Referee’s opinion on the paper of Borwein, Montesinos, and Vanderwerff

**Bondedness, ... Convex Functions**

The paper evolved from Borwein’s previous result that in Asplund spaces the weak Hadamard and Fréchet differentiability of convex functions are the same.

The first half of the paper organizes, in a transparent way, many results mostly already appeared elsewhere, and mostly authored by Borwein, Fitzpatrick and Vanderwerff. They are usually formulated as series of equivalent statements, putting together: coincidence of two different types of convergence of sequences, two different types of boundedness of functions, two different types of differentiability, etc. The above concepts are considered with respect to three topologies/bornologies: weak∗ convergence, uniform convergence of weakly compact sets, and norm convergence.

Proofs are usually not difficult, yet they use clever and rather original constructions of convex functions. The second half deals with a possibility of extending a convex function from a subspace to the whole space. It should be noted that, contrary to norms/Lipschitz functions, for general convex functions this may not be the case. Again, it is shown that this is equivalent to an extension of weak∗ convergent sequences in the dual from the subspace to the whole space. The authors show a broad expertise from Banach space theory when looking for suitable classes of Banach spaces where such an extension exist.

I think that the topic of the paper, and the results therein, may attract a broader audience.

The paper is written in a telegraphic, concise language. Sometimes, the notation is not completely exact and unambiguous, yet it is still understandable and not confusing much. So I consider the style acceptable though not perfect.

The proofs are correct – I checked all of them in details.

Below, I attach a few suggestions how to improve, I think, the overall shape of the paper.

Because of the above, I strongly recommend the paper for publication in your journal.

**List of suggestions**

1. To add “f : X → IR”
2. and in many other places below: “τW → τW”
3. The symbol “A(x)” looks Russian. I suggest A(·) or just A.
4. It took me a while to understand these lines. I suggest to replace “If ...τW to 0.” by ”We observe that ϕk do not converge to 0 in τW by (a). Find a weakly compact set C ⊂ X so that supC ϕk > K for infinitely many k. As fn → 0 uniformly on C, we know that supC fnk < ϵ for infinitely many k, and so C ⊂ Ck for infinitely many k. Then for these k we have K < supC ϕk ≤ supCk ϕk < (ϕk(xnk) ≤ K), a contradiction.
5. “true.” → “true. Find a continuous convex function f on X which is bounded on weakly compact sets and which is not bounded on a bounded set.”
6. ”to the proof” → ”from the proof”?
7. “[10]” → ”[10, page ??]”

Please, diminish the room between single lines in Theorems and in Propositions.
Please provide a reference to the fact that $\ell_\infty$ has DP* property.

A personal question of the referee: Can you prove that weak Hadamard differentiability coincides with Fréchet differentiability for convex functions in Asplund spaces without Rosenthal?

"supercoercive" $\rightarrow$ "supercoercive"
"cofinite" $\rightarrow$ "cofinite"

"[1, p. 623,624]" $\rightarrow$ "[1, pp. 623, 624]"

"weak Hadamard directionally differentiable" is not defined. The referee does not like this name much. Other names can be: "one-sided weak Hadamard differentiable", "weak Hadamard subdifferentiable". Godefroy, Montesinos and Zizler in Comment. Math. Univ. Caroliniae (some 8 years ago) use "SSD – strongly subdifferentiable".

Actually you do not need to prove (e), (f), or (h). You are rather proving the equivalence of these three assertions with another (which one(s))?.

I was confused a bit while reading these lines. I would welcome a more detailed text here.

"f(0) = 0 and" $\rightarrow$ "f(0) = 0, f si bounded on weakly compact sets, and"
"infinitely many n, and get that f is unbounded on W." $\rightarrow$ "infinitely many n. As inf$_W f < +\infty$, we have W $\subset$ C$_n$" for all large n, and so for these n we have 1 < sup$_W x_n \leq$ sup$_{C_n} x_n \leq 1$, a contradiction."

To write a formula like "$\tilde{f}(x) = \inf \{ f(y) + (L + 1)\|y - x\|; y \in Y \}$", where L is the Lipschitz constant of f.

I think that the text "any existing ... Question 4.1" is confusing. Actually, Example 4.2 is such a result. Probably you wanted to say something else here.

"n $\geq$ n$_0$" $\rightarrow$ "n $\geq$ n$_0$ and $\alpha \in A_n$"
"such that" $\rightarrow$ "of $\phi$ such that"
"f(0)" $\rightarrow$ "f(y$_0)$"
"I did not see that $a \geq 1$. I have a picture showing that $a = \epsilon$."

"for (\phi, -1) $\in$ Y$^* \times IR$" $\rightarrow$ "considering (\phi, -1) as en element of (Y $\times IR)^*$"
"sup" $\rightarrow$ "sup$_{\alpha,n}$"
"f(x)" $\rightarrow$ "$\tilde{f}(x)$"
"p$_n(x)$" $\rightarrow$ "p$_n$"
"$\cap$" $\rightarrow$ "$\cap$"

Is [22, Theorem 3.5] a correct reference?
"c$_0$(Z$_1, Z_2,\ldots$)" $\rightarrow$ "($\sum_{i=1}^{\infty} Z_i$)$_{c_0}$"
"$\ell_\infty(A_n))$_{c_0}" $\rightarrow$ "($\sum_{n=1}^{\infty} \ell_\infty(A_n))$_{c_0}"

I did not understand the proof: The space in 4.13 is proclaimed to be injective, while the proof says that X is not.

"now forces" $\rightarrow$ "now forces the Gâteaux differentiability of sup$_{\alpha,n}$\,$\tilde{\phi}$... and hence"

Do you really need $\frac{1}{k}$? (i.e., that $a_{n,k} \geq \frac{1}{k}$?)