Fixed point results for multimaps in CAT(0) spaces

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Common fixed point results for families of single-valued nonexpansive or quasi-nonexpansive mappings and multivalued upper semicontinuous, almost lower semicontinuous or nonexpansive mappings are proved either in CAT(0) spaces or \( \mathbb{R} \)-trees. It is also shown that the fixed point set of quasi-nonexpansive self-mapping of a nonempty closed convex subset of a CAT(0) space is always nonempty closed and convex.

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1. Introduction

The study of metric spaces without linear structure has played a vital role in various branches of pure and applied sciences. One of such space is a CAT(0) space. A useful example of it is an \( \mathbb{R} \)-tree (see, e.g., [7,21]), whose study found applications in mathematics, biology/medicine and computer science (see, e.g., [3,12,19]). Fixed point results in CAT(0) spaces have been proved by a number of authors, see, e.g., [1,2,6,9,11–18,20].

In 2005, Dhompongsa, Kaewkhao, and Panyanak [6] proved the following fixed point result for commuting mappings.

**Theorem DKP.** Let \( X \) be a nonempty closed bounded convex subset of a complete CAT(0) space \( M \), \( f \) a nonexpansive self-mapping of \( X \) and \( T : X \to 2^X \) is nonexpansive, where for any \( x \in X \), \( Tx \) is nonempty compact convex. Assume that for some \( p \in \text{Fix}(f) \)

\[
\alpha p \oplus (1 - \alpha)Tx \text{ is convex for all } x \in X \text{ and } \alpha \in [0, 1].
\]

If \( f \) and \( T \) commute, then there exists an element \( z \in X \) such that \( z = f(z) \in T(z). \)

Recently, Shahzad and Markin [20] extended and improved Theorem DKP. On the other hand, Espinola and Kirk [9] established that a commutative family of nonexpansive self-mappings of a geodesically bounded closed convex subset of a complete \( \mathbb{R} \)-tree has a nonempty common fixed point set.
In this paper, we prove some fixed point results either in CAT(0) spaces or $\mathbb{R}$-trees for families of single-valued nonexpansive or quasi-nonexpansive mappings and multivalued upper semicontinuous, almost lower semicontinuous or nonexpansive mappings which are weakly commuting. We also establish a result which implies that the fixed point set of quasi-nonexpansive self-mapping of a nonempty closed convex subset of a CAT(0) space is always nonempty closed and convex.

2. Preliminaries

For any pair of points $x, y$ in a metric space $(M, d)$, a geodesic path joining these points is a map $c$ from a closed interval $[0, r] \subseteq \mathbb{R}$ to $M$ such that $c(0) = x$, $c(r) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, r]$. The mapping $c$ is an isometry and $d(x, y) = r$. The image of $c$ is called a geodesic segment joining $x$ and $y$ which when unique is denoted by $[x, y]$. For any $x, y \in M$, denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$, where $0 \leq \alpha \leq 1$. The space $(M, d)$ is called a geodesic space if any two points of $M$ are joined by a geodesic, and $M$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in M$. A subset $X$ of $M$ is called convex if $X$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(M, d)$ consists of three points in $M$ (the vertices of $\Delta$) and a geodesic segment between each pair of points (the edges of $\Delta$). A comparison triangle for $\Delta(x_1, x_2, x_3)$ in $(M, d)$ is a triangle $\Delta' \Delta(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space $M$ is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom [4]:

Let $\Delta$ be a geodesic triangle in $M$ and let $\Delta$ in $\mathbb{R}^2$ be its comparison triangle. Then $\Delta$ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y}$ in $\Delta$, $d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$.

A subset $X$ of $(M, d)$ is said to be gated [8] if for any point $x \notin X$ there exists a unique $z \in X$ such that for any $y \in X$,

$$d(x, y) = d(x, z) + d(z, y).$$

The point $z$ is called the gate of $x$ in $X$.

It is known that gated sets in a complete geodesic space are always closed and convex, and gated subsets of a complete geodesic space $(M, d)$ are proximinal nonexpansive retracts of $M$. It is also known the family of gated sets in a complete geodesic space $(M, d)$ has the Helly property, that is, if $X_1, \ldots, X_n$ is a collection of gated sets in $M$ with pairwise nonempty intersection, then $\bigcap_{i=1}^n X_i \neq \emptyset$.

There are many equivalent definitions of $\mathbb{R}$-tree. Here we include the following definition.

An $\mathbb{R}$-tree is a metric space $M$ such that:

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in M$;

(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

It follows from (i) and (ii) that

(iii) if $u, v, w \in M$, then $[u, v] \cap [u, w] = [u, z]$ for some $z \in M$.

Examples of $\mathbb{R}$-trees can be found in [12].

The following properties of $\mathbb{R}$-tree are useful [4,9,12].

(i) An $\mathbb{R}$-tree is a CAT(0) space.

(ii) The metric $d$ in an $\mathbb{R}$-tree is convex, that is, it satisfies the inequality

$$d(ax \oplus (1 - \alpha)y, au \oplus (1 - \alpha)v) \leq \alpha d(x, u) + (1 - \alpha)d(y, v)$$

for any points $x, y, u, v \in M$.

(iii) A metric space is a complete $\mathbb{R}$-tree if and only if it is hyperconvex and has unique geodesic segments.

For any subset $X$ in a metric space $M$, we define $d(x, X) = \inf_{y \in X} d(x, y)$. The mappings $f : X \to X$ and $T : X \to 2^X$ are said to commute if $f(T(x)) \subseteq T(f(x))$ for all $x \in X$. $f$ and $T$ are said to commute weakly [10] if $f(\partial_X T(x)) \subseteq T(f(x))$ for all $x \in X$, where $\partial_X Y$ denotes the relative boundary of $Y \subseteq X$ with respect to $X$. Let $\mathcal{F}$ be a family of self-mappings of $X$. Then $\mathcal{F}$ and $T$ are said to commute weakly (resp. commute) if each $f \in \mathcal{F}$ and $T$ commute weakly (resp. commute). We denote by $Fix(\mathcal{F})$ the sets of common fixed points of $\mathcal{F}$ in $X$. A mapping $f : X \to M$ is called a contraction if for any $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$ for some $k \in (0, 1)$. $f$ is said to be nonexpansive with respect to $Y$ (a nonempty subset of $X$) if for any $x \in X, y \in Y$, $d(f(x), f(y)) \leq d(x, y)$. If $Y = X$, $f$ is called nonexpansive and if $Y = Fix(f)$, $f$ is called quasi-nonexpansive.

Denote the nonempty subsets of a metric space $M$ by $2^M$. In a metric space $M$, a mapping $T : X \to 2^M$ with closed bounded values is called nonexpansive if $H(T(x), T(y)) \leq d(x, y)$ for any pair $x, y \in X$, where $H$ denotes the Hausdorff
metric derived from the metric $d$. The mapping $T$ is said to be upper semicontinuous if for any $\varepsilon > 0$ and any sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n = x$, there exists a $\delta > 0$ such that

$$d(x_0, x) < \delta \quad \text{implies} \quad T(x_0) \subset N_\varepsilon(T(x)),$$

where $N_\varepsilon(T(x)) = \{ y \in M : d(y, T(x)) \leq \varepsilon \}$. $T$ is called almost lower semicontinuous if given $\varepsilon > 0$, for each $x \in X$ there is a neighborhood $U(x)$ of $x$ such that $\bigcap_{y \in U(x)} N_\varepsilon(T(y)) \neq \emptyset$. Any mapping that is lower semicontinuous or quasi-lower semicontinuous is almost lower semicontinuous (see, e.g., [17]).

3. Main results

The following result is immediate.

**Theorem 3.1.** Let $X$ be a nonempty closed convex subset of a complete $\text{CAT}(0)$ space, $f$ a mapping of $X$ into $M$ such that $A = \{ x \in X : d(f(x), x) = d(f(x), X) \}$ is nonempty and $f$ is nonexpansive with respect to $A$. Then $A$ is a closed set on which $f$ is continuous. In addition, suppose that $f$ is isometric on $A$. If for $u, v \in A$, $x \in [u, v]$, then $f(x) \in [f(u), f(v)]$.

If $f(X) \subset X$, then $\text{Fix}(f) = A$ and so we have the following result of Chaoha and Phon-on [5] as a corollary.

**Corollary 3.2.** Let $X$ be a closed convex subset of a complete $\text{CAT}(0)$ space, $f$ a quasi-nonexpansive self-mapping of $X$. Then $A = \text{Fix}(f)$ is a nonempty closed convex set on which $f$ is continuous.

The following result extends and improves Theorem DKP. It basically shows that the assumption that the mapping $f$ is nonexpansive in Theorem DKP of [6] can be replaced by the assumption that $f$ is only quasi-nonexpansive.

**Theorem 3.3.** Let $X$ be a nonempty closed bounded convex subset of a complete $\text{CAT}(0)$ space $M$, $f$ a quasi-nonexpansive self-mapping of $X$, and $T : X \to 2^X$ a nonexpansive mapping, where for any $x \in X$, $Tx$ is nonempty compact convex. If $f$ and $T$ commute weakly, then there exists an element $z \in X$ such that $z = f(z) \in T(z)$.

**Proof.** By Corollary 3.2, $\text{Fix}(f)$ is nonempty closed convex. Let $x \in \text{Fix}(f)$. Then $f(\partial_X T(x)) \subset T(f(x)) = T(x)$. Let $u \in \partial_X T(x)$ be a unique closest point to $x$. Since $f$ is nonexpansive with respect to $\text{Fix}(f)$, we have $d(f(u), x) \leq d(u, x)$ and so $f(u) = u$ by uniqueness of the closest point $u$. Thus

$$T(x) \cap \text{Fix}(f) \neq \emptyset.$$

Let $F(x) = T(x) \cap \text{Fix}(f)$. Then $F$ is a mapping of $\text{Fix}(f)$ into $2^{\text{Fix}(f)}$. Notice that for any $x, y \in \text{Fix}(f)$,

$$H(F(x), F(y)) = \max \left\{ \sup_{u \in F(x)} d(u, F(y)), \sup_{v \in F(y)} d(v, F(x)) \right\} \leq \max \left\{ \sup_{u \in F(x)} d(u, T(y)), \sup_{v \in F(y)} d(v, T(x)) \right\} \leq d(T(x), T(y)) \leq d(x, y).$$

Now Corollary 3.5 of Dhompongsaa, Kaewkhao and Panyanak [6] guarantees the existence of $z \in \text{Fix}(f)$ such that $z \in F(z)$. As a result, we have $f(z) = z \in T(z)$. $\square$

**Corollary 3.4.** Let $X$ be a nonempty closed bounded convex subset of a complete $\text{CAT}(0)$ space $M$, $f$ a nonexpansive self-mapping of $X$ and $T : X \to 2^X$ a nonexpansive mapping, where for any $x \in X$, $Tx$ is nonempty compact convex. If $f$ and $T$ commute weakly, then there exists an element $z \in X$ such that $z = f(z) \in T(z)$.

**Proof.** By Theorem 12 of Kirk [14], $\text{Fix}(f)$ is nonempty closed bounded convex. So the result follows from Theorem 3.3. $\square$

**Theorem 3.5.** Let $X$ be a nonempty closed bounded convex subset of a complete $\text{CAT}(0)$ space $M$ and $f$ a nonexpansive self-mapping of $X$. Then for any closed convex subset $Y$ of $X$ such that $f(\partial_Y X) \subset Y$, we have $P_{\text{Fix}(f)}(Y) \subset Y$.

**Proof.** Fix $u \in Y$, and define the mapping $f_t : Y \to X$ by taking $f_t(x)$ to be the point of $[u, f(x)]$ at distance $t d(u, f(x))$ from $u$. Then by convexity of the metric

$$d(f_t(x), f_t(y)) \leq td(x, y)$$

for any $x, y \in Y$. Since $f_t$ is a nonexpansive mapping for each $t$, $f_t(Y)$ is a nonempty closed convex subset of $Y$. Moreover, $f_t(Y)$ is also nonempty closed convex for each $t$. Thus $f_t(Y)$ is nonempty closed convex for each $t$.

Now Corollary 3.5 of Dhompongsaa, Kaewkhao and Panyanak [6] guarantees the existence of $z \in f_t(Y)$ such that $z \in f_t(z)$. As a result, we have $f_t(z) = z \in T(z)$. $\square$
for all \( x, y \in Y \). This shows that \( f_1 : Y \to X \) is a contraction. Let \( P_Y \) be the proximinal nonexpansive retraction of \( X \) into \( Y \). Then \( P_Y f_1 \) is a contraction self-mapping of \( Y \). By the Banach Contraction Principle, there exists a unique fixed point \( y_1 \in Y \) of \( P_Y f_1 \). Thus

\[
    d(f_1 y_1, y_1) = \inf \{d(f_1 y_1, z) : z \in Y \}.
\]

Since \( f(\partial_X Y) \subset Y \), we have \( f_1(\partial_X Y) \subset Y \) and so we have \( f_1(y_1) = y_1 \in [u, f(y_1)] \). Note that \( A = \text{Fix}(f) \) is nonempty closed bounded convex by Theorem 12 of Kirk [14]. Now Theorem 26 of Kirk [15] guarantees that \( \lim_{t \to 1^-} y_t \) converges to the unique fixed point of \( f \) which is nearest \( u \). As a result, \( \lim_{t \to 1^-} y_t = P_{\mathcal{A}}(u) \in Y \). Since \( M \) is a CAT(0) space, \( P_{\mathcal{A}} \) is nonexpansive and \( P_{\mathcal{A}}(Y) \subset Y \). □

**Remark 3.6.** Let \( X \) be a nonempty closed bounded convex subset of a complete CAT(0) space \( M \), \( f : X \to M \) a nonexpansive mapping. Then there exists an element \( z \in X \) such that

\[
    d(f(z), z) = d(f(z), X).
\]

To see this, let \( P_X \) be the proximinal nonexpansive retraction of \( M \) into \( X \). Then \( P_X f \) is a nonexpansive self-mapping of \( X \) and so has a fixed point \( z \). Hence

\[
    d(f(z), z) = d(f(z), X).
\]

The following result also follows from Theorem 3.3 but the proof given here is constructive one.

**Theorem 3.7.** Let \( X \) be a nonempty closed bounded convex subset of a complete CAT(0) space \( M \), \( f : X \to M \) a nonexpansive mapping, and \( T : X \to \mathcal{P}(X) \) a nonexpansive mapping, where for any \( x \in X \), \( Tx \) is nonempty compact convex. If for each \( x \in X \), \( P_X f(\partial_X T(x)) \subset T(P_X f(x)) \), where \( P_X \) is the proximinal nonexpansive retraction of \( M \) into \( X \), then there exists an element \( z \in X \) with \( z \in T(z) \) such that

\[
    d(f(z), z) = d(f(z), X).
\]

**Proof.** Clearly

\[
    A = \{x \in X : d(f(x), x) = d(f(x), X)\}
\]

is nonempty by Remark 3.6 and \( A = \text{Fix}(P_X f) \). Define \( F : X \to 2^X \) by \( F(x) = T(P_A(x)) \). Then \( F \) is nonexpansive and has a fixed point \( v \in X \) by Corollary 3.5 of Dhompansua, Kaewkhao and Panyanak [6]. Notice that

\[
    P_X f(\partial_X F(v)) = P_X f(\partial_X T(P_A(v))) \subset T(P_X f(P_A v)) = T(P_A(v)) = F(v).
\]

Also \( A = \text{Fix}(P_X f) \). Now Theorem 3.5 guarantees that \( P_A F(v) \subset F(v) \). In particular, \( P_A v \in F(v) \). Let \( z = P_A v \). Then \( P_X f(z) = z \in T(z) \) and

\[
    d(f(z), z) = d(f(z), X). \quad \Box
\]

**Theorem 3.8.** Let \( X \) be a nonempty geodesically bounded closed convex subset of a complete \( \mathbb{R} \)-tree \( M \), \( \mathcal{F} \) a family of self-mappings of \( X \) for which \( \text{Fix}(\mathcal{F}) \) is nonempty and each \( f \in \mathcal{F} \) is nonexpansive with respect to \( \text{Fix}(\mathcal{F}) \), and \( T : X \to 2^X \) upper semicontinuous, where for any \( x \in X \), \( Tx \) is nonempty closed and convex. If \( \mathcal{F} \) and \( T \) commute weakly, then there exists an element \( z \in X \) such that \( z = f(z) \in T(z) \) for all \( f \in \mathcal{F} \).

**Proof.** By Corollary 3.2, \( \text{Fix}(\mathcal{F}) \) is nonempty closed convex. Let \( x \in \text{Fix}(\mathcal{F}) \). Then for any \( f \in \mathcal{F} \), \( f(\partial_X T(x)) \subset T(f(x)) \). Let \( u = \partial_X T(x) \) be a unique closest point to \( x \). Since \( f \) is nonexpansive with respect to \( \text{Fix}(\mathcal{F}) \), we have \( d(f(u), x) \leq d(u, x) \) and so \( f(u) = u \). Thus \( u \) is a common fixed point of \( \mathcal{F} \), which implies

\[
    T(x) \cap \text{Fix}(\mathcal{F}) \neq \emptyset.
\]

Let \( F(x) = T(x) \cap \text{Fix}(\mathcal{F}) \). Then \( F \) is an upper semicontinuous mapping of \( \text{Fix}(\mathcal{F}) \) into \( 2^{\text{Fix}(\mathcal{F})} \). Now Theorem 2.1 of Kirk and Panyanak [11] guarantees the existence of \( z \in \text{Fix}(\mathcal{F}) \) such that \( z \in F(z) \). As a result, we have \( f(z) = z \in T(z) \) for all \( f \in \mathcal{F} \). □

**Corollary 3.9.** Let \( X \) be a nonempty geodesically bounded closed convex subset of a complete \( \mathbb{R} \)-tree \( M \), \( \mathcal{F} \) a commuting family of nonexpansive self-mappings of \( X \), and \( T : X \to 2^X \) upper semicontinuous, where for any \( x \in X \), \( Tx \) is nonempty closed and convex. If \( \mathcal{F} \) and \( T \) commute weakly, then there exists an element \( z \in X \) such that \( z = f(z) \in T(z) \) for all \( f \in \mathcal{F} \).

**Proof.** By Theorem 4.3 of Espinola and Kirk [9], \( \text{Fix}(\mathcal{F}) \) is nonempty. So the result follows from Theorem 3.8. □
Theorem 3.10. Let $X$ be a nonempty geodesically bounded closed convex subset of a complete $\mathbb{R}$-tree $M$, $\mathcal{F}$ a family of self-mappings of $X$ for which $\text{Fix}(\mathcal{F})$ is nonempty and each $f \in \mathcal{F}$ is nonexpansive with respect to $\text{Fix}(\mathcal{F})$, and $T : X \to 2^X$ almost lower semicontinuous, where for any $x \in X$, $Tx$ is nonempty closed bounded and convex. If $\mathcal{F}$ and $T$ commute weakly, then there exists an element $z \in X$ such that $z = f(z) \in T(z)$ for all $f \in \mathcal{F}$.

Proof. By Corollary 3.2, $\text{Fix}(\mathcal{F})$ is nonempty closed convex. Let $x \in \text{Fix}(\mathcal{F})$. Then for any $f \in \mathcal{F}$, $f(\partial_x T(x)) \subset T(f(x)) = T(x)$. Let $u \in \partial_x T(x)$ be a unique closest point to $x$. Since $f$ is nonexpansive with respect to $\text{Fix}(\mathcal{F})$, we have $d(f(u), x) \leq d(u, x)$ and so $f(u) = u$. Thus $u$ is a common fixed point of $\mathcal{F}$, which implies $T(x) \cap \text{Fix}(\mathcal{F}) \neq \emptyset$.

Let $F = T(x) \cap \text{Fix}(\mathcal{F})$. We claim that $F$ is an almost lower semicontinuous mapping of $\text{Fix}(\mathcal{F})$ into $2^{\text{Fix}(\mathcal{F})}$. Since $T : \text{Fix}(\mathcal{F}) \to 2^X$ is almost lower semicontinuous, for any $x \in \text{Fix}(\mathcal{F})$ and $\epsilon > 0$ there exist a neighborhood $U(x)$ of $x$ and a point $w \in x$ such that $T(y) \cap B(w, \epsilon) \neq \emptyset$ for $y \in U(x)$. Since $w \in N_\epsilon(T(y))$ and $N_\epsilon(T(y)) \cap \text{Fix}(\mathcal{F}) \neq \emptyset$, it follows from Lemma 4.1 of Markin [17] that $P_{\text{Fix}(\mathcal{F})}(w) \in N_\epsilon(T(y)) \cap \text{Fix}(\mathcal{F})$ for $y \in U(x)$. Thus, $B(P_{\text{Fix}(\mathcal{F})}(w), \epsilon) \cap T(y) \cap \text{Fix}(\mathcal{F}) \neq \emptyset$ for $y \in U(x)$. This proves our claim. Now Theorem 4.4 of Markin [17] guarantees the existence of $z \in \text{Fix}(\mathcal{F})$ such that $z = f(z) \in T(z)$ for all $f \in \mathcal{F}$. □

Corollary 3.11. Let $X$ be a nonempty geodesically bounded closed convex subset of a complete $\mathbb{R}$-tree $M$, $\mathcal{F}$ a commuting family of nonexpansive self-mappings of $X$, and $T : X \to 2^X$ almost lower semicontinuous, where for any $x \in X$, $Tx$ is nonempty closed bounded and convex. If $\mathcal{F}$ and $T$ commute weakly, then there exists an element $z \in X$ such that $z = f(z) \in T(z)$ for all $f \in \mathcal{F}$.

Proof. By Theorem 4.3 of Espinola and Kirk [9], $\text{Fix}(\mathcal{F})$ is nonempty closed convex and geodesically bounded. So the result follows from Theorem 3.10. □

References