}$. Since $\nu$ is semistable we conclude that all measures $\nu|_{H\eta}$ are supported on $H\eta$, i.e. $\nu|_{H\eta}(\overline{H\eta} - H\eta) = 0$.

Choose an antipode $\hat{\xi}$ for $\xi$ so that $b_{\mu}$ assumes its minimum on the parallel set $P(\xi, \hat{\xi})$. Then $\nu$ is supported on $\partial_{\infty}P(\xi, \hat{\xi})$. We denote by

$$\text{Fold} : \partial_{\infty}X \to \partial_{\infty}P(\xi, \hat{\xi})$$

the folding map for the one parameter group translating from $\hat{\xi}$ towards $\xi$. Observe that $\text{Fold}$ is continuous on every $H$-orbit (however in general not on their closures). Therefore $\text{Fold}_*(g_n \cdot \mu|_{H\eta}) \to \text{Fold}_*(\nu|_{H\eta})$ and, by summing over the vertex orbits, we obtain

$$\text{Fold}_*(g_n \cdot \mu) \to \text{Fold}_*\nu = \nu. \quad (13)$$

The last equation follows from the fact that $\text{Fold}$ fixes $\text{supp}(\nu)$ pointwise.

There exist isometries $\tilde{g}_n \in \text{Stab}_G(\xi, \hat{\xi})_o$ which act as $g_n$ on the space of strong asymptote classes $X_\xi$. They satisfy $\tilde{g}_n \text{Fold} = \text{Fold}g_n$. With (13) we get

$$\tilde{g}_n \text{Fold}_*\mu \to \nu.$$

Assume now in addition that $\xi$ was chosen to be a maximally regular (i.e. smooth) point in $\{\text{slope}_{\mu}(\xi) = 0\}$. Then $\text{Fold}_*\mu$ is nice semistable and 2.16 implies that $\nu \in G \cdot \text{Fold}_*\mu$. Note that $H$ acts transitively on antipodes for $\xi$ and hence $\text{Fold}_*\mu$ stays in the same $G$-orbit as $\hat{\xi}$ varies. Thus we have shown that all nice semistable degenerations $\nu \in \overline{G\mu}$ of $\mu$ lie in the same $G$-orbit. □

Consider again an ideal point $\xi$ of slope zero: $\text{slope}_{\mu}(\xi) = 0$. If $b_{\mu}$ happens to be bounded below along rays asymptotic to $\xi$ then folding may be carried out already on the level of Busemann functions: $b_{T_{\xi,\mu}} = b_{\mu} \circ T_{\xi}$ converges (uniformly on compacta) to $b_{\text{Fold}_*\mu}$ and we have $b_{\mu} \geq b_{\text{Fold}_*\mu}$ (modulo additive constants). This observation can be applied to prove the following semistability criterion for asymptotically semistable measures:
Lemma 2.23  $b_\mu$ is bounded below (i.e. $\mu$ is semistable) iff $b_\mu$ is bounded below along all rays.

Proof. Suppose that $b_\mu$ is bounded below along every ray. The assertion is clear if $\mu$ is stable. We therefore assume that $\{\text{slope}_\mu = 0\}$ is non-empty, and we choose inside it a maximally regular point $\xi$. We now fold at $\xi$. Since $b_\mu$ is bounded below along rays, we have convergence already on the level of Busemann functions: $b_{T_\xi t_\mu} = b_\mu \circ T_\xi t_\mu \to b_{\text{Fold}, \mu}$, and $b_\mu \geq b_{\text{Fold}, \mu}$. Observe that $b_{\text{Fold}, \mu}$ is nice semistable and therefore has a minimum. It follows that $b_\mu$ is bounded below.  

2.2.6 Measures with finite support

We specialize to the case that our measures are finitely supported. For single atoms, we have:

Lemma 2.24 Let $\xi \in \partial_\infty X$ and $\rho : [0, \infty) \to X$ be a ray parametrised by unit speed. Then $b_\xi \circ \rho(t) + \cos \angle_{\text{Tit}}(\xi, \rho(\infty)) \cdot t$ is convex and converges to a finite limit as $t \to \infty$.

Proof. There is a ray asymptotic to $\rho$ on which $b_\xi$ is linear, namely a ray in a flat which contains $\xi$ and $\rho(\infty)$ in its ideal boundary. The slope of $b_\xi$ along such rays is $-\cos \angle_{\text{Tit}}(\xi, \rho(\infty))$. This shows that the difference (14) remains bounded as $t \to \infty$. It is clearly convex, because the Busemann function is convex.

Corollary 2.25 Let $\mu$ be finite atomic. Then $b_\mu$ is bounded below along the ray $\rho$ iff $\text{slope}_\rho(\infty) \geq 0$.

As a consequence, we show that semistability and asymptotic semistability agree for measures with finite support.

Proposition 2.26 Let $\mu$ be a finite atomic measure on $\partial_\infty X$. Then $\mu$ is semistable if and only if it is asymptotically semistable.

Proof. Assume that $\mu$ is asymptotically semistable, i.e. $\text{slope}_\mu \geq 0$ everywhere. According to 2.25, $b_\mu$ is bounded below along every ray. By 2.23, this implies that $b_\mu$ is bounded below, i.e. $\mu$ is semistable.

3 Moduli spaces

3.1 Moduli spaces of weighted configurations of flags

We will study quotients of certain configuration spaces which we will call moduli spaces.
3.1.1 Definition of the moduli spaces

**Configuration spaces.** Recall that the accordion map \( \text{acc} : \partial_\infty X \to \Delta^{\text{sp}}_{\text{model}} \) is the quotient map for the natural action of \( G \) on \( \partial_\infty X \), i.e. \( \Delta^{\text{sp}}_{\text{model}} \) parametrizes \( G \)-orbits at infinity. The point stabilizers are parabolic subgroups, and the \( G \)-orbits are thus identified with (partial) flag manifolds.

Consider a finite measure space \( F \) with underlying set \( \mathbb{Z}/n\mathbb{Z} \) and masses \( r_i \geq 0 \) on the points \( i \). We fix a type map \( \tau : F \to \Delta^{\text{sp}}_{\text{model}} \) and define a configuration of type \( \tau \) as a map

\[
F \to \partial_\infty X
\]

such that the composition with \( \text{acc} \) yields \( \tau \). The information of the type map \( \tau \) is equivalent to a map \( \tau^{\text{euc}} : F \to \Delta^{\text{euc}}_{\text{model}} \) where \( h_i := \tau^{\text{euc}}(i) \) has direction \( \tau_i := \tau(i) \) and length equal to the mass \( r_i \). We write \( h = (h_1, \ldots, h_n) \) and denote by \( \text{Conf}_h \) the space of all such configurations. Let \( \mathcal{O}_i^\infty \) be the \( G \)-orbit of type \( \tau_i \). Configurations of type \( \tau \) can be thought of as \( n \)-tuples \( (\xi_1, \ldots, \xi_n) \in \mathcal{O}_1^\infty \times \cdots \times \mathcal{O}_n^\infty \) with weights \( r_i \) assigned to the points \( \xi_i \). Note that we do not assume that the image of \( \tau \) is contained in the interior of \( \Delta^{\text{sp}}_{\text{model}} \).

**Moduli spaces.** \( G \) acts naturally on configurations and we want to form a quotient. Since \( G \) is non-compact, the naive quotient is in general non-Hausdorff. Motivated by geometric invariant theory, we will form a Mumford type quotient by throwing away enough orbits to make the quotient Hausdorff while keeping enough orbits to make the quotient compact. This is achieved, as we expect (cf. 3.4), by our notion of stability in 2.4. We proceed as follows:

The closure of any orbit will contain unstable orbits, and we therefore dispense with all unstable orbits. On the semistable part \( \text{Conf}_{h}^{\text{ss}} \) of \( \text{Conf}_{h} \) we introduce an equivalence relation by defining two configurations to be equivalent if and only if their orbit closures intersect. This is indeed an equivalence relation because, according to 2.21, every semistable orbit contains in its closure a unique nice semistable orbit. This equivalence relation is called *extended orbit equivalence*.

**Definition 3.1 (The moduli space of weighted configurations)** We define the moduli space of configurations with type \( h \) to be the quotient

\[
\mathcal{M}_h = \text{Conf}_{h}^{\text{ss}} / G
\]

of the semistable configurations \( \text{Conf}_{h}^{\text{ss}} \) by the equivalence relation of extended orbit equivalence.

Two natural questions arise:

**Question 3.2**

1. For which choices of \( h \in (\Delta^{\text{euc}}_{\text{model}})^n \) is \( \mathcal{M}_h \) non-empty, i.e. are there semistable configurations?

2. Which natural structure does \( \mathcal{M}_h \) carry on top of its topology? (If \( G \) is complex then it is an analytic space, cf. 3.3, in fact a projective variety if the \( r_i \) are integral.)
If \( G \) is a complex group and if the weights \( r_i \) are natural numbers (which implies that the configuration space is a projective variety), our construction coincides with the Mumford quotient in geometric invariant theory. If the weights are arbitrary real numbers, the configuration space still carries a Kähler structure and general results of Heinzer-Loose and Sjamaar imply:

**Theorem 3.3 ([HL94, Sj95])** The moduli space \( \mathcal{M}_h \) is compact Hausdorff and has a natural structure as a complex analytic space.

**Remark 3.4** We expect that in the case when \( G \) is a real semisimple group, the configuration spaces are compact and Hausdorff.

### 3.1.2 Nonemptyness of the moduli spaces

We will now answer the first question above by giving a system of linear inequalities that are necessary and sufficient in order that there exist a semistable point. We will need the

**Schubert calculus.** Think of \( \Delta^\text{sph}_\text{model} \) as sitting in the spherical Coxeter complex \( a_{\text{model}} \). For a vertex \( \zeta \) of \( \Delta^\text{sph}_\text{model} \), we denote by \( \text{Grass}_\zeta \) the corresponding maximally singular \( G \)-orbit in \( \partial_\infty X \), a "generalized Grassmannian". The action of a Borel subgroup \( B \) stratifies each \( \text{Grass}_\zeta \) into Schubert cells, one cell \( C_{\eta} \) corresponding to each vertex \( \eta \) in the orbit \( W\zeta \) of \( \zeta \) under the Weyl group \( W \). Hence, if we denote \( W_\zeta := \text{Stab}_W(\zeta) \) then the Schubert cells correspond to cosets in \( W/W_\zeta \). The Schubert cycles are defined as the closures \( \overline{C}_\eta \) of the Schubert cells; they are unions of Schubert cells. \(^\text{13}\) Note that as real algebraic varieties, the Schubert cycles represent homology classes \( [\overline{C}_\eta] \in H^*(\text{Grass}_\zeta, \mathbb{Z}/2\mathbb{Z}) \) which we abbreviate to \( [C_\eta] \); in the complex case they even represent integral homology classes.

It will be useful to have another description of the Schubert cells and Schubert cycles. We recall the definition of the relative position of a spherical Weyl chamber \( \sigma \) and a vertex \( \eta \) of \( \partial_{\text{Tit}_s} X \). Choose \( g \in G \) such that \( g\sigma \) coincides with \( \Delta^\text{sph}_\text{model} \). Now choose \( b \in B \) such that \( bg\eta = \eta^* \) lies in the standard apartment \( a_{\text{model}} \). We then define the relative position \( (\sigma, \eta) \) to be \( \eta^* \). We first prove

**Lemma 3.5** Suppose there exist \( b \in B \) and vertices \( \eta_1, \eta_2 \in a_{\text{model}} \) such that \( b\eta_1 = \eta_2 \). Then \( \eta_1 = \eta_2 \).

**Proof.** Since \( \eta_1 \in G_\eta_2 \) we may write \( \eta_1 = w_1\zeta \) and \( \eta_2 = w_2\zeta \) for some \( w_1, w_2 \in W \) and \( \zeta \) a vertex of \( \Delta^\text{sph}_\text{model} \). Hence, \( w_2^{-1}bw_1\zeta = \zeta \) whence \( w_1 \in Bw_2P \) where \( P \) is the stabilizer of \( \zeta \) in \( G \). By the Bruhat decomposition \( w_1 \in w_2W_\zeta \) whence \( w_1\zeta = w_2\zeta \) and \( \eta_1 = \eta_2 \).

Now we can prove

**Lemma 3.6** The relative position \( (\sigma, \eta) \) is well-defined.

\(^{13}\)One may elaborate how the order of Schubert cells relates to the word metric on the Weyl group. There is one top-dimensional Schubert cell corresponding to vertices \( \eta \) at maximal distance from \( \zeta \).
Proof. Suppose \( \sigma \) and \( \eta \) as above are given and we have \( g_1, b_1 \) and \( g_2, b_2 \) with \( g_1 \sigma = g_2 \sigma = \Delta_{\text{model}}^{\text{sph}} \) and \( b_1 g_1 \eta_1 = \eta_1^* \in a_0 \) and \( b_2 g_2 \eta_2 = \eta_2^* \in a_0 \). We must prove that \( \eta_1^* = \eta_2^* \).

Since \( \text{Stab}(\Delta_{\text{model}}^{\text{sph}}) = B \), we have \( g_2 = bg_1 \) for some \( b \in B \). Hence,

\[
\eta_2^* = b_2 g_2 \eta = b_2 b g_1 \eta = b_2 b b_1^{-1} b_1 g_1 \eta = b_2 b b_1^{-1}(\eta_1^*).
\]

By the previous lemma we have \( \eta_1^* = \eta_2^* \). \( \square \)

We leave the proof of the following lemma to the reader.

**Lemma 3.7** \( (\sigma_1, \eta_1) = (\sigma_2, \eta_2) \) if and only if there exists \( g \in G \) such that \( g \sigma_2 = \sigma_1 \) and \( g \eta_2 = \eta_1 \).

We next relate Tits distance to relative position.

**Lemma 3.8** Suppose the \( \sigma_1, \eta_1 \) and \( \sigma_2, \eta_2 \) are given with \( (\sigma_1, \eta_1) = (\sigma_2, \eta_2) \). Suppose further we are given \( \xi_1 \in \sigma_1 \) and \( \xi_2 \in \sigma_2 \) with \( \text{acc}(\xi_1) = \text{acc}(\xi_2) \). Then

\[
\angle_{\text{Tits}}(\xi_1, \eta_1) = \angle_{\text{Tits}}(\xi_2, \eta_2).
\]

**Proof.** Put \( \tau = \text{acc}(\xi_1) = \text{acc}(\xi_2) \). We choose \( g_1, g_2 \) such that \( g_1 \sigma_1 = g_2 \sigma_2 = \Delta_{\text{model}}^{\text{sph}} \). Hence, \( g_1 \xi_1 = g_2 \xi_2 = \tau \). By assumption there exist \( b_1, b_2 \in B \) such that \( b_1 g_1 \eta_1 = b_2 g_2 \eta_2 \). Hence

\[
\angle_{\text{Tits}}(\xi_1, \eta_1) = \angle_{\text{Tits}}(g_1 \xi_1, g_\eta_1) = \angle_{\text{Tits}}(b_2^{-1} b_1 g_1 \xi_1, b_2^{-1} b_1 g_1 \eta_1) = \angle_{\text{Tits}}(g_1 \xi_1, g_2 \eta_2)
\]

\( = \angle_{\text{Tits}}(g_2 \xi_2, g_2 \eta_2) = \angle_{\text{Tits}}(\xi_2, \eta_2) \). \( \square \)

We now have another description of the Schubert cells as mentioned in the introduction. As above we assume we have chosen a spherical Weyl chamber \( \sigma \) and a vertex \( \zeta \) of \( \Delta_{\text{model}}^{\text{sph}} \). Now let \( \eta_i \in \text{W} \). We then have

**Lemma 3.9** The Schubert cell \( C_{\eta_i} \) is given by

\[
C_{\eta_i} = \{ \eta \in \text{Grass}_{\zeta} : (\sigma, \eta) = \eta_i \}.
\]

**Non-emptiness of the moduli space.** For all vertices \( \zeta \) of \( \Delta_{\text{model}}^{\text{sph}} \) and all \( n \)-tuples of vertices \( \eta_1, \ldots, \eta_n \in \text{W} \) consider the inequality

\[
\sum_{i} r_i \cdot \cos \angle(\tau_i, \eta_i) \leq 0, \tag{17}
\]

where \( \angle \) measures the spherical distance in \( a_{\text{model}} \) and \( \{ \tau_i \} = \mathcal{Q}_i^\infty \cap \Delta_{\text{model}}^{\text{sph}} \). We may rewrite the inequality as follows using standard terminology of Lie theory: Let \( \lambda_{\zeta} \in \Delta_{\text{model}}^{\text{exc}} \) be the fundamental coweight contained in the edge with direction \( \zeta \), and let \( \lambda_i := w_i \lambda_{\zeta} \) where \( [w_i] \in \text{W}/\text{W} \) such that \( w_i, \zeta = \eta_i \). Then (17) becomes the linear homogeneous inequality

\[
\sum_{i} \langle h_i, \lambda_i \rangle \leq 0. \tag{18}
\]

The following result describes, in terms of the Schubert calculus, a subset of these inequalities which is equivalent to the moduli space being non-empty. This generalizes earlier results of Helmke-Rosenthal [HR95], Klyachko [Kly98] and Belkale [Bel].
**Theorem 3.10** \( \mathcal{M}_h \neq \emptyset \) iff (17) or, equivalently, (18) holds whenever the intersection of the Schubert classes \([C_{\eta_1}], \ldots, [C_{\eta_n}]\) in \( H_*(\text{Grass}_\zeta, \mathbb{Z}/2\mathbb{Z})\) equals \([pt]\).

*Proof.* Assume \( \mathcal{M}_h = \emptyset \), i.e. all configurations are unstable. Due to the transversality result 3.13, there exist chambers \( \sigma_1, \ldots, \sigma_n \subset \partial_{\infty}X \) so that the corresponding \( n \) stratifications of the Grassmannians \( \text{Grass}_\zeta \) by orbits of the Borel subgroups \( B_i = \text{Stab}_G(\sigma_i) \) are transversal. (This transversality is actually generic.) We choose our configuration so that the atoms are located on the \( \sigma_i \). Unstability of the configuration and the Harder-Narasimhan lemma 2.9 yield a vertex \( \eta_{\text{sing}} \) of smallest (negative) slope. Moreover, within its Grassmannian the vertex is unique with this slope. Therefore it is the unique intersection point of the \( n \) Schubert cells (\( B_i \)-orbits) passing through it. Transversality implies that the corresponding Schubert cycles intersect transversally in the unique point \( \eta_{\text{sing}} \), and hence one of the inequalities in our set is violated.

Conversely, assume that \( \mathcal{M}_h \neq \emptyset \), i.e. there is a semistable configuration \( \xi = (\xi_1, \ldots, \xi_n) \in \text{Conf}_{\text{ss}}^h \) and assume further that we have a homologically nontrivial product of \( n \) Schubert classes: \([C_{\eta_1}], \ldots, [C_{\eta_n}]\) in a generalized Grassmannian \( \text{Grass}_\zeta \). Here the \( \eta_i \) are vertices of the model spherical Coxeter complex in the Weyl group orbit \( W \zeta \). Now choose a chamber \( \sigma_i \) containing \( \xi_i \) in its closure; \( \sigma_i \) is unique iff \( \xi_i \) is regular. The choice of chambers determines cycles \( \tilde{C}_{\eta_i} \) representing the Schubert classes. We have \( \tilde{C}_{\eta_i} \cap \cdots \cap \tilde{C}_{\eta_n} \neq \emptyset \) and choose a point \( \theta \) in the intersection. Since, for all \( \eta \in C_{\eta_i} \) we have \( (\sigma_i, \eta) = (\tau_i, \eta_i) \) it follows from (3.8) that \( \angle_{\text{Tits}}(\xi_i, \cdot) = \angle_{\text{Tits}}(\tau_i, \eta_i) \) on the Schubert cell \( C_{\eta_i} \). The semicontinuity of the Tits distance then implies that \( \angle_{\text{Tits}}(\xi_i, \cdot) \leq \angle_{\text{Tits}}(\tau_i, \eta_i) \) on the cycle \( \tilde{C}_{\eta_i} \). Thus \( \angle_{\text{Tits}}(\xi_i, \theta) \leq \angle_{\text{Tits}}(\tau_i, \eta_i) \). It follows that \( \sum_i r_i \cos \angle_{\text{Tits}}(\tau_i, \eta_i) \leq \sum_i r_i \cos \angle_{\text{Tits}}(\xi_i, \theta) = -\text{slope}_\mu(\theta) \leq 0 \) where \( \mu \) is the measure on \( \partial_{\infty}X \) given by the weighted configuration \( \xi \). Hence the inequality (17) holds whenever the corresponding Schubert classes have non-zero homological intersection. \( \square \)

**Addendum 3.11** If \( G \) is complex, then \( \mathcal{M}_h \neq \emptyset \) iff (17) or, equivalently, (18) holds whenever the intersection of the integral Schubert classes \([C_{\eta_1}], \ldots, [C_{\eta_n}]\) in \( H_*(\text{Grass}_\zeta, \mathbb{Z})\) equals \([pt]\).

*Proof.* Same proof. \( \square \)

**Remark 3.12** Note that, if \( G \) is complex, the set of necessary and sufficient inequalities parametrized by Schubert calculus in integral cohomology is smaller.

### 3.1.3 A transversality result for homogeneous spaces

**Proposition 3.13** Let \( Y \) be a homogeneous space for the Lie group \( G \), and let \( Z_1, \ldots, Z_n \) be embedded submanifolds. Then, for almost all \((g_1, \ldots, g_n) \in G^n\), the submanifolds \( g_1Z_1, \ldots, g_nZ_n \) intersect transversally.

*Proof.* The maps \( G \times Z_i \rightarrow Y \) are submersions, and hence the inverse image

\[
N := \{(g_1, z_1, \ldots, g_n, z_n) : g_1z_1 = \cdots = g_nz_n\}
\]
of the small diagonal in \( Y^n \) under the canonical map \( G \times Z_1 \times \cdots \times G \times Z_n \to Y^n \) is a submanifold. We consider the natural projection \( N \to G^n \). According to Sard’s theorem, the regular values form a subset of full measure in \( G^n \) (i.e., the set of singular values has zero measure). Let \( (g_1^0, \ldots, g_n^0) \) be a regular value. Then the intersection \( \bigcap g_i^0 Z_i \) is a submanifold. It remains to verify that the \( g_i^0 Z_i \) intersect transversally.

Assume that this is not the case. It means that there exist \( z_i^0 \in Z_i \) so that \( g_i^0 z_i^0 = \cdots = g_n^0 z_n^0 = y \), and that all \( g_i^0 Z_i \) are tangent in \( y \) to some hypersurface \( S = \{ f = 0 \} \) with a smooth function \( f : Y \to \mathbb{R} \). Define \( \psi_i : G \to \mathbb{R} \) by \( \psi_i(g) := f(g z_i^0) \). Then the composed maps

\[
N \to Y^n \xrightarrow{(f, \ldots, f)} \mathbb{R}^n
\]

and

\[
N \to G^n \xrightarrow{(\psi_1, \ldots, \psi_n)} \mathbb{R}^n
\]

have the same differential in the point \( p = (g_1^0, z_1^0, \ldots, g_n^0, z_n^0) \). However, the image of the first map is at \( p \) tangent to the diagonal of \( \mathbb{R}^n \), while the second map has maximal rank in \( p \). This is a contradiction, and it follows that the intersection is transversal.

\[\square\]

**Remark 3.14** In the algebraic category one can prove a more precise result, namely that the intersection is transversal for a Zariski open subset of tuples \((g_1, \ldots, g_n)\), compare Kleiman’s transversality theorem [KL74].

### 3.2 Moduli spaces of polygons

#### 3.2.1 Definition of moduli spaces

Let \( G \) be a semisimple real Lie group of noncompact type and \( K \) be a maximal compact subgroup. Decompose the Lie algebra \( \mathfrak{g} \) of \( G \) according to \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) where \( \mathfrak{k} \) is the Lie algebra of \( K \) and \( \mathfrak{p} \) is the orthogonal complement of \( \mathfrak{k} \) relative to the Killing form. Then \( K \) acts on \( \mathfrak{p} \) by the restriction of the adjoint representation. Let \( \mathfrak{a} \) be a maximal (abelian) subalgebra of \( \mathfrak{p} \). Think of the model Euclidean Weyl chamber \( \Delta_{\text{model}}^{\text{euc}} \) as sitting in \( \mathfrak{a} \), \( \Delta_{\text{model}}^{\text{euc}} \subset \mathfrak{a} \) by identifying it with a Weyl chamber. Then each \( K \)-orbit \( \mathcal{O} \subset \mathfrak{p} \) meets \( \Delta_{\text{model}}^{\text{euc}} \) in a unique point \( h(\mathcal{O}) \).

The space \( OPol_n(\mathfrak{g}) \) of open \( n \)-gons in \( \mathfrak{p} \) is defined to be \( \mathfrak{p}^{\mathbb{Z}/n\mathbb{Z}} \). We denote by \( CPol_n(\mathfrak{g}) \) the subspace of closed polygons, 14 i.e., of the \((e_1, \ldots, e_n) \in OPol_n(\mathfrak{g})\) satisfying the closing condition

\[
\sum_{i=1}^{n} e_i = 0. \quad (19)
\]

We next define spaces of polygonal linkages, i.e., polygons with fixed side lengths. In general, a vector in \( \mathfrak{p} \) has more invariants under the \( K \)-action than just its length; the full invariant is the intersection point of its orbit with \( \Delta_{\text{model}}^{\text{euc}} \). It is therefore natural

\[14\]The reader may excuse that, in contrast to our notation, the letter “O” closes up whereas “C” does not.
to assign $\Delta_{\text{model}}$-valued side lengths to polygons. Let $n$ $K$-orbits $O_1, \ldots, O_n$ be given. Let $h_i = h(O_i)$ be the corresponding points in $\Delta_{\text{model}}$. We denote by $\text{Link}_h(g)$ and $C\text{Link}_h(g)$ the subspaces of open, respectively, closed polygons with fixed side lengths $h = (h_1, \ldots, h_n)$. In other words, $\text{Link}_h(g) = O_1 \times \cdots \times O_n$, and $C\text{Link}_h(g)$ is the set of solutions to (19) in $\prod_i O_i$. We denote by

$$\mathcal{M}\text{Link}_h(g) := \text{Link}_h(g)/K$$

(20)

the moduli space of closed $n$-gons with fixed side lengths $h$ modulo the $K$-action. Finally, let $P_n(p)$ be the set of $n$-tuples $h \in (\Delta_{\text{model}}) \in \mathbb{R}^n$ such that $C\text{Link}_h(g) \neq \emptyset$, that is, there exist closed $n$-gons with side lengths $h_1, \ldots, h_n$. Later, we will suppress $g$ in our notation if no confusion is caused. Note that all spaces which we just defined are, up to canonical isomorphism, independent of the splitting $g = \mathfrak{t} + \mathfrak{p}$ and the choice of $a$.

There is a canonical correspondence

$$\text{Link}_h(g) \leftrightarrow \text{ConG}_h(X)$$

between linkages and weighted configurations via the radial projection $\phi : p - \{0\} \to \partial_X$ to infinity. $\phi$ identifies each $K$-orbit $O_i$ in $p - \{0\}$ with a $G$-orbit $O_i^\infty$ in $\partial_X$. Given a linkage $e : \mathbb{Z}/n\mathbb{Z} \to p$, we obtain a configuration by composing with $\phi$; the weights $r_i$ attached to the ideal points $\xi_i := \phi(e(i))$ are defined as the lengths of the edges $e(i)$. We write $\mu_e$ for the resulting measure on $\partial_X$.

It is worth noting that only $K$ acts naturally on linkages, whereas on configurations the action extends to $G$.

### 3.2.2 The closing condition and $\nabla b_{\mu_e}(0)$

We now relate the closing condition for the linkage $e$ to the weighted Busemann function for the measure $\mu_e$ on $\partial_X$.

We identify the unit sphere $S(p)$ in $p$ with $\partial_X$ again using the radial projection $\phi$. We can then think of $\mu_e$ as a measure on $S(p)$ and define its *Euclidean center of mass* $Z(\mu_e)$ by

$$Z(\mu_e) = \int_{S(p)} u \, d\mu_e(u).$$

Clearly we have

$$Z(\mu_e) = \sum_{i=1}^n e_i.$$

**Lemma 3.15** $Z(\mu_e) = -\nabla b_{\mu_e}(0)$.

**Proof.**

$$\nabla b_{\mu_e}(0) = \int_{\partial_X} \nabla b_{\xi}(0) \, d\mu_e(\xi) = \int_{S(p)} \nabla b_{\phi(u)}(0) \, d\mu_e(u) = \int_{S(p)} (-u) \, d\mu_e(u) = -Z(\mu_e)$$

As a consequence, the existence of semistable configurations is equivalent to the existence of closed $n$-gon linkages.
Proposition 3.16 $\mathcal{M}_{Conf_h} \neq \emptyset \Leftrightarrow \mathcal{M}_{Link_h} \neq \emptyset$.

Proof. By Proposition 2.19, the moduli space $\mathcal{M}_{Conf_h}$ is non-empty if and only if there exists a nice semistable configuration $\xi = \phi(e) \in Conf^n_h$, i.e. if $MIN(\mu_e) \neq \emptyset$. Since we can move the minima around by the $G$-action, this is in turn equivalent to the existence of $\xi' = \phi(e') \in Conf^n_h$ with $0 \in MIN(\mu_{e'})$. This amounts to $\nabla b_{\mu_e}(0) = 0$ and $\mathcal{M}_{Link_h} \neq \emptyset$. \qed

3.2.3 For $G$ complex, the moduli space of linkages is a symplectic quotient

Assume that $G$ is complex semisimple. We will show that the closing condition $\nabla b_{\mu_e}(0) = 0$ is the momentum zero condition from symplectic geometry.

Since $G$ is complex we have $\mathfrak{p} = i\mathfrak{k}$, and we may identify $\mathfrak{k}$ and $\mathfrak{p}$ as $K$-modules. Thus we may consider $Link_h$ as a product of $Ad_k$-orbits in $\mathfrak{k}$ and consequently $Link_h$ has a symplectic structure. On the other hand, $Conf_h$ is a product of $G$-orbits in $\partial_{X^\infty} X$ and hence carries a natural complex structure. Via the identification $Link_h \leftrightarrow Conf_h$ both structures fit together to a Kähler structure.

The following lemma is standard.

Lemma 3.17 The diagonal action of $K$ on $Link_h$ is Hamiltonian with momentum map given by

$$m(e) = \sum_{i=1}^n e_i = Z(\mu_e).$$

We obtain

Proposition 3.18 If $G$ is complex semisimple, the moduli space $\mathcal{M}_{Link_h}$ of $n$-gon linkages in $\mathfrak{p} = i\mathfrak{k}$ is canonically isomorphic to a symplectic quotient of a product of $K$-orbits in $\mathfrak{k}$.

3.2.4 Polygons in the symmetric space

Define the space $CPol_n(X)$ of closed $n$-gons in $X$ as $X^\mathbb{Z}/n\mathbb{Z}$. There is a natural notion of vector-valued side lengths because the quotient of the $G$-action on $X \times X$ is canonically identified with $\Delta_{\text{model}}^{\text{euc}}$. Let $C_{Link_h}(X)$ denote the space of closed $n$-gons with side lengths $h \in (\Delta_{\text{model}}^{\text{euc}})_n$. We show in [KLM]:

Theorem 3.19 ([KLM]) $C_{Link_h}(X)$ is non-empty if and only if $C_{Link_h}(g)$ is.

This is a generalization of the Thompson Conjecture [Th88] which was stated for the case of $GL(n, \mathbb{C})$. Related results were obtained by Klyachko [Kly99] and Alekseev-Meinrenken-Woodward [AMW].
4 Explicit examples

4.1 Explicit semistability inequalities for the classical groups

In this section we make explicit the semistability inequalities (18) for classical complex simple Lie algebras $\mathfrak{g}$ which we derived in section 3.1. As an application we will solve the problem of constructing closed $n$-gon linkages with edges real skew-symmetric matrices posed by Steinberg, cf. page 245 in [Fu00]. For us this amounts to the case $G = SO(m, \mathbb{C})$, $K = SO(m)$ and $\mathfrak{p} = \mathfrak{iso}(m)$.

We will identify the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with its dual $\mathfrak{h}^*$ via the Killing form. Below, we will regard all functionals in $\mathfrak{h}^*$ as vectors in $\mathfrak{h}$.

We will use the following notation. $\omega_i : \mathbb{R}^m \to \mathbb{R}$ will denote the projection on the $i$-th coordinate, $(\mathbb{R}^m)^+ := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 \geq \cdots \geq x_m\}$, $(\mathbb{R}_{\geq 0}^m)^+ := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 \geq \cdots \geq x_m \geq 0\}$, $(\mathbb{R}_{= 0}^m)^+ := \{x \in (\mathbb{R}^m)^+ : \sum_{i=1}^m x_i = 0\}$, and $(\mathbb{R}_{\geq 0}^m)^+ := \{x \in (\mathbb{R}^m)^+ : \sum_{i=1}^m x_i = 0\}$.

Since the fundamental weights and the fundamental coweights are positive multiples of each other we will use the fundamental weights in place of the fundamental coweights in the semistability inequalities. The fundamental weights for the simple complex Lie algebras may be found in [Sa69], pages 115-118. For each fundamental weight $\lambda$ we will identify the stabilizer $W_\lambda$ in the Weyl group $W$. We will parametrize the relative Weyl set $W^\lambda = W/W_\lambda$ by pairs $I, \epsilon$ where $I$ is a subset of $\{1, 2, \cdots, n\}$ and $\epsilon$ is an assignment of signs to the elements of $I$. We will call such a pair a signed Schubert symbol. For each signed Schubert symbol $I, \epsilon$ we will associate a coset representative $w_{I, \epsilon} \in W/W_\lambda$ and describe the weight $w_{I, \epsilon} \lambda$. We will then write down the corresponding semistability inequalities in terms of signed sums of eigenvalues $h^I_{I, \epsilon}$ determined by signed Schubert symbols $I, \epsilon$ and the intersections of the Schubert classes $[C_{I, \epsilon}]$. This latter class is the homology class represented by the closure of the orbit of $w_{I, \epsilon}$ under the minimal parabolic subgroup $B$ in $G/P_\lambda$ where $P_\lambda$ is the maximal parabolic subgroup stabilizing the vertex $\zeta = \zeta_\lambda$ of $\Delta_{\text{model}}^{\text{sph}}$ corresponding to $\lambda$.

4.1.1 $sl(m)$

The Euclidean model Weyl chamber $\Delta_{\text{model}}^{\text{exc}}$ is canonically identified with $(\mathbb{R}^m)_0^+$. The Weyl group. The Weyl group is $S_m$ and $\lambda_p$ is fixed by $S_p \times S_{m-p}$.

Schubert symbols and the relative Weyl set. $I = \{i_1, \ldots, i_p\}$ with $1 \leq i_1 < \cdots < i_p \leq m$. The coset $[w_I] \in S_m/S_p \times S_{m-p}$ is represented by any permutation satisfying $w_I(k) = i_k$ for $1 \leq k \leq p$.

Translates of $\lambda_p$, $w_I \lambda_p = \sum_{k=1}^p \omega_{i_k}$. We obtain the inequalities of Klyachko for $G = SL(m, \mathbb{C})$ as refined by Belkale.

Theorem A. Let $h^j \in \Delta_{\text{model}}^{\text{exc}} \cong (\mathbb{R}_0^m)^+$, $1 \leq j \leq n$. For each $p = 1, \ldots, m - 1$, list all $n$-fold intersections of Schubert classes $[C_{I_1}], \ldots, [C_{I_n}]$ in $H_*(\text{Grass}_p(\mathbb{C}^n), \mathbb{Z})$ such that

$$[C_{I_1}] \cdots [C_{I_n}] = \{pt\}.$$
Then the system of inequalities
\[ \sum_{j=1}^{n} h_{I,j}^{i} \leq 0, \]
where \( h_{I,j}^{i} := \sum_{k=1}^{p} h_{k,j}^{i} \), is necessary and sufficient for there to exist \( n \) Hermitian \( m \times m \)-matrices \( A_1, \ldots, A_n \) with eigenvalues \( h^1, \ldots, h^n \) and \( A_1 + \cdots + A_n = 0 \).

Equivalently, the above inequalities give necessary and sufficient conditions for the existence of a semistable configuration of complex weighted flags with weights corresponding to \( h^1, \ldots, h^n \).

### 4.1.2 \( so(2m+1) \)

The Euclidean model Weyl chamber \( \Delta_{\text{model}}^{\text{eucl}} \) is canonically identified with \( (\mathbb{R}^m_{\geq 0})^+ \).

Fundamental weights. \( \lambda_p = \sum_{i=1}^{p} \omega_i \) for \( 1 \leq p \leq m - 1 \), \( \lambda_m = \frac{1}{2} \sum_{i=1}^{m} \omega_i \).

The Weyl group. The Weyl group is \( O_m \), the hyperoctahedral group, i.e. the group of signed permutations (the symmetry group of the \( m \)-dimensional cube). We realize \( O_m \) as acting on the set of non-zero integers \( \{-m, \ldots, -1, 1, \ldots, m\} \). The stabiliser of \( \lambda_p \) is the subgroup \( S_p \times O_{m-p} \).

Signed Schubert symbols and the relative Weyl set. \( (I, \epsilon) = (\epsilon_1 i_1, \ldots, \epsilon_p i_p) \) with \( 1 \leq i_1 < \cdots < i_p \leq m \) and \( \epsilon_i \in \{\pm 1\} \). The coset \( w_{I,\epsilon} \in O_m / S_p \times O_{m-p} \) is represented by any signed permutation \( w_{I,\epsilon} \) satisfying \( w_{I,\epsilon}(k) = \epsilon_k i_k \) for \( 1 \leq k \leq p \). We note the special case \( p = m \). A set of representatives for the relative Weyl set \( W / W_{\lambda_m} \) is the normal subgroup \( E \) of “pure sign changes”. Precisely \( E = \{w_{I,\epsilon} : \epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{\pm 1\}^m \} \) where \( w_{I,\epsilon}(k) = \epsilon_k i_k, 1 \leq k \leq m \).

Translates of \( \lambda_p \). \( w_{I,\epsilon} \lambda_p = \sum_{k=1}^{p} \epsilon_k \omega_{i_k} \). We note that for the special case \( p = m \) we have \( w_{\epsilon} \lambda_m = 1/2 \sum_{k=1}^{m} \epsilon_k \omega_k \).

Taking \( G = SO(2m+1, \mathbb{C}), K = SO(2m+1) \) and \( \mathfrak{p} = i \cdot so(2m+1) \). we solve half of the “eigenvalues of a sum” problem for real skew-symmetric matrices posed by Steinberg, cf. page 245 in [Fu00].

**Theorem B.** Let \( h^j \in \Delta_{\text{model}}^{\text{eucl}} \approx (\mathbb{R}_{\geq 0}^m)^+ \), \( 1 \leq j \leq n \). For each \( p = 1, \ldots, m \), let \( \text{Grass}_p^{i\alpha}(\mathbb{C}^{m+1}) \) denote the space of isotropic \( p \)-planes. List all \( n \)-fold intersections of Schubert classes \([C_{I_{1,\epsilon}}, \ldots, C_{I_{n,\epsilon}}] \) in \( H_*(\text{Grass}_p^{i\alpha}(\mathbb{C}^{m+1}), \mathbb{Z}) \) such that
\[
[C_{I_{1,\epsilon}}] \cdots [C_{I_{n,\epsilon}}] = \{pt\}.
\]
Then the system of inequalities
\[ \sum_{j=1}^{n} h_{I,j,\epsilon}^{j} \leq 0, \]
where \( \epsilon^j = (\epsilon_1^j, \ldots, \epsilon_n^j) \) with \( \epsilon^j_k \in \{\pm 1\} \), \( h^j = (h_1^j, \ldots, h_n^j) \in \mathbb{R}_{\geq 0}^m \) and \( h_{I,\epsilon}^{j} := \sum_{k=1}^{p} \epsilon^j_k h_{k,i}^{j} \), give necessary and sufficient conditions in order that there exist \( n \) real skew-symmetric \( 2m+1 \times 2m+1 \)-matrices \( A_1, \ldots, A_n \) with eigenvalues \( i \cdot h^1, \ldots, i \cdot h^n \) such that \( A_1 + \cdots + A_n = 0 \).
Remark 4.1 We note that for the special case \( p = m \) we have
\[
\sum_{j=1}^{n} h_{ij}^j \leq 0,
\]
where \( h_{ij} = \sum_{k=1}^{m} \epsilon_k h_{ik}^j \).

There are corresponding theorems for the real forms \( SO(p,q) \) with \( p+q = 2m+1 \) involving mod 2 Schubert calculus, cf. section 4.1.5.

4.1.3 \( sp(2m) \)

Since \( sp(2m) \) and \( so(2m+1) \) have the same Weyl group, the Tits boundaries of \( SO(2m+1, \mathbb{C})/SO(2m+1) \) and \( Sp(2m, \mathbb{C})/Sp(2m) \) are modelled on the same Coxeter complex and have isometric chambers. Thus the form of the individual inequalities for \( sp(2m) \) is the same as for \( so(2m+1) \). However, the system will be different, because the Schubert calculus will be different in the corresponding isotropic Grassmannians.

Theorem C (Symplectic case). Same statement as Theorem B.

4.1.4 \( so(2m) \)

The Euclidean model Weyl chamber is given by \( \Delta_{\text{model}}^{\text{euc}} := \{ x \in (\mathbb{R}^m)^{+} : x_i \geq 0 \text{ for } 1 \leq i \leq m-1, x_{m-1} \geq |x_m| \} \).

Fundamental weights, \( \lambda_p = \sum_{i=1}^{p} \omega_i \) for \( p = 1, \ldots, m-2 \), \( \lambda_{m-1} = \frac{1}{2}(\omega_1 + \cdots + \omega_{m-1} - \omega_m) \), \( \lambda_m = \frac{1}{2}(\omega_1 + \cdots + \omega_{m-1} + \omega_m) \).

The Weyl group. The Weyl group is the special hyperoctahedral group \( O_m^+ \), the subgroup of the hyperoctahedral group consisting of those signed permutations with an even number of sign changes. The stabilizer of the fundamental weight \( \lambda_p \) for \( 1 \leq p \leq m-2 \) is the subgroup \( S_p \times O_{m-p}^+ \). The stabilizer of the fundamental weight \( \lambda_m \) is the subgroup \( S_m \). In order to describe the stabilizer of \( \lambda_{m-1} \) we let \( \iota \) be the element of \( O_m \) given by \( \iota(k) = k, 1 \leq k \leq m-1 \) and \( \iota(m) = -m \). Then the stabilizer of \( \lambda_{m-1} \) is the conjugate of \( S_m \) by \( \iota \). Since \( \iota \lambda_m = \lambda_{m-1} \), the operation of conjugation by \( \iota \) induces a bijection from the relative Weyl set \( W/W_{\lambda_m} \) to the relative Weyl set \( W/W_{\lambda_{m-1}} \). Hence a set of representatives for both coset spaces is given by the normal subgroup \( E \) of “pure sign changes”. Precisely \( E = \{ w_\epsilon : \epsilon = (\epsilon_1, \cdots, \epsilon_m) \in \{ \pm 1 \}^m \wedge \prod_{i=1}^{m} \epsilon_i = 1 \} \) where \( w_\epsilon(k) = \epsilon_k k, 1 \leq k \leq m \).

Signed Schubert symbols. We first write down the signed Schubert symbols corresponding to the fundamental weights \( \lambda_p \) for \( 1 \leq p \leq m-2 \). They consist of pairs \((I, \epsilon)\), \( I = \{ i_1, \ldots, i_p \} \) with \( 1 \leq i_1 < \cdots < i_p \leq m \) and \( \epsilon = (\epsilon_1, \cdots, \epsilon_p) \in \{ \pm 1 \}^p \) with \( \prod_{i=1}^{p} \epsilon_i = 1 \). Then \( [w_{I,\epsilon}] \in O_m^+/S_p \times O_{m-p}^+ \) is represented by any signed permutation \( w_{I,\epsilon}(k) = \epsilon_k i_k \) for \( 1 \leq k \leq p \).

We next write down the signed Schubert symbols corresponding to the fundamental weight \( \lambda_m \). They consist of \( m \)-tuples \( \epsilon = (\epsilon_1, \cdots, \epsilon_m) \in \{ \pm 1 \}^m \) such that \( \prod_{i=1}^{m} \epsilon_i = 1 \). Then \( [w_\epsilon] \in W_{\lambda_{m-1}} = O_m^+/S_m \subseteq \{ \pm 1 \}^m \) is represented by the “pure sign change” \( w_\epsilon \) where \( w_\epsilon(k) = \epsilon_k k, 1 \leq k \leq m \).

Finally, the Schubert symbols corresponding to \( \lambda_{m-1} \) consist of \( m \)-tuples \( \epsilon = (\epsilon_1, \cdots, \epsilon_m) \in \{ \pm 1 \}^m \) such that \( \prod_{i=1}^{m} \epsilon_i = -1 \). The coset representative \( w_\epsilon \) is defined by \( w_\epsilon(k) = \epsilon_k k, 1 \leq k \leq m \) where \( \epsilon_k = \epsilon_k, 1 \leq k \leq m-1 \) and \( \epsilon_m = -\epsilon_m m \).
Translates of $\lambda_p$.

For $1 \leq p \leq m - 1$ we have $w_{e,\lambda_p} = \sum_{k=1}^{p} \epsilon_k \omega_k$.

$w_{e,\lambda_{m-1}} = \sum_{k=1}^{m} \epsilon_k \omega_k$. There are an odd number of $-1$'s in the sum.

$w_{e,\lambda_m} = \sum_{k=1}^{m} \epsilon_k \omega_k$. There are an even number of $-1$'s in the sum.

We take $G = SO(2m, \mathbb{C})$, $K = SO(2m)$ and $\mathfrak{p} = i \cdot so(2m)$. Here $so(2m)$ is the Lie algebra of skew-symmetric $2m \times 2m$-matrices.

Thus we obtain the other half of the “eigenvalues of a sum” problem for real skew-symmetric matrices.

**Theorem D.** Let $h^j \in \Delta^{\text{cusp}}_{\text{mod}}, 1 \leq j \leq n$. For each $p$ with $1 \leq p \leq m - 2$, let $\text{Grass}^{i \omega}(\mathbb{C}^{m+1})$ denote the space of all totally-isotropic $p$-planes. List all $n$-fold intersections of Schubert classes $[C_{1,\epsilon_1}], \ldots, [C_{n,\epsilon_n}]$ in $H_s(\text{Grass}^{i \omega}(\mathbb{C}^{m+1}), \mathbb{Z})$ such that

$$[C_{1,\epsilon_1}] \cdots [C_{n,\epsilon_n}] = \{pt\}.$$ 

We obtain the corresponding inequalities

$$\sum_{j=1}^{n} h^j_{1,\epsilon_j} \leq 0,$$

In addition we have two more families of inequalities corresponding to the fundamental weights $\lambda_{m-1}$ and $\lambda_m$. First we list the inequalities corresponding to $\lambda_{m-1}$. We recall that a Schubert symbol is now an $m$-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{-1, 1\}^m$ such that $\prod_{i=1}^{m} = -1$. We obtain one inequality of type $\lambda_{m-1}$ for each $n$-tuple of Schubert symbols $\epsilon$ such that $[C_{\epsilon_1}] \cdots [C_{\epsilon_n}] = \{pt\}$. The corresponding inequality is

$$\sum_{j=1}^{n} h^j_{\epsilon_j} \leq 0,$$

where $h^j_{\epsilon_j} = \sum_{k=1}^{m} \epsilon_k \omega_k$.

The family of inequalities corresponding to the fundamental weight $\lambda_m$ are analogous except now the Schubert symbols $\epsilon$ contain an even number of $-1$'s.

The resulting system of inequalities gives necessary and sufficient conditions in order that there exist $n$ skew-symmetric real $2m \times 2m$-matrices $A_1, \ldots, A_n$ with eigenvalues $i \cdot h^1, \ldots, i \cdot h^n$ and satisfying $A_1 + \cdots + A_n = 0$.

**Remark 4.2** For the fundamental weights $\lambda_{m-1}$ and $\lambda_m$ the orbits of the corresponding vertices in $\partial_{Tt\mu} X$ are no longer Grassmannians of isotropic subspaces. This is because the group $SO(2m, \mathbb{C})$ does not act transitively on the space of isotropic $m$-planes. There are two orbits of isotropic $m$-planes corresponding to the two eigenvalues of the Hodge star acting on $\Lambda^m \mathbb{C}^{2m}$. The intersection products of Schubert classes in the last two statements of the previous theorem are computed in these two orbits.

### 4.1.5 Rank-one symmetric spaces

The $Ad_k$-orbits in $\mathfrak{p}$ are spheres and the problem is to find necessary and sufficient conditions to construct closed $n$-gons with given side-lengths $r_1, \ldots, r_n$ in $\mathfrak{p}$. These are just the usual triangle inequalities

$$r_i \leq r_1 + \cdots + r_n.$$  \hfill (21)
This is reflected in the simplicity of the Schubert calculus: The chambers in $\partial_{\text{Tits}} X$ are now points, and the only Schubert cycles are points and the whole sphere $\partial_{\infty} X$. $n$ Schubert cycles intersect transversally in one point if and only if one of the cycles is a point and the remaining ones are spheres. (21) results if the point is in the $i$-th position. See the introduction for a more detailed discussion.

4.2 Stability inequalities for measures on Grassmannians

We conclude with an analysis of stability and (asymptotic) semistability for the special case where the $h^3$'s are maximally singular. Although the semistability inequalities follow from Theorem 3.10, we will deduce the required inequalities directly.

4.2.1 Grassmannians

Our first goal is to compute the slope in the case when $G = SL_n(\mathbb{C})$ and the measure $\mu$ is supported on an orbit of vertices $G_\eta$ in $\partial_{\infty} X$. Such an orbit is identified with the Grassmannian $G_q(\mathbb{C}^n)$ of $q$-planes for some $q$, $1 \leq q \leq n - 1$. We introduce the auxiliary function

$$\dim_U(V) := \frac{\dim(U \cap V)}{\dim(U)}$$

where $U, V \subset \mathbb{C}^n$ are non-trivial linear subspaces. These subspaces can be viewed as vertices in $\partial_{\text{Tits}} X$ and their Tits distance is given by

$$\cos \angle_{\text{Tits}}([U], [V]) = (\dim_U(V) - \frac{1}{n} \dim(V)) \cdot f(\dim(U), \dim(V), n) \tag{22}$$

where $f$ is some real-valued function in 3 variables which we don’t care about.

Proof of (22): Let $p = \dim(U)$, $q = \dim(V)$, $s = \dim(U \cap V)$ and choose a basis $e_1 \ldots e_n$ of $\mathbb{C}^n$ so that $e_1 \ldots e_p$ is a basis of $U$ and $e_{p+1} \ldots e_{p+q-s}$ is a basis of $V$. Then the splitting $\mathbb{C}^n = \{e_1\} \oplus \cdots \oplus \{e_n\}$ determines a maximal flat $F$ in $X$ (of dimension $n - 1$). The vertices of the apartment $\partial_{\infty} F \subset \partial_{\text{Tits}} X$ correspond to the non-trivial linear subspaces $L \subset \mathbb{C}^n$ spanned by some of the $e_i$. Let us denote by $\hat{e}_i$ the unit vector field in $F$ pointing towards $[e_i] \in \partial_{\infty} F$. Then, since $\hat{e}_1 \cdots \hat{e}_n = 0$, we find by symmetry that $\hat{e}_i \cdot \hat{e}_j = -\frac{1}{n-1}$ for $i \neq j$. Note that the vector field $\hat{e}_1 + \cdots + \hat{e}_k$ points towards the vertex $\{e_{i_1}, \ldots, e_{i_k}\} \in \partial_{\infty} F$. It follows that the Tits distance between the vertices $[U]$ and $[V]$ is given by the angle between the vector fields $\hat{e}_1 + \cdots + \hat{e}_p$ and $\hat{e}_{p+1} + \cdots + \hat{e}_{p+q-s}$ whose cosine equals

$$pq \frac{-1}{n-1} + s \frac{n}{n-1}$$

whence (22). \hfill \Box

Now, if $\mu$ is supported on the Grassmannian of $q$-planes $G_q(\mathbb{C}^n) \subset \partial_{\text{Tits}} X$ and $\xi_U \in \partial_{\text{Tits}} X$ is the vertex corresponding to a subspace $U \subset \mathbb{C}^n$ then (10) takes the form

$$\text{slope}_\mu(\xi_U) = - \int_{G_q(\mathbb{C}^n)} \cos \angle_{\text{Tits}}([U], [V]) d\mu([V]).$$
Hence \( \text{slope}_\mu(\xi_U) > 0 \) (resp. \( \text{slope}_\mu(\xi_U) \geq 0 \)) if and only if
\[
\int_{G_q(\mathbb{C}^n)} \dim_U(V) < \frac{q}{n} ||\mu|| \tag{23}
\]
respectively
\[
\int_{G_q(\mathbb{C}^n)} \dim_U(V) \leq \frac{q}{n} ||\mu||. \tag{24}
\]
We conclude:

**Proposition 4.3** A measure \( \mu \) supported on the Grassmannian \( G_q(\mathbb{C}^n) \subset \partial_{\text{Tits}} X \) is stable (resp. asymptotically semistable) if and only if (23) (resp. (24)) holds for all non-trivial linear subspaces \( U \subset \mathbb{C}^n \).

We now compute the analogue of the previous stability criterion for Grassmannians for the symmetric spaces of the other families of classical groups \( SO(n, \mathbb{C}) \) and \( Sp(2n, \mathbb{C}) \). To streamline the notation we note that in either case the group is the group of orientation preserving isometries of a non-degenerate bilinear form \( b \) on a complex vector space \( V \). The orbits of vertices in \( \partial_{\text{Tits}} X \) for the above groups are then the *isotropic* Grassmannians \( G_q^o(V) \subset G_q(V) \) consisting of the \( q \)-dimensional subspaces \( U \) on which \( b \) restricts to zero except for the case \( G = SO(2n, \mathbb{C}) \) and \( q = n \). To obtain the vertex orbits in this case we have to replace the full Grassmannians of isotropic \( n \)-planes by the proper subspaces described in 4.1.4. In all cases these vertex orbits are compact symmetric spaces associated to the classical groups but they do not exhaust the set of such symmetric spaces. This will not be important for the theorem below since the full Grassmannian of isotropic \( n \)-planes is still a subset of \( \partial_{\text{Tits}} X \).

**Theorem 4.4** Let \( G_q^o(V) \) be the Grassmannian \( M \) of all \( q \)-dimensional \( b \)-isotropic subspaces in \( V \) for the classical complex semisimple Lie group \( G = SL(V), SO(V, b) \) or \( Sp(V, b) \), (we make the convention \( b \equiv 0 \) in the case of \( SL(V) \).) Then a finite measure on \( M \) is stable (resp. asymptotically semistable) if and only if for every totally-isotropic subspace \( U \subset V \)
\[
\int_{G_q^o(V)} \dim_U d\mu < \frac{q}{\dim(V)} ||\mu||. \tag{25}
\]
respectively
\[
\int_{G_q^o(V)} \dim_U d\mu \leq \frac{q}{\dim(V)} ||\mu||. \tag{26}
\]

**Proof.** By restriction using Lemma 4.5 below.

**Lemma 4.5** Assume that \( Y \subset X \) is a closed convex subset of the Hadamard space \( X \), and that the measure \( \mu \) is supported on \( \partial_{\infty} Y \subset \partial_{\infty} X \). Then \( b_\mu \) is bounded below on \( Y \) if and only if it is bounded below on \( X \).

**Proof.** \( b_\mu \geq b_\mu \circ \pi_Y \) where \( \pi_Y : X \rightarrow Y \) denotes the nearest point projection.

**Remark 4.6** The theorem remains true with the same proof in the exceptional case of \( G = SO(2n, \mathbb{C}) \) and \( q = n \) even though \( G_n^o(V) \) is the union of two orbits of vertices in \( \partial_{\text{Tits}} X \).
4.2.2 Existence of stable configurations of points on $\mathbb{P}^m$

We will now specialise the above formula to the case in which $\mu$ is the positive measure

$$\mu = \sum_{i=1}^{n} r_i \delta_{x_i}$$

on real or complex projective space $\mathbb{P}^m$. We assume that the points $x_1, \ldots, x_n \in \mathbb{P}^m$ are in general position. As a consequence of Theorem 4.4 we have now

**Theorem 4.7** $b_{\mu}$ is stable (resp. semistable) if and only if $r = (r_1, \ldots, r_n)$ satisfies the strong triangle inequalities

$$mr_i < r_1 + \ldots + r_n, \quad 1 \leq i \leq n$$

respectively

$$mr_i \leq r_1 + \ldots + r_n, \quad 1 \leq i \leq n.$$ 

**Proof.** First some notation. For $I \subseteq \{1, \ldots, n\}$ let $r_I$ denote the sum $\sum_{i \in I} r_i$. Let $r = \sum_{i=1}^{n} r_i$. It is then an immediate consequence of Theorem 4.4 that $b_{\mu}$ is stable if and only if

$$r_I < \frac{|I|}{m+1}r, \quad \forall I.$$ 

Clearly the above inequalities are equivalent to the system

$$r_i < \frac{1}{m+1}r, \quad 1 \leq i \leq n.$$ 

This last system is equivalent to the strong triangle inequalities. \qed

**References**


Bernhard Leeb, Mathematisches Institut, Universität Tübingen Auf der Morgenstelle 10, D-72076 Tübingen, Germany bernhard@riemann.mathematik.uni-tuebingen.de

John Millson, Department of Mathematics, University of Maryland, College Park, MD 20742, USA, jjm@math.umd.edu