Analytical models of optical refraction in the troposphere

Brett D. Nener, Neville Fowkes, and Laurent Borredon
The University of Western Australia, 35 Stirling Highway, Crawley 6009, Australia

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An extremely accurate but simple asymptotic description (with known error) is obtained for the path of a ray propagating over a curved Earth with radial variations in refractive index. The result is sufficiently simple that analytic solutions for the path can be obtained for linear and quadratic index profiles. As well as rendering the inverse problem trivial for these profiles, this formulation shows that images are uniformly magnified in the vertical direction when viewed through a quadratic refractive-index profile. Nonuniform vertical distortions occur for higher-order refractive-index profiles. © 2003 Optical Society of America

1. INTRODUCTION

The propagation of radiation through the atmosphere is of interest to radar and electro-optic systems, and, theoretically at least, it can be predicted from Maxwell’s equations. The solution of these equations is problematic in real atmospheres, and hence various approximations and techniques are necessary. There is much current interest in the parabolic equation model (see Barrios1) and its application to horizontally inhomogeneous atmospheres and diffraction over terrain. However, the approximation of interest in this paper is the method of geometrical optics (see, for example, Kravtsov and Orlov2) and is applied to horizontally homogeneous but radially inhomogeneous atmospheres. Further, we are concerned here with terrestrial refraction where both observer and source are totally immersed within the atmosphere. We therefore do not consider the astronomical case in which rays enter the atmosphere from space (see, for example, Garfinkel3 and Mahan4). We are also not concerned here with the paraxial propagation of rays over short distances as in the methods used in optical instrument design. The results of this paper are applicable to the analysis of image distortion and mirages of objects on the horizon and the inversion of the vertical atmospheric temperature profiles causing these distortions.

An accurate determination of the path of light rays propagating a few meters above the ocean/land over distances of the order of 10–20 km is required if one wishes to determine the true location and shape of objects on the horizon as seen by an observer above the ocean/land. The refractive-index variations arise because the speed of ray propagation is (weakly) dependent on the temperature, pressure, and humidity of the atmosphere through which the ray is propagating. The refractive-index variations are relatively small $O(10^{-6})$, so that the deviations from a straight-line path are small. Such small variations, however, lead to deflections of the order of meters over the propagation distances of interest and give rise to optical distortions and, in some situations, even inversions, i.e., mirages. Over the distances of interest, the curvature of the Earth cannot be ignored; indeed, a determination of the viewing horizon is of importance. Temporal variations in refractive index due to turbulence result in variations in the image position (shimmering or dancing) and intensity (twinkling or scintillation). Additionally, scattering effects are of importance in context; these effects will be addressed in subsequent papers.

Following a brief discussion of the physical origin of refractive-index variations, we will examine an exact implicit integral equation that determines a ray path in an atmosphere with prescribed radial variations of refractive index; see Section 2. Although concise, this description is not useful; simple; numerical evaluation is required. Scaling arguments are then used to justify an extremely accurate approximation scheme leading to a much simpler description of the ray path; see Section 3. Exact solutions for the approximate equation are then obtained for particular refractive-index profiles (linear and quadratic), the inverse problem is discussed, and the implications in terms of the viewed image are described in Section 4. It is also noteworthy that exact solutions have been obtained for exponential profiles; this research will be presented in a follow-up paper.

A. Refractive-Index Variations in the Atmosphere

Given the smallness of variations in refractive index $n$, it is conventional and convenient to describe such variations in terms of the refractivity $n’$ defined by

$$n = 1 + \delta n’, \quad \delta = 10^{-6};$$

(1)

variations in $n’$ of unit order are to be expected in the atmosphere of interest. The refractivity varies with the temperature, pressure, and humidity of the medium through which the ray is passing and also with the wavelength of the wave (see Edlen5 for the optical case). However, in the boundary-layer atmosphere context, temperature effects dominate, and the optical refractivity

$$n’ = \frac{P}{T}, \quad \gamma \approx 78,$$


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Following a brief discussion of the physical origin of refractive-index variations, we will examine an exact implicit integral equation that determines a ray path in an atmosphere with prescribed radial variations of refractive index; see Section 2. Although concise, this description is not useful; simple; numerical evaluation is required. Scaling arguments are then used to justify an extremely accurate approximation scheme leading to a much simpler description of the ray path; see Section 3. Exact solutions for the approximate equation are then obtained for particular refractive-index profiles (linear and quadratic), the inverse problem is discussed, and the implications in terms of the viewed image are described in Section 4. It is also noteworthy that exact solutions have been obtained for exponential profiles; this research will be presented in a follow-up paper.

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$$n’ = \frac{P}{T}, \quad \gamma \approx 78,$$
where \( P \) is the pressure in millibars and \( T \) is the absolute temperature in Kelvin, provides a description accurate enough for the conditions studied here. In the absence of net radiative input (isothermal atmosphere), pressure variations are hydrostatic, and the refractivity varies (exponentially) from around 274 at the surface (\( P = 1013 \text{ mbar} \) and \( T = 288 \text{ K} \)) to around 93 at an altitude of 10 km (\( P = 265 \text{ mbar} \) and \( T = 223 \text{ K} \); see Fleagle and Bussinger\(^6\)). Much of the radiative input from the Sun is absorbed on the Earth's surface, and buoyancy-generated turbulent eddies cause mixing over a layer of depth approximately 300 m. A typical diurnal cycle starts in the early morning with air temperature increasing with height above the Earth and stable atmosphere, which give rise to refractive-index levels that decrease with height; rays are deflected toward the Earth's surface. Later in the day, it is possible for the temperature on the Earth's surface to exceed that of the atmosphere above it, and the atmosphere is unstable; rays have a tendency to deflect away from the Earth's surface.

The above description is simplistic, with temperature variations determined by local and global air exchanges (ocean to land, for example), which sometimes give rise to thermal layers that can trap light rays and permit propagation well beyond the expected horizon. Close to the ocean's surface, wind-generated eddies (and ocean waves) further complicate the picture, giving rise to relatively large temperature variations (and thus refractivity changes) through a boundary layer typically of thickness 5 m. Although the results we obtain are broadly applicable, we focus our attention on rays propagating within a zone of depth 0–20 m above the ocean, so that such boundary-layer effects are a primary interest.

**B. Previous Research**

Although the basic physics of propagation is well understood (see, for example, Born and Wolf\(^7\)), the practical implications have not been adequately explored. Snell's law \( \sin(\pi/2 - \gamma) = C \), as found in standard textbooks, is modified for spherical symmetry as \( n(R)R \cos \gamma(R) = C \) along a ray, where \( \gamma \) is the angle of the ray with respect to the local horizontal and \( C \) is a constant for the ray. This equation determines the deviation of the propagating ray locally, so that in theory the complete path can be determined by patching together local solutions numerically\(^8\) or by integrating Snell's law. An exact integral is, however, unavailable even for simple profiles. The practical difficulty is that highly accurate evaluations are required for quantitative interpretation of visual observations. Earth curvature effects introduce a further source of numerical inaccuracy. In short, numerical ray tracing and integration calculations run into accuracy problems of both a truncation type and a round-off type. Even if such calculations are performed to the required accuracy, they are unlikely to lead to a practical understanding of the effect of atmospheric variations on the quality of the image. Such difficulties are compounded when one attempts to solve the inverse problem by using forward solutions to home in on the required refractive-index profile.

Early investigations of atmospheric propagation were made by Nölke\(^9\) using a flat Earth model valid for short ranges. He used the locus of the ray turning points or a vertex curve (Scheitelkurven) as a tool to predict mirage events. The use of vertex curves was first published by Tait\(^10,11\) in 1882. Using Nölke's research, Liljequist\(^12\) analyzed some mirage effects observed on an expedition to the Antarctic (1949–1952).

Freehafer\(^13\) used the concept of modified index of refraction \( N \) (originally put forth by Schelleng et al.)\(^14\) to take into account the Earth's curvature effect, reducing the problem again to rectangular coordinates but with extended range of applicability. We can see this relationship if we take the spherical form of Snell's law and rewrite it in the following way:

\[
\begin{align*}
n(z)(a_e + z)\cos \gamma &= C, \\
n(z)(1 + z/a_e)\cos \gamma &= C/a_e, \\
N \cos \gamma &= C',
\end{align*}
\]

where \( z \) is the height above the Earth of radius \( a_e \). The accuracy of this flat-Earth technique was studied by Pekeris\(^15\).

A number of approaches to the solution of the forward problem have been taken; see Lehn and co-workers\(^8,16–19\) White,\(^20\) Fraser,\(^21,22\) Mach and Fraser,\(^23\) Sozou,\(^24\) Rees et al.,\(^25\) and Rees.\(^26\) Of particular note is the extension by Kropla and Lehn\(^27\) of the parabolic ray curvature approach of Lehn,\(^8\) applicable to spherical layers containing constant refractive-index gradients, to the general case to give a geodesic description of the problem. These approaches have the difficulty that the approximations are local and require accurate numerical patching to realize an acceptable output. The present paper determines uniformly valid accurate asymptotic descriptions for ray paths, although admittedly just for selected index profiles.

**2. RAY PATHS**

In the particular case of interest, the refractive index varies only with distance \( R \) from the center of the Earth, and Snell's law invariant for spherical coordinates \( n(R)R \cos \gamma(R) = q_0 \) (a constant for a specific ray) can be used to describe the ray path locally, where \( \gamma \) is the angle of the ray with respect to the local horizontal. In terms of the angular displacement \( \theta \) of the path measured from the center of the Earth away from a convenient datum (chosen here to be the ray source) (see Fig. 1), this can be written in the form

\[
\tan \gamma = \frac{dR(\theta)}{R d\theta} = \pm \frac{R[n^2 - (q_0/R)^2]^{1/2}}{q_0}, \tag{2}
\]

where the sign needs to be chosen appropriately. This can be integrated to give the following well-known implicit result for a ray path \( R(\theta) \),

\[
\int_{R_0}^{R} \frac{q_0 dR}{R^2[n^2(R) - q_0^2/R^2]^{1/2}} = \pm \theta, \tag{3}
\]

where \((R_0,0)\) defines the location of the ray source and the Snell's law parameter

\[
q_0 = n(R_0)R_0 \cos \gamma_0 = [1 + n'(R_0)]R_0 \cos \gamma_0 \tag{4}
\]
[see Eq. (1)] conveniently identifies the initial direction of propagation of the ray; $\gamma_0$ is the launch angle (see Fig. 1), and $n'$ is the refractivity.

It should be noted that a prescribed ray, identified by $(R_0, q_0)$, cannot propagate into a region defined by $[n(R) - q_0/R] < 0$; see Eq. (2). A ray propagating toward such a region will normally bend back on itself at the boundary $R = R_s$ of this domain defined by

$$\frac{dR(R_s)}{d\theta} = 0,$$

i.e.,

$$n(R_s) = q_0/R_s.$$  

An explicit evaluation of the integral in Eq. (3) is not possible except in the constant refractive index $n = 1$ case; this yields the expected straight-line solution $R^1(\theta)$ given by

$$q_0/R^1 = \cos(\gamma_0 \pm \theta),$$

which provides a useful first estimate for the ray path.

Although exact, the above ray path description (3) is not suitable for evaluation because of large and small variables and parameters sprinkled throughout the expression.

**A. Scaling**

Typically, we are interested in waves propagating over distances $L \approx 10$ km, at heights $H = R_0 - a_e \approx 10$ m above the Earth; a sphere of radius $a_e \approx 6 \times 10^7$ km; and, as indicated earlier, within an atmosphere with refractive-index variations of the order of $10^{-6}$. With this in mind, we introduce scaled coordinates $(x', z')$: $x'$ is the distance from the source location measured around the surface of the Earth and scaled so that $x' = 1$ is the location of the target, and $z'$ is the height above the surface, scaled so that $(0, 1)$ is the location of the source (see Fig. 2). To do this, we write

$$R = a_e[1 + (hl)x'], \quad \theta = lx', \quad n = 1 + \delta n'(z'),$$

where

$$l = \frac{L}{a_e} \mathcal{O}(10^{-3}), \quad h = \frac{H}{L} \mathcal{O}(10^{-3}), \quad \delta = 10^{-6}$$

are the dimensionless groups of the problem.

The trajectories of interest reach the target of typical height $H$ at distance $L$, so the initial launch angle is typically of order $(H/L) = h$. We therefore scale $\gamma_0$ so that

$$\gamma_0 = h \gamma'_0$$

and use Eq. (4) to obtain

$$\frac{q_0}{R_0} = 1 - \frac{h^2 \gamma_0^2}{2} + \delta n'(1) + \mathcal{O}(h^2 \delta, h^4),$$

thus enabling us to eliminate $q_0/R_0$ in favor of the scaled launch angle $\gamma'_0$ in path description (3). After changing to scaled variables, utilizing Eq. (9), and expanding in terms of the small parameters $(h, l, \delta)$, we find that ray path equation (3) reduces to

$$x' = \text{sgn}(\gamma'_0) \int_1^{x'} \left[ \frac{2 \eta[n'(u) - n'(1)] + 2 \kappa(u - 1) + (\gamma'_0)^2}{2} \right]^{1/2} du + \mathcal{O}(hl, \delta),$$

where

$$\eta = \delta/h^2, \quad \kappa = l/h$$

and where the choice of sign needs to be made to match the launch condition at the source. This result for the
ray path represents a very significant advance on the earlier result [Eq. (3)] for both analytic and numeric research and will be used as a basis for all that follows. The following should be noted:

- Both \( hl \) and \( \delta \) are of the order of \( 10^{-6} \), so the first-order description (obtained by ignoring the \( hl, \delta \) terms) is extremely accurate. Ignoring this term results in an error of the order of \( 10^{-5} \) m in the predicted ray height at all locations along the path and, in particular, at the target end \( x' = 1 \). Higher-order estimates are possible [a perturbation expansion with error \( O(\delta^3) \) can be set up by use of an algebraic package] but are hardly appropriate in the present context.

- For circumstances of interest the parameters \( \eta \) and \( \gamma \) are of unit order, so, unlike the earlier ray description (3), all the parameters are of unit order with variables ranging over unit order in the present description. Standard numerical techniques (Simpson’s or better) can thus be used to numerically evaluate the path with high precision with few intervals. Thus, even in the absence of analytic results, path description (10) represents a significant improvement on the exact description (3), being effectively as accurate but much more computable.

- The three dimensionless groups \( \eta, \kappa, \) and \( \gamma_0 \) determine the relative importance of refractive-index variations, the Earth’s curvature, and the launching angle on the ray’s path. It is noteworthy that all three effects are of the same order of magnitude according to our scaling; given that our concern is with rays propagating out to the horizon, this is necessarily the case.

- It can be seen from path description (10) that curvature effects can be accurately accounted for by replacing the refractivity \( n'(u) \) by a modified or effective refractivity \( n'(u) + (\kappa/\eta)u \) and treating the Earth as flat or, equivalently, working with a modified refractive index \( n(R)[1 + (\kappa/\eta)(\delta u/n)] \), i.e., \( n(R) = (R - a_z)/a_z \) [this result represents an approximation to the exact modified index defined by \( a_z n_{\text{eff}}(R) = R n(R) \)].

For future purposes it should also be noted that the path description depends on the refractive-index difference \( n'(z) - n'(1) \), so that a uniform shift in the refractive-index profile will not affect ray paths. One implication is that such shifts cannot be detected by a simple ray path observation, so that the inverse problem is ill posed; an independent measurement of the datum, \( n'(1) \), for example, is necessary for a complete profile determination.

### B. Turning Point Behavior

As indicated earlier, at a turning point on a ray’s path \( n(R_1) = q_0/R_1 \) [see Eqs. (5)], and the integral [Eq. (3)] defining the exact ray path is singular; a careful examination of the path’s local behavior is necessary. In scaled variables such turning points are located at \( z' = z'_s \) defined by

\[
2 \eta n'(z'_s) - n'(1) + 2 \kappa(z'_s - 1) + (\gamma'_0)^2 = 0
\]

[to \( O(h^2 \delta, h^4) \)], where the corresponding defining integral [Eq. (10)] is again singular. Since the ray path deflects locally toward a region of higher refractive index, such turning points will occur for upward traveling rays entering into a lower index region \( \{dz'/dx' > 0, \{dn'(z'_s)/dz' < 0 \} \) or rays propagating downward into a lower refractive-index region \( \{dz'/dx' < 0, \{dn'(z'_s)/dz' > 0 \). In such cases the integral in Eq. (10) needs to be split into near-turning-point and ordinary contributions. We will present results in the latter minimum case. Close to the turning point, the refractive-index profile can be replaced by a local approximation:

\[
n'(z') = n'(z'_s) + (z' - z'_s) \frac{dn'(z'_s)}{dz'} + \cdots \quad (12)
\]

(assuming a smooth profile), and Eq. (10) then gives

\[
x' - x'_{es} = z'_s + \int_{z'_s}^{z'} \frac{1}{\xi(u - z'_s)}^{1/2} du + \cdots
\]

\[
\xi = \left[ \frac{dn'(z'_s)}{dz'} + 2 \kappa \right]^{1/2}
\]

and \( z'_{es} = z'_s(1 + \varepsilon) ; \quad \varepsilon \ll 1 \) is a convenient point to split the integral, and \( x'_{es} \approx x'_s \) is the corresponding \( x' \) location \( [z'(x'_s) = z'_s] \), obtained by integrating Eq. (10) up to \( z'_{es} \). After integrating and rearranging, we obtain

\[
z' - z'_s = \left[ \sqrt{\varepsilon z'_s} + \sqrt{\frac{\xi}{2}} (x' - x'_{es}) \right]^2 + \cdots \quad (14)
\]

The path is locally quadratic with local curvature \( \xi/2 = \{\eta dn'(z'_s)/dz' + \kappa \} \), the distance required for the path to bend depends on the index profile slope locally. The ray path is symmetric about \( (x'_s, z'_s) \), so that computation of the ray path past \( x'_s \) is unnecessary. This result can also be used to trace the \( \gamma'_0 = 0 \) ray.

Evidently, the above analysis can be extended to examine special circumstances. For example, if Earth curvature and refractive-index effects balance for the ray of interest at \( z'_s \) so that \( \{\eta dn'(z'_s)/dz' + \kappa \approx 0 \), then higher-order terms in the \( n' \) expansion [Eq. (12)] need to be retained; in this case the ray path asymptotically approaches, but does not reach, \( z'_s \); the ray does not bend back on itself.

### 3. Exact Solutions

A further advantage of the approximate ray path description (10) is that exact solutions are available for simple index profiles of practical importance. This is especially useful for the inverse problem and for visual reconstructions.

#### A. Constant Refractive-Index Profile

In the case of \( n'(z') = n'(1) \), the integral in Eq. (10) can be evaluated to give, after rearrangement,

\[
z'_0 - 1 = \frac{1}{2} (\kappa x'^2 + 2 \gamma'_0 x'), \quad (15)
\]

where, for later comparison purposes, the notation \( z'_0(x') \) is introduced to denote this ideal ray path over the Earth through a vacuum (\( n' = 0 \)) or, equivalently, through any constant refractive-index medium. The ray’s path is thus parabolic (bending away from the Earth) as seen from the Earth’s surface [the Earth is circular, and the ray path is (exactly) a straight line; the above result represents an \( O(hl) \) approximation]. Note that on a flat Earth (\( \kappa \)}
In the case in which the scaled profile is given by $B$. Linear Profile

the integral in Eq. (10) can again be evaluated, yielding the implicit path description

$$x' = \frac{\text{sgn}(\gamma_0) a_1}{a_1 \eta + \kappa} \left( (\gamma_0'^2 + 2(\eta a_1 + \kappa)(z' - 1))^{1/2} - \gamma_0' \right),$$

which, after rearrangement, gives the explicit result of

$$z' - 1 = \frac{1}{2} ((\kappa + \eta a_1) x'^2 + 2 \gamma_0' x'),$$

with the deviation from the ideal path given by [see Eq. (15)]

$$z' - z_0' = \frac{1}{2} \eta a_1 x'^2.$$  \hfill (18)

Again, the profile is quadratic, with additive refractive-index and curvature effects on the ray height. It is interesting to note that curvature effects can be accounted for by working with an effective index profile $(a_1 + \kappa/\eta)x'$. The path further deflects from the Earth’s surface if $a_1 > 0$, and the deflection is reduced in the $a_1 < 0$ case. Trajectories are shown for different launch angles in cases in which $a_1 < 0$ and $a_1 > 0$ (see Fig. 3). The apparent horizon as viewed from the source is determined by the ray that just grazes the Earth’s surface. This ray leaves the source at an angle $\gamma_0' = \pm \left[ 2(\kappa + a_1 \eta) \right]^{1/2}$ with the horizon located at a distance $\left[ 2(\kappa + a_1 \eta) \right]^{1/2}$ from the source. In theory the horizon extends indefinitely if refractive-index effects balance curvature effects so $a_1 = -\kappa/\eta$, but, in practice, absorption effects will place a limit on the range of observation.

Note that the difference $z' - \gamma_0' x'$ is independent of $\gamma_0'$, so that all rays from a single source (with different $\gamma_0'$’s) deviate from a straight-line path by the same amount in $(x', z')$ space, irrespective of whether there is a turning point within the domain; this is quite surprising!

In Fig. 4 the paths of rays initially launched at an angle $\gamma_0' = 1$ into various linear refractive-index profiles are plotted. Also plotted is

$$z'(1, \gamma_0') = 1 + \frac{1}{2} \left[ (\kappa + \eta a_1) + 2 \gamma_0' \right],$$

which is the destination point corresponding to a source ray projected at angle $\gamma_0'$, for a range of profile gradients. Evidently, all such graphs are of unit slope but with offset (intercept) increasing linearly with the index gradient $a_1$.

C. Quadratic Profile

If the profile is quadratic, then after scaling,

$$n'(z') - n'(1) = a_2(z' - 1)^2 + a_1(z' - 1),$$

(20)

and path description (10) becomes

$$x' = \text{sgn}(\gamma_0') \frac{du}{\sqrt{2 \eta a_2}}$$

$$\int_{z'}^{\gamma_0'} du = \frac{1}{2} \left[ (u - (1 - \xi))^2 + \frac{\xi}{(2 \eta a_2)^2} \right]^{1/2},$$

(21)

where

$$\xi = \frac{\eta a_1 + \kappa}{2 \eta a_2}, \quad \xi = 2 \eta a_2 \gamma_0'^2 - (\eta a_1 + \kappa)^2.$$  \hfill (22)

This can be evaluated to give

$$z' - (1 - \xi) = \frac{\sin(\sqrt{-2 \eta a_2 x'})}{\sqrt{-2 \eta a_2}} \gamma_0'$$

$$+ \xi \cos(\sqrt{-2 \eta a_2 x'}).$$

Note that in the limit as $a_2 \to 0$ the linear profile result [Eq. (17)] is recovered. Although valid over the complete parameter range, this (complex) result does not ad-
equately display the physics. The character of the solution changes dramatically with both the medium and the launching angle; it is clear from Eq. (21) that the signs of $a_2$ and $\xi$ are critical.

1. Convex Profile Case ($a_2 > 0$)
If $a_2 > 0$, then the refractive index decreases away from an effective maximum located at $1 - a_1/(2a_2)$ [see Eq. (20)], so that (almost all) rays (that deflect toward higher-index regions) eventually radiate out to $z' = \pm \infty$ (see Fig. 5). Ray path (22) in this case is best expressed in the (real) form

$$z' - (1 - \xi) = \frac{\sinh(\sqrt{2\eta a_2} z')}{{\sqrt{2\eta a_2}}} \gamma_0' + \xi \cosh(\sqrt{2\eta a_2} z').$$

The ray will bend at a turning point when the argument of the square root in Eq. (21) vanishes, so

$$[z' - (1 - \xi)]^2 + \frac{\xi}{(2\eta a_2)^2} = 0,$$

and this will occur only if $\xi < 0$. All other rays are monotonic. Thus the turning point will occur if the ray is launched into a locally decreasing effective refractive-index region with the initial angle insufficient for the ray to reach the increasing refractive-index zone; this requires $|\gamma_0'| < \gamma_{\text{crit}} = (\eta a_1 + \kappa)/\sqrt{2\eta a_2}$. Note, however, that ray paths do not intersect.

In the Earth’s context, global convex refractive-index profiles are unlikely; however, the above results may be useful for describing propagation in refractive-index pockets associated with inversions and turbulent jets.

2. Concave Profile Case ($a_2 < 0$)
If $a_2 < 0$, then the refractive index decreases away from an effective maximum located at $1 - a_1/(2a_2)$, and rays will bend back toward this maximum, leading to oscillatory behavior, with period $\sqrt{-2\eta a_2}/2\pi$ (see Fig. 6). These rays are well represented by Eq. (22).

In this case (unlike the $a_2 > 0$ case), rays intersect; in fact, all rays intersect at common intersection points (see Fig. 6). Note that if observer and object are located on opposite sides of an intersection point then the observer will see an inverted image of the object; mirages are possible.

Of particular interest to us is the parametric dependence of the image height at the target end $x' = 1$. Equations (22) and (23) give

$$z'(1) - 1 = \begin{cases} 
-\xi + \xi \cos(\sqrt{-2\eta a_2}) + \frac{\sin(\sqrt{-2\eta a_2})}{\sqrt{-2\eta a_2}} \gamma_0' & \text{for } a_2 < 0 \\
-\xi + \xi \cosh(\sqrt{2\eta a_2}) + \frac{\sinh(\sqrt{2\eta a_2})}{\sqrt{2\eta a_2}} \gamma_0' & \text{for } a_2 > 0
\end{cases}.$$

D. Higher-Order Profiles
Exact solutions can be obtained for up to quartic refractive-index profiles, but, for practical purposes, such results are not likely to be of interest. Exact results are
also available in the exponential profile case, which is of particular importance in boundary-layer phenomena, for example, propagation over the ocean. That research will be presented in a later paper.

It is, of course, possible to represent complex observed refractive-index profiles in terms of patched-together quadratics, and the above exact results for ray paths within the zones can be matched across the interfaces. In this way, very accurate results can be obtained for realistic profiles with little computational effort. In view of the results obtained in Section 4, this seems to be a most appropriate way to determine realistic ray paths.

4. IMAGE RECONSTRUCTION

One objective of this paper is to determine the vertical location and shape of an object, given its image viewed through a medium with an unknown radial refractive-index profile. Before the image is deconstructed, the index profile needs to be determined; this is the inverse problem. We have already seen that a uniform shift in the profile (i.e., a change in \( n'(1) \)) cannot be detected; however, such a shift does not affect the image, and so there is no need to make such a determination in context.

The apparatus used to view an object consists of a lens or mirror to produce a real image and a light-sensitive plate at the image plane. For purposes of exposition, the object that is viewed is a lighthouse with equally spaced markings up its side, located at a distance far enough so that the image occurs in the focal plane of the optical system (see Fig. 7). All rays from a particular marking on the lighthouse converge to a single point after passing through the lens. Hence the lens simply converts the angle \( \gamma_0 \) subtended by the ray passing through the center of the lens into a corresponding height \( H' \) in the focal plane given by \( \tan \gamma_0 = H'/f \), where \( f \) is the focal length of the lens. If we equate the incoming angle at the lens to the source launch angle as defined in Section 2, then all our previous results carry across immediately to this situation, with \( z'(1, \gamma_0) \) now representing the (scaled) height above the datum [the height of the source is determined by the height of the axis of the optical system; in our case, \( z'(0, \gamma_0) = 1 \)] at the object position of the ray received at a (scaled) angle \( \gamma_0 \) at the lens (the distance between the lens and the lighthouse corresponds to the \( L \) used in Section 2, and the analysis proceeds as before).

In an uniform atmosphere (i.e., one with constant refractive index) on a flat Earth, all rays are straight lines, so the image is contracted but remains undistorted in the vertical direction; i.e., the markings remain equally spaced in the image plane. In a variable refractive-index medium, the incoming ray with angle \( \gamma_0 \) is curved and comes from \( z'(1, \gamma_0) \) instead of following the straight-line path from \( z'(1, \gamma_0) \), as it would in a uniform medium. The image will thus, in general, be both relocated and distorted (see Figs. 8 and 9).

The constant refractive-index situation provides a convenient datum for measuring the relocation and distortion of the image. In this case the ray entering the lens at \( \gamma_0 = 0 \) arrives from the location \( z'(1, 0) = \kappa / 2 \), representing the scaled offset that is due to the Earth’s curvature alone. Thus the offset that is due to refractive-index variations alone is given by

\[
\mathcal{O} = z'(1, 0) - z'_0(1, 0) = z'(1, 0) - \kappa / 2, \tag{25}
\]

which will be used as a measure of the relocation. In the constant refractive-index case, \( dz'(1, \gamma_0) / d\gamma_0 = 1 \) [see Eq. (15)] (as a consequence of the scaling we have adopted), so that equally spaced markings on the lighthouse will be seen as equally spaced markings in the image plane; the image is uniformly magnified in the image plane. Note that if \( dz'(1, \gamma_0) / d\gamma_0 = c \), i.e., independent of \( \gamma' \), then the image will be uniformly magnified in the vertical direction, but the magnification of the image will differ from that in the constant refractive-index case if \( c \neq 1 \).

More generally, if we write

\[
z'(1, \gamma_0') - z'_0(1, \gamma_0') = \mathcal{O}(n', \eta, \kappa) + \mathcal{M}(n', \gamma_0', \eta, \kappa) \gamma_0',
\]

then an experimental determination of the offset \( \mathcal{O} \) and magnification \( \mathcal{M} \) should enable one to determine the profile \( n'(x') \) and permit a deconstruction of the image.

Explicit results obtained for the profiles examined earlier are summarized in Table 1, where

![Fig. 7. Observation system used in the discussion of inversion.](image)

The example is of a lighthouse observed through a telescope at a position such that the image occurs in the focal plane.

![Fig. 8. Ideal or straight-line path that would result from a uniform atmosphere and the actual ray path for a radically varying refractive index.](image)

![Fig. 9. This figure depicts equally spaced markings on the lighthouse to measure the vertical distorting of the image.](image)
Table 1. Image Characteristics for Constant, Linear, Quadratic, and Polynomial Refractive-Index Profiles

<table>
<thead>
<tr>
<th>Refractive Index Profile</th>
<th>C</th>
<th>M</th>
<th>(R?, D?)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n' = n'(1) )</td>
<td>0</td>
<td>1</td>
<td>(N, N)</td>
</tr>
<tr>
<td>( n' = n'(1) + a_1(z' - 1) )</td>
<td>( \frac{\eta a_1}{2} )</td>
<td>1</td>
<td>(Y, N)</td>
</tr>
<tr>
<td>( n' = n'(1) + a_2(z' - 1)^2 + a_1(z' - 1) )</td>
<td>( G(a_1, a_2, \eta, \kappa) )</td>
<td>( F(a_2, \eta) )</td>
<td>(Y, N)</td>
</tr>
<tr>
<td>poly ( &gt; 2 ), ( n' = n'(1) + \Sigma a_n(z' - 1)^n )</td>
<td>( C(a_n, \gamma_0, \eta, \kappa) )</td>
<td>( M(a_n, \gamma_0, \eta, \kappa) )</td>
<td>(Y, Y)</td>
</tr>
</tbody>
</table>

\( ^* \) C and M are the offset and magnification, respectively. \( R? \) is the question “Does shifting of the origin occur?” and \( D? \) is the question “Is the image magnified nonuniformly with height?”

\[
F(a_2) = \frac{\sin(\sqrt{-2\eta a_2})}{\sqrt{-2\eta a_2}},
\]

\[
G(a_1, a_2) = \left(1 - \frac{\eta a_1 + \kappa}{2\eta a_2}\right) + \frac{\eta a_1 + \kappa}{2\eta a_2} \cos(\sqrt{2\eta a_2}).
\]

It is not surprising that there is no image distortion in the vertical direction (uniform magnification) in the constant \( n' \) case, but it is somewhat surprising that this is also true in the linear profile case and is quite remarkable that this is also true for quadratic profiles. This result was pointed out by Kropka and Lehn in an interesting theoretical paper in which, by introducing an optical path-length metric, they were able to identify surfaces on which light rays were geodesics. The present paper explicitly determines the solution.

Table 1 indicates that:

- If there is neither distortion in the vertical direction nor relocation of the image, then the refractive index is constant with height.
- If there is no vertical distortion of the image, then the profile is either linear or quadratic. If the magnification is constant, \( c \), say, then \( F(a_2) = c \) determines \( a_2 \), and, with \( a_2 \) now known, the offset \( C \) can be used to determine \( a_1 \). It should be noted in the \( a_2 < 0 \) oscillatory path case there are several choices for \( a_2 \) that give identical magnification, so some care is required. Based on previous history, a particular choice would normally be obvious.

Using the above, one can determine the refractive-index profile for quadratic index profiles and thus reconstruct the undistorted image of the object viewed. Although the above discussion is limited to quadratic index profiles, the extension to more general profiles is evident, and the above distortion results greatly facilitate the process. Vertical sections of a viewed image can be investigated for distortion, and thus local quadratic approximations of the index profile can be determined; in this way, a complete refractive-index profile can be determined, at least in theory. These ideas will be developed in later papers.

5. CONCLUSIONS

Scaling arguments have been used to justify an accurate approximation scheme leading to a simple description of the ray path in an atmosphere with prescribed radial variations of refractive index. This approximation is particularly useful for both analytic and numeric research. Exact solutions for the approximate equation are obtained for particular refractive-index profiles (linear and quadratic), and the inverse problem is discussed. A discussion of how the results may be employed to determine the refractive-index profile from visual data and hence permit a determination of the shape and location of an object, given its image, has been presented. The results described in this paper can also be obtained by applying perturbation techniques to the eikonal equation in the manner described in Refs. 2 and 28.

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The corresponding author, B. D. Nener, can be reached by phone, 61 89380 3111; fax, 61 89380 1095; or e-mail, brett@ee.uwa.edu.au.

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