Conjectured Bessel Moment Integrals

Problem 08-003, by David H. Bailey\textsuperscript{1} (Lawrence Berkeley National Laboratory) and Jonathan Borwein\textsuperscript{2} (Dalhousie University, Halifax, NS, Canada).

1. Background. A recent paper by the present authors, together with mathematical physicists David Broadhurst and M. Larry Glasser, explored Bessel moment integrals, namely definite integrals of the general form \( \int_0^\infty t^n f^n(t) \, dt \), where the function \( f(t) \) is one of the classical Bessel functions \([2]\). In that paper, numerous previously unknown analytic evaluations were obtained, using a combination of analytic methods together with some fairly high-powered numerical computations, often performed on highly parallel computers.

In several instances, while we were able to numerically discover what appears to be a solid analytic identity, based on extremely high-precision numerical computations, we were unable to find a rigorous proof. Thus we present here a brief list of some of these unproven but numerically confirmed identities. In the following, the functions \( I_0(t) \) and \( K_0(t) \) are the classical Bessel functions, as defined in \([1, \text{ Chap. 15}]\), while the function \( K(x) \) is the complete elliptic integral of the first kind, namely

\[
K(x) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - x^2 \sin^2 \phi}}.
\]

These formulas also employ constants \( K_3 := K(k_3), \ K'_3 = \sqrt{3}K_3, \ K_{15} := K(k_{15}), \ K_{5/3} = K(k_{5/3}), \) and \( C \), where

\[
k_3 = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \sin\left(\frac{\pi}{12}\right),
\]

\[
k_{15} = \frac{(2 - \sqrt{3})(\sqrt{5} - \sqrt{3})(3 - \sqrt{5})}{8\sqrt{2}},
\]

\[
k_{5/3} = \frac{(2 - \sqrt{3})(\sqrt{5} + \sqrt{3})(3 + \sqrt{5})}{8\sqrt{2}},
\]

\[
C := \frac{\pi}{16} \left(1 - \frac{1}{\sqrt{5}}\right) \left(1 + 2 \sum_{n=1}^{\infty} \exp(-n^2\pi \sqrt{15})\right)^4.
\]

Alternatively

\[
C = \frac{\sqrt{5} - 1}{4\sqrt{5}\pi} K_{15}^2 = \frac{1}{2\sqrt{15}\pi} K_{15} K_{5/3}.
\]

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2. Conjectured Identities. Here are our selected conjectures. Can you find proofs for any (or all!) of these?

\[
\begin{align*}
(1) & \quad \frac{1}{\pi^2} \int_0^\infty t I_0(t) K_0^4(t) \; dt \equiv C,
(2) & \quad \frac{1}{\pi^2} \int_0^\infty t^3 I_0(t) K_0^4(t) \; dt \equiv \left( \frac{2}{15} \right)^2 \left( 13C - \frac{1}{10C} \right), \\
(3) & \quad \frac{1}{\pi^2} \int_0^\infty t^5 I_0(t) K_0^4(t) \; dt \equiv \left( \frac{4}{15} \right)^3 \left( 43C - \frac{19}{40C} \right), \\
(4) & \quad \frac{2}{\pi \sqrt{15}} \int_0^\infty t I_0^2(t) K_0^3(t) \; dt \equiv C, \\
(5) & \quad \frac{2}{\pi \sqrt{15}} \int_0^\infty t^3 I_0^2(t) K_0^3(t) \; dt \equiv \left( \frac{2}{15} \right)^2 \left( 13C + \frac{1}{10C} \right), \\
(6) & \quad \frac{2}{\pi \sqrt{15}} \int_0^\infty t^5 I_0^2(t) K_0^3(t) \; dt \equiv \left( \frac{4}{15} \right)^3 \left( 43C + \frac{19}{40C} \right), \\
(7) & \quad \int_0^\infty t I_0^2(t) K_0^3(t) K_0(2t) \; dt \equiv \frac{1}{12} K_3 K'_3.
\end{align*}
\]

A number of other related identities that are experimentally discovered but as yet unproven are mentioned in [2]. A discussion of the relative difficulty of each one on our list is discussed in [2].

REFERENCES


Status. This problem is open.