Integrals of the Ising class

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Abstract

From an experimental-mathematical perspective we analyse ‘Ising-class’ integrals. These are structurally related $n$-dimensional integrals we call $C_n, D_n, E_n$, where $D_n$ is a magnetic susceptibility integral central to the Ising theory of solid-state physics. We first analyse

$$C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{j=1}^n (u_j + 1/u_j))^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}.$$

We had conjectured—on the basis of extreme-precision numerical quadrature—that $C_n$ has a finite large-$n$ limit, namely $C_\infty = 2e^{-2\gamma}$, with $\gamma$ being the Euler constant. On such a numerological clue we are able to prove the conjecture.

We then show that integrals $D_n$ and $E_n$ both decay exponentially with $n$, in a certain rigorous sense. While $C_n, D_n$ remain unresolved for $n \geq 5$, we were able to conjecture a closed form for $E_5$. Our experimental results involved extreme-precision, multidimensional quadrature on intricate integrands; thus, a highly parallel computation was required.

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**Part I. Experimental-mathematics approaches**

1. Background and nomenclature

This research began as a quest for a numerical scheme for high-precision values of Ising susceptibility integrals, in our preferred normalization being defined as

\[ D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} (u_i-u_j)^2}{(\sum_{j=1}^n (u_j + 1/u_j))^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}. \]  

(1)

The integrals \( D_n \) appear in susceptibility expansions from Ising theory, as detailed in the literature including works on field-theoretic and form-factor approaches [4, 19, 20, 24–28]. Very briefly, the importance of \( D_n \) in Ising physics runs as follows [22]. Magnetic susceptibility \( \chi(T) \)—essentially a spin–spin correlation in the 2D Ising model—depends asymptotically on temperature \( T \) as

\[ \chi_{\pm}(T) \sim C_{0,\pm} \left( 1 - \frac{T}{T_c} \right)^{-7/4}, \]

where \( T_c \) is the critical temperature and the subscript \( \pm \) indicates whether \( T > T_c \) (plus) or \( T < T_c \) (minus). The connection with our present analysis is that the so-called susceptibility amplitudes

\[ C_{0+} = C_+ \sum_{n=0}^{\infty} I_{2n+1}, \quad C_{0-} = C_- \sum_{n=1}^{\infty} I_{2n}, \]

where \( C_{\pm} \) are explicitly known constants [24], involve integrals \( I_n \) proportional to our \( D_n \); specifically

\[ I_n := 2^{-n} n^{1-n} D_n. \]

We have taken the \( D_n \) integral, therefore, as a prime candidate for experimental-mathematics research; i.e. knowing a \( D_n \) in a closed form traces immediately back to an important term from a susceptibility expansion.

It was suggested to us by Tracy [22] and emphasized by Maillard [16] that evaluation of the \( D_n \) susceptibility integrals—to sufficient precision—could well lead to experimental-mathematical capture for some \( n > 4 \). In fact, the appearance of Riemann-zeta evaluations is already a known phenomenon in related nonlinear physics [10]. Now, because closed forms for the \( D_n \) are difficult, as are numerical evaluations for large \( n \), we elected to study first some related but simpler integrals. This was our initial motive for defining the entities

\[ C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{j=1}^n (u_j + 1/u_j))^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}. \]  

(2)

(not to be confused with the \( C \) amplitudes of Ising theory), and later \( C_{n,k} \) as discussed in part II of this paper.

Because these \( C_n \) are relatively easy to resolve to extreme\(^6\) precision, we remain hopeful that finding closed forms experimentally for some \( C_n \) will suggest, at least qualitatively, what fundamental constants might appear in the higher \( D_n \). Indeed, a mere glance at similarities between closed forms at a given level \( n \) vindicates this expectation (see table 1). In the sense that we are taking not a physics-oriented but an experimental-mathematics approach, the present work is reminiscent of [12, pp 312–3] and [7–9]. Moreover, as enunciated in our

\(^6\) By ‘extreme precision’ we mean, loosely, ‘precision sufficient for reasonable confidence in experimental detection’, which in our experience means between 100 and 1000 digits.
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Table 1. What is known of Ising-class integrals: the symbols ‘=’ and ‘∼’ connote, respectively, ‘proven’ and ‘detected experimentally’. The asymptote $C_\infty = 2 e^{-2\gamma}$ is also proven.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$C_n$</th>
<th>$D_n$</th>
<th>$E_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$L_{-1}(2)$</td>
<td>1/3</td>
<td>6 – 8 log 2</td>
</tr>
<tr>
<td>3</td>
<td>$7\zeta(3)/12$</td>
<td>$8 + 4\pi^2/3 - 27L_{-1}(2)$</td>
<td>10 – $2\pi^2 - 8 \log 2 + 32\log^2 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2 \log 2 - 22\pi^2/3$</td>
</tr>
<tr>
<td>4</td>
<td>$2$</td>
<td>$2\sqrt[3]{2} / (3\log 2 + 32\log 2 - 40 \log 2)$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$0.665 759 8001 \ldots$</td>
<td>$0.002 484 6057 \ldots$</td>
<td>$22 - 82\zeta(3) - 24 \log 2 + 16\pi^2 \log 2 - 22\pi^2/3$</td>
</tr>
<tr>
<td>6</td>
<td>$0.648 634 2090 \ldots$</td>
<td>$0.000 489 1417 \ldots$</td>
<td>$0.000 687 832 87 \ldots$</td>
</tr>
</tbody>
</table>
... | $\sim 2 e^{-2\gamma}$ | $\Omega \left( \frac{1}{\sqrt{n}} \right), O \left( \frac{1}{\sqrt{n}} \right)$ | $\Omega \left( \frac{1}{\sqrt{n}} \right), O \left( \frac{1}{\sqrt{n}} \right)$ |

Abstract, these $C_n$ for large $n$ appeared to approach a positive constant, in fact rather rapidly. The natural conjecture and proof of same are given in a later section.

Even though our introduction of the $C_n, E_n$ integrals is thus ‘symbolically motivated’, it turns out in retrospect that the $C_n$ do have relevance in Ising physics. Namely, these integrals appear naturally in the analysis of bounds on certain amplitude transforms [22], [20, lemma 5.1, and p 384].

We have found the following symbolic machinations particularly useful. For either integral (1) or (2), consider the simplex with constraint $u_1 > u_2 > \cdots > u_n$. We may then use the change of variables $u_k = \prod_{i=2}^{k} t_i$, with $t_1 \in (0, \infty)$ and all other $t_i \in (0, 1)$, to transform the integration domain into a finite one. Define

$$ u_k := \prod_{i=2}^{k} t_i, \quad v_k := \prod_{i=k}^{n} t_i. $$

and the functions

$$ A_n(t_2,t_3,\ldots,t_n) := \left( \prod_{n \geq k > j \geq 1} \frac{u_k / u_j - 1}{u_k / u_j + 1} \right)^2 $$

$$ B_n(t_2,t_3,\ldots,t_n) := \frac{1}{(1 + \sum_{k=2}^{n} u_k)(1 + \sum_{k=2}^{n} v_k)}.$$

Then the relevant integrals can be cast like so:

$$ D_n = 2 \int_0^1 \cdots \int_0^1 A B \, dt_2 \, dt_3 \cdots \, dt_n, \quad (3) $$

$$ C_n = 2 \int_0^1 \cdots \int_0^1 B \, dt_2 \, dt_3 \cdots \, dt_n, \quad (4) $$

Here, the $1/n!$ normalization has disappeared due to the $n!$ ways of ordering the simplex indices, and we have symbolically integrated over $t_1$. It will turn out to be useful to define
also an integral
\[
E_n := 2 \int_0^1 \cdots \int_0^1 A \, d \tau_2 \, d \tau_1 \cdots d \tau_n.
\] (5)

It transpires that, for all \( n \geq 1 \), we have
\[
D_n \leq E_n \leq C_n.
\] (6)
The first inequality is trivial, and also trivial is the implicit relation
\( D_n \leq C_n \), since by their very definitions \( A, B \in [0, 1] \) on the domain of integration. Almost as obvious is the inequality
\( E_n \leq n^2 D_n \). But it will require more work to establish the hardest branch \( E_n \leq C_n \) (see the text after theorem 3).

Beyond such inequalities, one can go yet further in the matter of asymptotic analysis. Using representations (3), (5) we shall be able to establish that \((D_n), (E_n)\) sequences are both strictly monotone decreasing and genuinely exponentially decaying in the sense that for positive constants \( a, b, A, B \) we have
\[
a \frac{b^n}{n^a} \leq D_n \leq E_n \leq A \frac{B^n}{n^a}.
\]
In section 7 we shall not only prove this (theorem 3) but also give effective \( a, b, A, B \) values.

2. Tabulation of results

Table 1 exhibits known evaluations of \( D_n \) and the structurally related Ising-class integrals \( C_n, E_n \). The reader should beware of varying normalizations in the physics literature; yet every Ising-susceptibility integrand involves, as do our \( D_n \) from (1), some manner of combinatorial entity constructed over \((i, j)\) index pairs. (For \( n = 1 \) we interpret the \((i < j)\) product in the definition (1) as unity.) Our particular normalization for \( D_n \) versus \( I_n : = D_n/(2^n \pi^{n-1}) \) means, in reference to our table 1, that \( I_1 = 1, I_2 = 1/(12 \pi) \) and so on. The constants \( I_3 = D_3/(8 \pi^2) \approx 0.00081446 \) and \( I_4 = D_4/(16 \pi^3) \approx 0.0000025448 \) were resolved in closed form c. 1977 [21, 24], while \( D_5 \), though still algebraically elusive, was resolved to 30 decimal places by Nickel in 1999 [17]—these respective symbolic and numerical achievements being remarkable for their eras. Though \( I_1 \) is sometimes known in the literature as the ferromagnetic constant, it looms appropriate to honor the pioneering work of [24], by referring to the collections \((I_n)\) and \((I_1, \ldots, I_4)\) as the McCoy–Tracy–Wu (MTW) integrals and constants, respectively. Indeed, our section 9 provides a synopsis of their historical analysis, while section 14 and appendix B contain our recent extreme-precision rendition of
\( D_5 = 32 \pi^4 I_4 \) and also \( D_6 = 64 \pi^5 I_5 \).

In the construction of table 1, we have invoked a Dirichlet L-function that occurs frequently in mathematical physics (see [11, section 2.6], [12, chapter 3]) namely\(^7\)
\[
L_{-3}(2) := \sum_{n \geq 0} \left( \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right),
\]
and also the standard polylogarithm
\[
\text{Li}_s(z) := \sum_{k \geq 1} \frac{z^k}{k^s}.
\]
All the closed forms in table 1 are proven, except for the one shown for \( E_5 \), which is an experimental result based on a 240-digit computation. This \( E_5 \) relation was found using
\(^7\) Note that some literature treatments (e.g. [21]) use the Clausen function [15] which is algebraically related to the stated L-function.
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PSLQ at a confidence level of 190 digits beyond the level that could reasonably be ascribed to numerical round-off error (we will describe the computation of $E_5$ in section 14). As for large-$n$ behaviour implied in table 1, we know $C_{\infty}$ rigorously as an exotic constant, while the $\Omega$, $O$ notation means both $D_n$, $E_n$ decay exponentially but no faster than that (see theorem 3). Numerical entries here are known to higher precision than is displayed—in fact we know many $C_n$, as well as some $D_n$, $E_n$, to extreme precision (see section 14 and appendix A).

3. Bessel-kernel representations for $C_n$

Let us first use the transformation $u_k \rightarrow e^{x_k}$ in (1), (2) to achieve the representations

$$D_n = \frac{1}{n!} \int \mathcal{D} \vec{x} \frac{\prod_{i<j} \tanh^2 \left( \frac{x_i - x_j}{2} \right)}{(\cosh x_1 + \cdots + \cosh x_n)^2},$$

and

$$C_n = \frac{1}{n!} \int \mathcal{D} \vec{x} \frac{\prod_{i<j} \tanh^2 \left( \frac{x_i - x_j}{2} \right)}{(\cosh x_1 + \cdots + \cosh x_n)^2}.$$

where here and elsewhere $\int \mathcal{D} \vec{x}$ is interpreted symbolically as the full-space operation $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n$. Now $C_n$ can be put in the form

$$C_n = \frac{1}{n!} \int_0^\infty p \int \mathcal{D} \vec{x} e^{-p \sum \cosh x_i} \, dp,$$

which leads to an attractive, one-dimensional integral

$$C_n = \frac{2^n}{n!} \int_0^\infty p K_0(p) \, dp,$$

where $K_0$ is the standard, modified Bessel function [1]

$$K_0(p) := \int_0^\infty e^{-p \cosh t} \, dt.$$

In anticipation of experiments and theorems to follow, we state ascending and asymptotic expansions of $K_0$, respectively:

$$K_0^{(\text{asc})}(t) = \sum_{k \geq 0} \frac{t^{2k}}{4^k k!^2} \left( H_k - \left( \gamma + \log \frac{t}{2} \right) \right),$$

and

$$K_0^{(\text{asy})}(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{m=0}^{\infty} \frac{(-1)^m ((2m)!)^2}{m!^3 (32t)^m}.$$

where $\gamma$ denotes the Euler constant and the $H_k := \sum_{m \leq k} 1/m$ are the harmonic numbers, with $H_0 := 0$. It is known [1] that the error accrued in taking terms through index $m = M$ in (12) is no larger than the first dropped term (and with sign of that dropped term). We also make use of the representation

$$K_v(x) = 2^v \frac{\Gamma(v+1/2)}{\sqrt{\pi}} \int_0^\infty \cos(x t) \, dt / (1 + t^2)^{(v+1/2)}.$$

8 It is both a convenience and a pleasure to invoke thus the ‘curly-$D$’ of Feynman path-integral lore, as the present research traces back to solid-state physics, not to mention that we contemplate at one juncture an infinite-dimensional limit.
valid for real \( x > 0 \) and \( \text{Re}(\nu) > -1/2 \) [1]. Observe that in the ascending series (11) the leading term is \( -\gamma - \log(t/2) \), revealing a logarithmic singularity at the origin. It will turn out to be lucrative to define a ‘pivot point’

\[
p_0 := 2 e^{-\gamma},
\]

such that the said leading term vanishes at \( t = p_0 \). To simplify our derivations to follow, we also adopt an ‘effective big-\( O \)’ notation, as

\[
\Theta(f) = g,
\]

meaning \( |f/g| \leq 1 \), equivalent to \( O(\cdot) \) notation but with implied big-\( O \) multiplier of unity.

Again in anticipation of experiment and theory, we state the next result.

**Lemma 1.** For the modified Bessel function \( K_\nu(x) \) with real \( \nu \geq 0 \) and real \( x > 0 \), with pivot point \( p_0 \), we have

\[
0 < K_\nu(p) < \Gamma(\nu) \frac{2^{\nu-1}}{p^\nu}; \quad \nu > 0, \quad (14)
\]

\[
K'_0 = -K_1, \quad (15)
\]

\[
K_0(p) = -\gamma - \log(p/2) + \Theta(p/3); \quad p \in (0, p_0), \quad (16)
\]

\[
K_0(p) < \sqrt{\frac{\pi}{2p}} e^{-p}. \quad (17)
\]

**Proof.** Relation (14) follows easily from integral (13), since \( |\cos| \leq 1 \). Relation (15) is standard [1]. Relation (16) follows from inspection of the ascending series (11) over the finite interval \((0, p_0)\). (Note that \( \Theta(p/3) \) is simply some function bounded by \( p/3 \) on the said interval, and could also be written as \( p\Theta(1/3) \).) Relation (17) either follows from the general asymptotic theory [1], or from the observation that \( \int_0^\infty e^{-p \cosh x} \, dx < e^{-p} \int_0^\infty e^{-p x^2/2} \, dx \). \( \Box \)

4. **Experiment leads to theory**

Later in section 11 we discuss numerical evaluation of \( C_n \) for large \( n \). Even a cursory examination of the high-precision numerical results displayed in appendix A suggests that \( C_n \) appears to approach a definite limit, namely

\[
C_\infty = 0.630 473 503 374 386 796 122 040 192 710 878 904 354 587 078 712 732 34 \ldots
\]

After inserting the numerical value we obtained for \( C_{1024} \) into the smart lookup facility of the CECM Inverse Symbolic Calculator at

http://oldweb.cecm.sfu.ca/cgi-bin/isc

we obtained the output:

Mixed constants, 2 with elementary transforms.

6304735033743867 = sr(2)\(^2\)/exp(gamma)\(^2\)

In fact, according to our calculations,

\[
0 < C_{1024} - 2 e^{-2\gamma} < 10^{-300}.
\]

On the basis of this and other observations, we were convinced of the truth of the following, experimentally motivated conjecture:
Conjecture 1. The sequence of integrals \((C_n : n = 1, 2, 3, \ldots)\) is strictly decreasing. Moreover, we have the finite limit

\[
\lim_{n \to \infty} C_n = 2e^{-2y}.
\]

Indeed, armed with confidence in the above conjecture, we may proceed to prove all aspects of the conjecture, starting with

Theorem 1. \((C_n : n = 1, 2, 3, \ldots)\) is strictly decreasing.

Proof. We may integrate by parts, starting with equation (9), to arrive, via lemma 1 (15), at

\[
C_n = \frac{2^{n-1}}{(n-1)!} \int_0^\infty p^2 K_1(p)K_0^{n-1}(p) \, dp.
\]

We may therefore express a difference

\[
C_{n-1} - C_n = \frac{2^{n-1}}{(n-1)!} \int_0^\infty p(1-p)K_1(p)K_0^{n-1}(p) \, dp.
\]

But, by lemma 1 (14), the integrand in (19) is nonnegative on \(p \in (0, \infty)\), whence

\[
C_{n-1} - C_n > 0.
\]

Our next observation is that certain generating functions can be used to extract limits of monotonic sequences. We have

Lemma 2. Let \((r_n : n = 1, 2, 3, \ldots)\) be a positive, strictly monotone-decreasing sequence. Denote, then, \(r = \lim_n r_n\), and define a generating function

\[
R(z) := \sum_{n=1}^\infty r_n z^n.
\]

Then \(r = \lim_{z \to 1} (1-z)R(z)\).

Proof. For \(z \in (0, 1)\), we have

\[
(1-z)R(z) := rz + T(z) \quad \text{where} \quad T(z) := (1-z) \sum_{n=1}^\infty (r_n - r)z^n.
\]

Now fix \(\epsilon > 0\), and observe that

\[
T(z) \leq r_1 N(1-z) + \frac{\epsilon}{2} z^{N+1},
\]

when \(N\) is chosen such that \(r_M - r < \epsilon /2\) for \(M > N\).

Set \(\delta := \min[\epsilon /2(r + r_1 N)], \epsilon /2\). It follows that \(|(1-z)R(z) - r| < \epsilon\) for \(1-z \leq \delta\).

Remark 1. Deeper such results obtain in Abelian–Tauberian theory, yet this lemma is quite sufficient for our present purpose.

Now we contemplate the generating function

\[
C(z) := \sum_{n=1}^\infty C_n z^n.
\]

and we use this construct to establish the large-\(n\) limit of our \(C_n\):
Theorem 2. The sequence \( (C_n : n = 1, 2, 3, \ldots) \) has
\[
\lim_{n \to \infty} C_n = 2 e^{-2y}.
\]

Proof. The generating function (21) at hand may be developed, via the representation (9) and then (16), (17) of lemma 1, like so:
\[
C(z) = \int_0^\infty p(e^{2zK_0(p)} - 1) dp
= \int_0^{p_0} p e^{2z(\gamma - \log(p/2) + \Theta(1/3))} dp + \Theta(c),
= e^{-2z\gamma} \int_0^{p_0} \frac{p}{(p/2)^{1/3}} e^{p\Theta(1/3)} dp + \Theta(c),
\]
where \( c \) is a constant independent of \( z \). Using the fact that for \( x \in [0, 1] \) we have
\[
e^x = 1 + \Theta(x + x^2),
\]
we obtain
\[
C(z) = e^{-2z\gamma} \frac{2z\gamma}{2 - 2z} + \Theta \left( e + \frac{c_1}{3 - 2z} + \frac{c_2}{4 - 2z} \right),
\]
where \( c_1, c_2 \) are again \( z \)-independent constants. It follows that
\[
\lim_{z \to 1^-} (1 - z) C(z) = 2 e^{-2y},
\]
and via lemma 2 the theorem follows. \( \square \)

It has become evident—largely on hindsight—that integration of (9) up to only the pivot point \( p_0 \) generally leaves an extremely small residual integral. Indeed, if we interpret the representation (9) as
\[
C_n = \frac{2^n}{n!} \left( \int_0^{p_0} + \int_0^\infty \right) \frac{p^{n-1}}{K_0^n(p)} dp
\]
then the second integral is easily seen—via lemma 1 (17)—to be factorially minuscule, in the sense that for any \( n > 1 \),
\[
C_n = \frac{2^n}{n!} \int_0^{p_0} p K_0^n(p) dp + \Theta \left( \frac{1}{n!} \right).
\]

By inserting the ascending series (11) into this pivot integral over \( p \in (0, p_0) \), we obtain—after various manipulations—the asymptotic expansion
\[
C_n \sim \frac{2^n}{n!} \sum_{J=1}^\infty \frac{e^{-2Jy}}{J} \sum_{k_1+\cdots+k_n=J} \int_0^\infty e^{-y} dy \prod_{i=1}^n \frac{2H_{k_i} + y/J}{k_i!^2},
\]
where the partitions are over nonnegative integers \( k_i \). This attractive expansion is in the spirit of mathematical physics—it is essentially a perturbation expansion with coupling parameter \( e^{-2y} \). Indeed, the first few terms go
\[
C_n \sim 2 e^{-2y} + \frac{n+4}{2} e^{-3y} + \frac{2n^2 + 23n + 57}{3n \cdot 6} e^{-6y} + \ldots
\]
(22)
Remarkably, just these displayed terms with \( n = 32 \) yield a \( C_{32} \) value to 17 good decimals—an efficient way to effect quadrature to reasonable precision on a 32-dimensional integral!
5. Further dimensional reduction for $C_n$

One way to proceed analytically is to invoke a scaled-coordinate system. Using the representation

$$C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \, du_1 \cdots du_n,$$

(23)

we let the first coordinate $u_1$ be an overall scale. This is much the same as using $n$-dimensional 'spherical coordinates' involving the scale (radius) $r$ and $(n-1)$ angular coordinates. Let us posit, for (5.1),

$$u_1 = r, \quad u_2 = rx_0, \quad u_3 = rx_1, \ldots, \quad u_n = rx_{n-2}. $$

It turns out that this scaled-coordinate transformation generally reduces the integral (23) by two dimensions, since one may easily integrate symbolically over $r$, then almost as easily over $x_0$. Inter alia we find, trivially, that $C_1 = 2$ and $C_2 = 1$, as start out our table 1 entries for $C_n$. Beyond this, the general procedure yields an $(n-2)$-dimensional form

$$C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \log P \, dx_1 \cdots dx_{n-2} / Q - 1,$$

(24)

for $n \geq 3$, where $P, Q$ are the interesting constructs (here and in what follows, $P, Q$ are to be written in terms of the available integration variables $x_1, \ldots$):

$$P := 1 + x_1 + \cdots + x_{n-2},$$

(25)

$$Q := P \cdot (1 + 1/x_1 + \cdots + 1/x_{n-2}).$$

(26)

Thus, for $n = 3$ we only need to evaluate a one-dimensional integral:

$$C_3 = \frac{2}{3} \int_0^\infty \frac{\log(1 + x)}{x^2 + x + 1} \, dx,$$

which, via the transformation $x \to 1/t - 1$ becomes

$$= -\frac{2}{3} \int_0^1 \frac{(1 + t) \log t}{1 + t} \, dt$$

$$= \frac{2}{3} \sum_{n \geq 0} (-1)^n \left( \frac{1}{(3n+1)^2} + \frac{1}{(3n+2)^2} \right)$$

$$= L_{-3}(2),$$

where the factor ‘2/3’ is removed from the final line on the observation that $1/1^2 + 1/2^2 - 1/4^2 - 1/5^2 + \cdots = (1 + 1/2)(1/1^2 - 1/2^2 + 1/4^2 - 1/5^2 + \cdots)$.

For $n = 4$ we had conjectured, on the basis of numerical values, such as those in appendix A, and PSLQ integer relation finding facilities [11], that $C_4 \approx 7/12 \zeta(3)$. This turns out to be true, derivable via the two-dimensional reduced integral

$$C_4 = \frac{1}{6} \int_0^\infty \int_0^\infty \frac{\log(1 + x + y)}{(1 + x + y)(1 + y + 1/y) - 1} \, dx \, dy.$$
Indeed performing the internal integration leads to

\[ C_4 = \frac{1}{6} \int_0^\infty \frac{\text{Li}_2(x^{-1}) - \text{Li}_2(x)}{x^2 - 1} \, dx \]

by transforming \( x \to 1/x \). Here \( \text{Li}_2(x) := \sum x^n/n^2 \) is the dilogarithm [11], analytically continued. Now, integrating by parts leads to

\[ 24C_4 = 8 \int_0^1 \frac{\ln^2 (x+1)}{x} \, dx - 8 \int_0^1 \frac{\ln(1+x) \log(1-x)}{x} \, dx \]

\[ - 4 \int_0^1 \frac{\log(x) \log(1+x)}{x} \, dx + 4 \int_0^1 \frac{\log(x) \log(1-x)}{x} \, dx \]

\[ = 2\zeta(3) + 5\zeta(3) + 3\zeta(3) + 4\zeta(3) = 14\zeta(3), \]

where each integral is an integral multiple of \( \zeta(3) \), as can be obtained from the analysis of the trilogarithm \( \text{Li}_3(x) := \sum x^n/n^3 \), in [15, section 6.4 and appendix A3.5].

For \( n \geq 5 \) we may continue the procedure at least once more and write an \((n - 3)\)-dimensional integral. One expresses the coordinates \((x_1, \ldots, x_{n-2})\) using \( x_1 \) as scale, to arrive at

\[ C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty M(Q) \frac{dt_1}{t_1} \cdots \frac{dt_{n-3}}{t_{n-3}} \]

where, here, \( Q := Q(t_1, \ldots, t_{n-3}) \) is the \( Q \)-form (25) for \((n - 3)\) dimensions, and

\[ M(Q) := \int_0^\infty \frac{\log(1+u)}{u^2 + Qu + Q} \, du. \]

Moreover, \( M(Q) \) is directly expressible in terms of logarithms and dilogarithms. In fact, with \( \alpha := \frac{Q}{2} - 1 - (\frac{Q}{2} - 1)^2 - 1)^{1/2} > 0 \) so that the larger quantity \( 1/\alpha = \frac{Q}{2} - 1 + (\frac{Q}{2} - 1)^2 - 1)^{1/2} \) we have

\[ (Q^2 - 4Q)^{1/2} M(Q) = \text{Li}_c(-\alpha) - \text{Li}_c(-\infty/\alpha) \]

\[ = 2\text{Li}_2(-\alpha) + \zeta(2) + \frac{1}{4} \log^2(\alpha) \]

where the last equality follows from [15, A.2.1. (5)]. This development, for example, represents \( C_5 \) as a double integral, namely

\[ C_5 = \frac{1}{30} \int_0^\infty \int_0^\infty M(Q) \frac{dx \, dy}{x \, y} \]

\[ = \frac{1}{10} \int_0^1 \int_0^1 M(Q) \frac{dx \, dy}{x \, y}, \]

where \( Q := (1 + x + y)(1 + 1/x + 1/y) \).

While the details are a bit foreboding, all of this suggests that in general \( C_n \) may well be a combination of polylogarithmic constants of order at most \( n - 1 \). In this language the results we have obtained are \( C_3 = (4/3) \text{Im} \text{Li}_2((-1)^{1/3})/\sqrt{3} \) and \( C_4 = -(56/3) \text{Re} \text{Li}_3((-1)^{1/3})/3 \).

On the other hand, there is some theoretical evidence in support of a possible ‘blockade’ against closed forms for \( C_5 \) and beyond. Namely, the Adamchik algorithm [2] for evaluating integrals of argument powers with Bessel-function powers does not extend beyond fourth powers of the Bessel terms [3]. Thus \( C_4 \) can be derived via the Adamchik method, but evidently \( C_5 \) cannot.
To summarize so far: we have rigorously established closed forms as in table 1 for $C_1$ through $C_4$. However, the higher $C_n$’s remain elusive. It is pleasing—and justifies our original research motivation—that the above closed forms for the $C_n$ involve, at least for these small $n$, similar fundamental constants as appear for the few known $D_n$ shown in table 1.

6. Symbolics for the susceptibility integrals $D_n$

A first approach to closed forms for $D_n$ is to exploit various advantages of integral representation (3). We have, with $A_nB_n$ denoting the integrand with the $(n - 1)$ variables $t_2, t_3, \ldots, t_n$, $A_1B_1 := 1$ and $A_2B_2 = (t_2 - 1)^2/(t_2 + 1)^4$, while

$$A_3B_3 = \frac{(t_2 - 1)^2 (t_2t_3 - 1)^2 (t_3 - 1)^2}{(t_2 + 1)^2 (t_2t_3 + 1)^2 (t_3 + 1)^2 (t_2 + t_2t_3 + 1)(t_2t_3 + t_3 + 1)}$$

Hence, $D_1 = 2$ while

$$D_2 = 2 \int_0^1 \frac{(x - 1)^2}{(x + 1)^4} \, dx = \frac{1}{3}$$

$$D_3 = \frac{1}{3} \int_0^1 \int_0^1 A_3B_3(x, y) \, dx \, dy$$

$$= \frac{2}{3} \int_0^1 \int_0^1 A_3B_3(x, y) \, dx \, dy,$$

which integral Maple can reduce to the exact value for $D_3$ given in our introduction, at least in the form

$$18i \text{Li}_2(1/2 - 1/2 i \sqrt{3}) \sqrt{3} - 18i \text{Li}_2(1/2 + 1/2 i \sqrt{3}) \sqrt{3} + 24 + 4 \pi^2.$$

As noted in our introduction, a closed form for $D_4$ is known (see our section 9), yet the status of higher values is open. The representation above for $D_4$ via $A_4B_4$ was sufficient to compute 14 decimal places in Maple and so to recover this constant with PSLQ. In principle, these methods and especially those of section 7 allow for a complete symbolic resolution of $D_4$ but the details are somewhat daunting.

For a second analytical foray, one may envision possible roles of the $C_n$ in $D_n$ analysis. Looking longingly at (7), one may write

$$D_n = \frac{1}{n!} \int D^n \prod_{1 < j} (1 - \text{sech}^2 \frac{\pi x_k}{\cosh x_1 + \cdots + \cosh x_n})$$

$$\frac{\cosh x_1 + \cdots + \cosh x_n}{(\cosh x_1 + \cdots + \cosh x_n)^2}.$$ (30)

This form reveals that in a specific sense, $C_n$ amounts to a first term in a finite sum of integrals. Indeed, one might expand the product into partial products of sech$^2$ terms, and furthermore employ the attractive Fourier identity

$$\text{sech}^2 \left( \frac{z}{2} \right) = 2 \int_{-\infty}^{\infty} \frac{k}{\sinh(\pi k)} e^{ikz} \, dk.$$ (31)

We also have the convenient integral representation

$$\int_{-\infty}^{\infty} e^{-p \cosh x + ikx} \, dx = 2K_{ik}(p).$$

$^9$ Adequate Maple code is

$p := \frac{1}{2} \times (x - 1)^2 \times (y - 1)^2 \times (x + y)^2 / (x + 1)^2 / (y + x)^2 / (y + 1)^2 / (1 + y + x) / (y + x + x + y)$;

d := Int(Int(p, x = 0..infinity), y = 0..infinity) : evalc(value(d));
Now for small $n$ one may extract closed forms for $D_n$ using a $(p, k)$-transform apparatus. For example, we have

\[ D_2 = C_2 - 4 \int_{-\infty}^{\infty} \frac{k \, dk}{\sinh \pi k} \int_{0}^{\infty} p K^2_{ik}(p) \, dp \]

\[ = C_2 - 2\pi \int_{-\infty}^{\infty} \frac{k^2 \, dk}{\sinh^2 \pi k} = \frac{1}{3}. \]

Note the direct involvement of the $C_2$ value as a first-order perturbation term.

For higher $n$, one can still evaluate the Bessel-$K$ integrals in terms of hypergeometric functions, but it is not clear how to handle the rapidly growing number of $k$ variables. Still, these $(p, k)$-transforms may conceivably give rise to high-precision numerical schemes. The problem with growing $k$-variable counts is that an appropriate term from the natural expansion of representation (30), say

\[ \int_{0}^{\infty} p \, dp \int_{0}^{D} e^{-p \sum \cosh x_i} \prod_{(a, b) \in P} \text{sech}^2((x_a - x_b)/2), \]

where $P$ is some set of index pairs, has the expansion

\[ \int_{0}^{\infty} p \, dp \int_{0}^{D} \prod_{q=1}^{c} \frac{k_q}{\sinh(\pi k_q)} K_{i\nu}(p), \]

where $c = \text{card}(P)$. Unfortunately, $c$ can be $O(n^2)$.

Still it may somehow be possible to somehow employ a higher-order sech-Fourier transform, namely a generalization of (31) [18]:

\[ \text{sech}^{2m}(x/2) = \frac{2^{2m-1}}{(2m-1)!} \int_{-\infty}^{\infty} k \, \frac{d}{dk} \prod_{h=1}^{m} \frac{1}{(k^2 + h^2)} \, dk. \]

Likewise, it would be good to know the Fourier transform of

\[ \prod_{(a, b) \in P} \text{sech}^2((x_a - x_b)/2) \]

in terms of at most $n$ spectral variables $k_q$, rather than $c = \text{card}(P) = O(n^2)$ such variables. In any case, it may well be that an appropriate $(k, p)$ transform would lead us back to the highly successful numerical approach that yielded results for the $C_n$. As interesting as these $(k, p)$ transforms may be, such an approach may be misdirected in the sense that a ‘perturbation series’ for $D_n$ starting with a leading term $C_n$ is unrealistic, due to the different asymptotic character of $D_n$, as we next discuss.

7. Asymptotic character of $D_n$ and $E_n$

With a view to proving that $D_n, E_n$ are genuinely exponentially decaying in a certain sense, we first note the examples

\[ E_1 := 2, \]

\[ E_2 = 2 \int_{0}^{1} A \, dy = 2 \int_{0}^{1} \left( \frac{1 - x}{1 + x} \right)^2 \, dx = 6 - 8 \log 2 \approx 0.454 \, 823, \]

\[ E_3 = 2 \int_{0}^{1} \int_{0}^{1} \left( \frac{(1 - x)(1 - xy)(1 - y)}{(1 + x)(1 + xy)(1 + y)} \right)^2 \, dx \, dy \]

\[ = 10 - 2\pi^2 - 8 \log 2 + 32 \log^2 2 \approx 0.090 \, 1102, \]
with $E_4$ also enjoying a more extended but similar closed form (see table 1). Just these few examples suggest exponential decay of the $E_n$ integrals, with a decay constant about 5 (see table 2 and section 13).

For convenience in the theorem to follow, we define

$$R(x) := \left( \frac{1 - x}{1 + x} \right)^2,$$

and let $m := n - 1$, so that $E_n$ is the integral over the unit $m$-cube of the product of (a triangular number) $m(m + 1)/2$ instances of $R$. Specifically, for $n > 1$,

$$E_n = 2 \int_{[0,1]^m} D \prod_{k=1}^m R(x_k) R(x_kx_{k+1}) \cdots R(x_k \cdots x_m).$$

Observe also that the reduced $D_n$ integrand is the same $R$-product multiplied by the extra factor $B_n(x_1, \ldots, x_m) := (1 + x_1 S)^{-1} (T + U x_1)^{-1}$, where

$$S := 1 + x_2 + x_2 x_3 + \cdots + x_2 \cdots x_m,$$

$$T := 1 + x_m + x_m x_{m-1} + \cdots + x_m \cdots x_2,$$

and $U := x_m \cdots x_2$.

**Theorem 3.** The sequences $(D_n)$ and $(E_n)$ are both strictly monotone decreasing for $n \geq 1$. Moreover, $D_n$ and $E_n$ enjoy genuine exponential decay; that is, there exist positive constants $a, b, A, B$ such that for all positive integers $n$

$$\frac{a}{B^n} \leq D_n \leq E_n \leq \frac{A}{B^n},$$

where effective values are $[a, b] = [19, 14]$ and $[A, B] = [12, 4]$.

**Remark 2.** The effective values may be further improved with more aggressive application of the following techniques. For example, $B$ can be $(2/E_p)^{1/(p-1)}$ for any $p > 1$, and so the approximate (nonrigorous) value for $E_6$ in table 2 yields effective constant $B \approx 4.97$. Likewise, more effort to enhance (32) will presumably improve the lower bound $b$, the remaining inequalities being quite tight.

**Proof.** First, monotonicity. By bounding the integral over the first coordinate $x_1$ we see that

$$E_n \leq \left( \int_0^1 R(x_1) \, dx_1 \right) E_{n-1} = \frac{E_2}{2} E_{n-1} < 0.26 E_{n-1}.$$

This establishes strict monotonicity for the sequence $(E_n)$; below we shall tighten this approach to yield a tighter effective constant. As for monotonicity of the $D_n$, note that for $m := n - 1$ the $R$-product involving the first coordinate $x_1$ can be bounded as

$$R(x_1) R(x_1 x_2) \cdots R(x_1 \cdots x_m) \leq e^{-2x_1 S'},$$

where $S'$ is given in the text prior to this theorem. This bound on the $x_1$-dependent part can be quickly obtained by taking the logarithm of the $R$-product, noting $\log R(z) = -2(z + z^3/3 + z^5/5 + \cdots) \leq -2z$. Now we obtain an upper bound for the integral over $x_1$, as

$$\int_0^1 \frac{e^{-2x_1 S}}{(1 + x_1 S)(T + U x_1)} \, dx_1 < \frac{0.37}{ST},$$

where we have used $\int_0^\infty e^{-z}/(1 + z) \, dz = e^2 \text{Ei}(1, 2) \approx 0.361$, an exponential integral, [1]. But $1/(ST)$ is precisely the $B_{n-1}$ factor in the integrand for $D_{n-1} = 2 \int_{[0,1]^{n-2}} A_{n-1} B_{n-1} D \, dx$; thus we establish monotonicity in the form $D_n < 0.37 D_{n-1}$.
Next, for a fundamentally tighter effective upper bound on \( E_n \) (and perforce \( D_n \)—recall the trivial inequality \( D_n \leq E_n \)). For a given \( n \), the integrand for \( E_n/2 \) has at least \( \lfloor(n - 1)/2 \rfloor \) disjoint triples of the form \( R(x_1)R(x_1x_2)R(x_2) \), as inspection of a few cases suggests. For example, the integrand for \( E_{5/2} \) with variables \( w, x, y, z \) is

\[
\]

from which one may read off six (underlined) \( R \)'s amounting to \( \lfloor(5 - 1)/2 \rfloor = 2 \) disjoint triples. Thus the integral for \( E_{n/2} \) is bounded above by the product of \( \lfloor(n - 1)/2 \rfloor \) copies of \( E_{5/2} \) and so

\[
\frac{1}{2} E_n \leq \left( \frac{2}{E_1} \right)^{\lfloor(n-1)/2 \rfloor}
\]

and the upper bound follows.

Now for the lower bound. The reduced \( D_n \) integrand is a product of \( m(m+1)/2 \) evaluations of \( R \) (where \( m := n - 1 \)) times the factor \( B_n \). Said integrand is monotone decreasing in all variables \( x_1, \ldots, x_m \). That is, the integrand \( \iota \) satisfies \( \iota(\vec{x}) \leq \iota(\vec{y}) \) whenever \( x_k \leq y_k \) for all coordinate indices \( k \). But this means that for any \( \alpha \in [0, 1] \) the integral is bounded below by a natural approximation of the integral over the sub-cube \( [0, \alpha]^m \). So, we evaluate all the \( R \) terms at the corner vector \( \vec{\alpha} := (\alpha, \alpha, \ldots, \alpha) \), observing also \( B_n(\vec{\alpha}) \geq (1 - \alpha)^2 \), and deduce

\[
D_n \geq 2(1 - \alpha)^2 \alpha^m \left( 1 - \frac{\alpha}{1 + \alpha} \right)^{2m} \left( 1 - \frac{\alpha^2}{1 + \alpha^2} \right)^{2m-2} \left( 1 - \frac{\alpha^3}{1 + \alpha^3} \right)^{2m-4} \cdots \left( 1 - \frac{\alpha^n}{1 + \alpha^n} \right)^2
\]

since \( \alpha^m \) is the volume of the reduced hyper-cube. Interestingly, this expression in \( \alpha \) may be bounded below by a theta-function term, as we may estimate

\[
D_n \geq 2(1 - \alpha)^2 \alpha^m \prod_{k=1}^{m} \left( 1 - \frac{\alpha^k}{1 + \alpha^k} \right)^{2m},
\]

where \( \theta_4(q) := \sum_{n \in \mathbb{Z}} (-q)^n \) is a Jacobi theta function, see \cite{12}. Now \( \alpha \theta_4(\alpha)^2 \) has a maximum greater than 0.074 at \( \alpha = \alpha_0 > 0.169 \) and we conclude that

\[
D_n \geq 2(1 - \alpha_0)^2 (0.074)^{n-1},
\]

leading immediately to the desired lower bound as well as effective constants. \( \square \)

Corollary 1. For all positive integers \( n \), we have \( E_n \leq C_n \).

Proof. This follows directly from the observation that even for \( n = 2 \), theorem 3 with \( A := 12, B := 4.71 \) gives us \( E_{n \geq 2} < 0.54 < 2e^{-2} \), the right-hand side being \( \inf_n C_n < \).

Theorem 3 suggests that \( D_n, E_n \) may both follow a truly exponential-decay asymptotic, and numerical work suggests further a universal decay constant, whence we posit:

Conjecture 2. \( D_n, E_n \) both decay exponentially, with the same decay constant. That is, there exist positive constants \( \delta, \Delta, \phi \) such that

\[
D_n \sim \frac{\delta}{\Delta^n} \text{ and } E_n \sim \frac{\phi}{\Delta^n},
\]

so that ratios behave as

\[
\lim_{n \to \infty} \frac{D_n}{D_{n+1}} = \lim_{n \to \infty} \frac{E_n}{E_{n+1}} = \Delta, \quad \text{and} \quad \lim_{n \to \infty} \frac{D_n}{E_n} = \delta/\phi.
\]

Remark 3. If this conjecture is true, we expect, based on the quasi-Monte Carlo (qMC) integrations of section 13, that \( \Delta \approx 5 \) and \( \delta/\phi \approx 0.7 \). Moreover, given our rigorous result theorem 3, is it perhaps reasonable anyway to expect \( \Delta \) to be of order \( b \approx 4.7 \).
8. Further dimensional reduction of $D_n$ and $E_n$

We have seen that $D_n, E_n$ can each be defined by an $(n - 1)$-dimensional integral, via relations (3), (5), and that $C_n$ can be reduced to an $(n - 2)$-dimensional integral, as in (24) and further to an $(n - 3)$-dimensional form (27). However, it turns out that $D_n, E_n$ can also be reduced to $(n - 2)$-dimensional forms, albeit with considerable combinatorial complications, as we shall now establish.

We begin by considering the integrand factor $A$ appearing in (3), (5), and noting the combinatorial recursion that results from an attempt to factor out terms involving only $t_2$:

$$A_n(t_2, \ldots, t_n) = \left( \frac{1 - t_2}{1 + t_2} \right)^2 \left( \frac{1 - t_2t_3}{1 + t_2t_3} \right)^2 \cdots \left( \frac{1 - t_2 \cdots t_n}{1 + t_2 \cdots t_n} \right)^2 A_{n-1}(t_3, \ldots, t_n).$$

Observe also that we may write

$$B_n(t_2, \ldots, t_n) = \frac{b^{-1}}{(1 + t_2(1 + t_3t_4 + \cdots + t_3 \cdots t_n)) \cdot (1 + (a/b)t_2)}$$

with

$$a := t_3 \cdots t_n, \quad b := 1 + t_n + t_n t_{n-1} + \cdots + t_n \cdots t_3.$$

Next, we observe a key formal identity

$$\left( \frac{1 - z}{1 + z} \right)^2 = \frac{\partial}{\partial \lambda} \bigg|_{\lambda = 1} \left( \lambda + \frac{4}{1 + \lambda z} \right)$$

which will allow us to create terms $(1 - z)^2/(1 + z)^2$ via partial differentiation. Now for a parameter vector $\vec{\lambda}$ of dimension $(n - 1)$, define

$$G_n(\vec{\lambda}; t_2, \ldots, t_n) := 2 \prod_{k=1}^{n-1} \left( \lambda_k + \frac{4}{1 + \lambda_k \prod_{j=2}^{k+1} t_j} \right).$$

Putting all this together yields

$$D_n = \int_0^1 \cdots \int_0^1 A_{n-1}(t_3, \ldots, t_n) \left( \frac{\partial^{n-1}}{\partial \lambda_1 \cdots \partial \lambda_{n-1}} \bigg|_{\lambda_k = 1} \int_0^1 G_n \, dr \right) \, dt_3 \cdots dt_n$$

$$E_n = \int_0^1 \cdots \int_0^1 A_{n-1}(t_3, \ldots, t_n) \left( \frac{\partial^{n-1}}{\partial \lambda_1 \cdots \partial \lambda_{n-1}} \bigg|_{\lambda_k = 1} \int_0^1 G_n \, dr \right) \, dt_3 \cdots dt_n.$$

Remarkably, as we shall presently show, $G_n$ and $G_n B_n$—for any $n$—can each be integrated in closed form with respect to the $t_2$ coordinate. Moreover, these closed forms may be differentiated with respect to the $\lambda_k$ and then evaluated at $\lambda_k = 1$ to provide a legitimate, $(n - 2)$-dimensional integral over $(t_3, \ldots, t_n)$. Indeed, we have a general reduction theorem:

**Theorem 4.** For every integer $n > 2$, each of $C_n, D_n, E_n$ can be written as an $(n - 2)$-dimensional integral with elementary integrand consisting of algebraic multivariate functions of logarithms.

**Proof.** For a parameter collection $(\sigma_k : k = 1, \ldots, M)$ we know from partial-fraction decomposition that

$$\int_0^1 \prod_{k=1}^M \frac{1}{1 + \sigma_k t} \, dt = \sum_{i=1}^M \frac{\sigma_i^{M-2} \log(1 + \sigma_i)}{\prod_{j \neq i} (\sigma_i - \sigma_j)}.$$
Now the $t_2$-dependent part of the product integrand $G_n B_n$ for $D_n$ can be written as a product of the type in the integral here, with $M = n + 1$, $t := t_2$, and the $\sigma_k$ involving subsets of variables taken only from $(t_3, \ldots, t_n)$, so immediately we have an algebraic function of logs for an integral over the one coordinate $t_2$. Then we differentiate inside with respect to $\lambda_1, \ldots, \lambda_{n-1}$ and arrive at an $(n-2)$-dimensional integral. The same argument goes through for the simpler integrand $G_n$ of $E_n$, with $M = n - 1$.

Note that if need be, $C_n$ can be processed as above, with integrand $2B_n$—see (4)—but the previous result (24) gives equivalent reduction. A specific manifestation of the reduction procedure is detailed in section 14, where we provide some numerical values for $D_5, E_5, D_6, E_6$.

We were able to reduce $E_4$ entirely to one-dimensional integrals and ultimately to evaluate it symbolically (as in table 1) but for higher dimensions this procedure becomes problematic and has not yet been rigorously pursued. The experimentally-detected form for $E_5$, described in section 14, appears not to have obvious higher-order analogues and perhaps represents the end of a polylogarithmic ladder.

9. Historical resolution of the MTW constants

In section 2 we describe the MTW constants as the currently known closed-form cases $I_1$-through-$I_4$ (in our present normalization, $D_1$-through-$D_4$). It is remarkable that McCoy, Tracy and Wu were able to resolve these constants in closed form some 30 years ago; moreover, it is likewise remarkable that no further closed forms for the $D_n$ have evolved in all that time.

We now summarize the historical MTW methods, based on some handwritten notes kindly provided to us for the purpose of finally casting those monumental results in a modern symbolic light [23]. The overall technique relies on three clever transforms; we believe it optimally instructive to describe these transforms first for the more tractable integrals $C_n$, then indicate how the previous researchers handled the $D_n$. Starting with (2) we write

$$C_n := \frac{4}{n!} \int_{[0,\infty]^n} \mathcal{D}\vec{u} \int_0^\infty p e^{-p(\sum u_{i} + \sum 1/u_{i})} \, dp.$$  \hfill (33)

First MTW transformation: $u_k \to v_k/p$. This leads to the representation

$$C_n := \frac{2}{n!} \int_{[0,\infty]^n} \mathcal{D}\vec{v} \prod_v e^{-\sum v_k}.$$  \hfill (34)

Second MTW transformation: $v_k \to \alpha_k \sum v_j$. This yields a finite domain of integration:

$$C_n := \frac{2}{n!} \int_{[0,1]^n} \mathcal{D}\vec{\alpha} \prod_\alpha \delta(1 - \sum \alpha_k) \sum \frac{1}{x_k}.$$  \hfill (35)

where $\delta$ is the Dirac delta-function.

Third MTW transformation. Now the key is to find a coordinate system of $(n-1)$ dimensions such that the $\alpha_k$ sum to unity automatically. For example, take $n = 3$ and write (here and beyond, we employ bar-notation, $\bar{x} := 1 - x$ for any variable $x$):

$$\alpha_1 = x, \quad \alpha_2 = \bar{x} \bar{y}, \quad \alpha_3 = \bar{x} \bar{y}.$$  

The beauty of such a transformation is that the three right-hand sides add up to 1, being as $\bar{x} + \bar{z} = 1$ always. For $n = 4$ one may take

$$\alpha_1 = xy, \quad \alpha_2 = x \bar{y}, \quad \alpha_3 = \bar{x} \bar{z}, \quad \alpha_4 = \bar{x} \bar{z}.$$
These two transforms are what McCoy, Tracy and Wu actually employed to resolve $D_3, D_4$, as we shall soon see. Generalization of these $n = 3, 4$ cases is ambiguous, but an example of a universal simplectic scheme having the property $\sum \alpha_k = 1$ is

$$\alpha_1 = x_1,$$
$$\alpha_2 = \bar{x}_1 x_2,$$
$$\alpha_3 = \bar{x}_1 \bar{x}_2 x_3,$$
$$\ldots$$
$$\alpha_{n-1} = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_{n-1} x_n,$$
$$\alpha_n = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_{n-1} \bar{x}_n,$$

so that the $\alpha_k$ is a product of $k$ terms, except the last, $\alpha_n$, is to have $(n - 1)$ terms. It is easy to see by adding from the bottom that the sum of the $\alpha_k$ here is unity. Note two things: first, that the historical MTW for $n = 4$ above is not this generalization, so that there are other ways to proceed for general $n$; and second, when doing the above integral with the Dirac delta-function, the rules are (a) drop the $\delta$ term altogether, and (b) introduce the Jacobian from the matrix of $(n - 1)^2$ derivatives ($\partial \alpha_k / \partial x_j : j, k \in [1, n - 1]$).

Now, a striking feature of the triple-MTW transformation scheme is that the Ising permutation products are invariant under the MTW transformations. That is to say, when confronting an Ising susceptibility integral $D_n$, we may casually insert any permutation product such as

$$\prod_{j<k} (\alpha_j - \alpha_k \alpha_j + \alpha_k)^2$$

into (35) and continue on with the third transformation.

Let us work some small-$n$ examples, then. For $n = 2$ we have, from the casual-insertion rule into (35),

$$D_2 = \frac{2}{2!} \int_{[0,1]^2} \frac{d\alpha_1 d\alpha_2}{\alpha_1 + \alpha_2} \delta(1 - \alpha_1 - \alpha_2) \left( \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \right)^2.$$

In this case we do not even need a third transformation, just the constraint $\alpha_1 + \alpha_2 = 1$, to obtain

$$D_2 = \int_{[0,1]} d\alpha_2 (1 - 2\alpha_2)^2 = 1/3,$$

consistent with table 1.

As for $C_3$, we use (35) with the above simplectic transform for $n = 3$ to get

$$C_3 = \frac{2}{3!} \int_{[0,1]^3} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_2 \alpha_3 + \alpha_2 \alpha_1 + \alpha_1 \alpha_2} \delta(1 - \alpha_1 - \alpha_2 - \alpha_3)$$
$$= \frac{1}{3} \int_{[0,1]^2} \frac{dx dy}{(1 - y + y^2) + y - y^2}$$
$$= -\frac{1}{3} \int_0^1 \frac{\log(y^2 - y)}{y^2 - y + 1}$$
$$= L_{-3}(2),$$

using at the end here the same kind of algebra as in section 5 for $C_3$.

$D_3$, in turn, takes the form (recall the rule that the permutation product may simply be inserted, with impunity, into a $C_n$ form to render a $D_n$)

$$D_3 = \frac{2}{3!} \int_{[0,1]^3} \frac{d\alpha_1 d\alpha_2 d\alpha_3 \prod_{j<k} (\frac{\alpha_j - \alpha_k}{\alpha_j + \alpha_k})^2}{\alpha_2 \alpha_3 + \alpha_2 \alpha_1 + \alpha_1 \alpha_2} \delta(1 - \alpha_1 - \alpha_2 - \alpha_3).$$
Under the same simplectic transformation as for $C_3$ above, we obtain

$$D_3 = \frac{1}{3} \int_{[0,1]^4} \frac{(1 - 2y)^2(x(y - 2) - y + 1)^2(yx + x - y)^2}{((y - 1)y^2 + (-2y^2 + 2y - 1)x + (y - 1)y)^2} \ dx \ dy.$$ 

This integral, recondite as it may be, can indeed be resolved and the closed form is given for $D_3$ in Table 1. Explicitly, integrating the rational function in Maple with respect to $x$ under the ‘assumption’ that $x > 0$, $y > 0$, $x < 1$, $y < 1$ and then integrating with respect to $y$ produces the evaluation in the form

$$6i \text{Li}_2\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right) \sqrt{3} - 6i \text{Li}_2\left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right) \sqrt{3} + \frac{\pi}{8} x^2 + 8.$$ 

At this juncture it is important to point out a refinement due to McCoy, Tracy Wu that actually simplifies the symbolic analysis for $D_3$. For example, in the $D_3$ case above, one may replace the permutation product $\prod$ of the integrand with

$$\prod \rightarrow \frac{\alpha_1 - \alpha_2 \alpha_2 - \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} + 3 \left(\frac{\alpha_2 - \alpha_3}{\alpha_2 + \alpha_3}\right)^2.$$ 

We are not saying this permutation form on the right is algebraically equivalent; we are saying that the integral for $D_3$ is invariant under this modification. At any rate, the $D_3$ integral with this modified permutation form is somewhat easier to handle, giving, of course, the correct closed form in Table 1.

Along such lines, the culmination of the MTW historical effort is that $D_4$ may be written as

$$D_4 = \frac{2}{4!} \int_{[0,1]^4} d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \frac{\delta(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \prod}{\alpha_2 \alpha_3 \alpha_4 + \alpha_1 \alpha_3 \alpha_4 + \alpha_1 \alpha_2 \alpha_4 + \alpha_1 \alpha_2 \alpha_3},$$

where again the profound knowledge of the underlying perturbation theory allowed those pioneering researchers to use

$$\prod \rightarrow \left(\frac{\alpha_1 - \alpha_2 \alpha_3 - \alpha_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} + \frac{\alpha_1 - \alpha_4 \alpha_2 - \alpha_3}{\alpha_1 + \alpha_4 + \alpha_2 + \alpha_3} - \frac{\alpha_1 - \alpha_3 \alpha_2 - \alpha_4}{\alpha_1 + \alpha_3 + \alpha_2 + \alpha_4}\right)^2$$

in place of $\prod := \prod_{1 \leq j < k \leq 4} (\alpha_j - \alpha_k)^2/(\alpha_j + \alpha_k)^2$ (although we presume that the latter transformation should go through, perhaps with more difficulty along the way). In this fashion the MTW constant $D_4$ in Table 1 was established, via the third MTW transformation above for $n = 4$, those decades ago.

10. Hypergeometric connections

It turns out that Ising-class integrals have a certain connection with hypergeometric functions and their powerful generalization, the Meijer $G$-functions. Such analysis gives rise to fascinating series representations, new closed forms and rational relations between certain pairs of integrals. We sketch such ideas here, with details to be found in our separate work [5].

This idea is to generalize Ising integrals by modifying intrinsic powers within integrands. Define for integers $k \geq 0$

$$C_n,k := \frac{1}{n!} \int \frac{D^n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}},$$

whence, per (8), the original $C_n$ integrals are $C_{n,1} := C_{n,1}$. Not surprisingly, the collection $(C_{n,k} : n, k \geq 0)$ provides yet more fertile ground for experimental-mathematical discovery,
not to mention clues as to what symbolic behaviour might be expected of Ising integrals in general. In addition, one can derive [5] some evidently new exact evaluations of Meijer G-functions themselves.

Now the Bessel-kernel representation (9) likewise generalizes to

\[ C_{n,k} := \frac{2^n}{n!} \frac{1}{k!} \int_0^\infty t^n K_0^n(t) \, dt. \]  

(37)

It is clear from the definition (36) that (i) for fixed \( n \), \( C_{n,k} \) is monotonic decreasing in \( k \). The arguments behind theorems 1 and 2 can be adapted to show first, that (ii) for fixed \( k \geq 1 \) the set \( (C_{n,k}) \) is monotonic decreasing in \( n \), and that (iii) for any fixed \( k \) we have the large-\( n \) asymptote

\[ C_{n,k} \sim \frac{1}{k!} \frac{2^{k+1+n}}{(k+1)^{n+1}} e^{-(k+1)\gamma}, \]

for which our original, canonical case reads \( C_n = C_{n,1} \sim 2 e^{-2\gamma} \). This can be verified experimentally.

We next proceed to summarize some closed forms for various \( C_{n,k} \), as proven in [5]. One has

\[ C_{1,k} = \frac{\sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right)}{2 \Gamma \left( \frac{k+1}{2} + 1 \right) \Gamma(k+1)}, \]

from which it is immediate that

\[ C_{1,k} = p_{1,k} + q_{1,k} \pi, \]

where the \( p, q \) coefficients are always rational, with \( q \) vanishing for odd \( k \) and \( p \) vanishing for even \( k \). Similarly, for \( n = 2 \), from relation (37) and some manipulations relevant to Meijer G-functions we obtained (see [5])

\[ C_{2,k} = \frac{\sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right)^3}{2 \Gamma \left( \frac{k}{2} + 1 \right) \Gamma(k+1)}, \]

and so

\[ C_{2,k} = p_{2,k} + q_{2,k} \pi^2, \]

with the same vanishing rule on the rational \( p, q \) multipliers as for \( n = 1 \).

The case \( n \geq 3 \) on \( C_{n,k} \) are problematic. We discovered experimentally the conjectures

\[ C_{3,3} = -\frac{4}{27} + \frac{2}{9} \text{L}_{-3}(2), \]

\[ C_{3,5} = -\frac{9}{125} + \frac{1}{5} \text{L}_{-3}(2), \]

and several more. We should mention that we found no rational relations whatever between pairs of \( C_{3,\text{even}} \) (however, we did find 3-term recurrence relations, as discussed below). Once again on the basis of what to expect, we were able to prove the suggested rational-relation conjecture in the form

**Theorem 5** (See [5]). For odd \( k \geq 1 \), we have

\[ C_{3,k} = p_{3,k} + q_{3,k} \text{L}_{-3}(2), \]

with the \( p, q \) coefficients always being rational.
Moreover, a finite form for the rationals $q_k$ can be written down. The method of proof also provides an algorithm for evaluating any $C_{3,\text{odd}}$ rather efficiently. One may arrive quickly at such instances as

$$C_{3,15} := \frac{1}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx \, dy \, dz}{(\cosh x + \cosh y + \cosh z)^{16}}$$

$$= \frac{11884272896}{837856594575} + \frac{4139008227988189}{227988189} \zeta(3)(2).$$


Continuing our summary, we conjectured pairwise rational relations also for the $C_{4,\text{odd}}$, and carried out an analysis in terms of Meijer $G$-functions, leading to (again, proof is in [5])

**Theorem 6** (See [5]). For odd $k \geq 1$, we have

$$C_{4,k} = p_{4,k} + q_{4,k} \zeta(3),$$

with the $p,q$ coefficients always being rational.

In these $C_{4,\text{odd}}$ cases, polynomial-remaindering and rational-arithmetic algorithms [5] quickly yield instances such as

$$C_{4,15} := \frac{1}{4!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dw \, dx \, dy \, dz}{(\cosh w + \cosh x + \cosh y + \cosh z)^{16}}$$

$$= -\frac{1744313209}{578605547520000} + \frac{6769726990346240}{26990346240} \zeta(3).$$

One important aspect of this separate work is the following. Beyond the above rational relations, we were not able to find any other relations whatsoever between any pair of $C_{n,k}$, regardless of the parity of $k$, for $n \geq 4$. However, we did find experimentally $m$-term relations, where $m = \lceil (n + 3)/2 \rceil$, involving $(C_{n,k}, C_{n,k+2}, \ldots, C_{n,k+2m-2})$. Subsequently, and again because we knew on the basis of experiment what to expect, we were able to prove that these universal recurrences do hold for all parameter pairs $(n,k)$ with $n = 1, 2, 3, 4$ and any complex $k$—these machinations amounting to an interesting application of Wilf–Zeilberger methods [5].

**Part II. Various numerical algorithms**

**11. Algorithm for Bessel-kernel evaluation of $C_n$**

As implied in our abstract and elsewhere, we first approached the $C_n$ integrals experimentally. Our central strategy for a high-precision numerical evaluation scheme for $F(t) = K_0(t)$ in relation (9) is to utilize a combination of an ascending series $F^{\text{asc}}(t)$ (which is well-suited for small $t$) and an asymptotic series $F^{\text{asy}}(t)$ (which is well-suited for large $t$), together with a chosen parameter $\lambda$ that is the boundary between the ‘small’ arguments and the ‘large’ $t$.

Given the formulae (11), (12) for the modified Bessel function $K_0$, there are two approaches to computing $C_n$ from (9). The first, suitable for those who have access to symbolic computing software, is simply to write the integral (9) as a sum of two integrals, one from 0 to $\lambda$, and the second from $\lambda$ to $\infty$, and then to symbolically expand suitably truncated versions of (11) and (12) and evaluate the numerous individual integrals that result. We have obtained reliable results by taking $\lambda = D/2$, where $D$ is the desired precision level in digits, and truncating the two series after $3n\lambda$ and $2\lambda$ terms, respectively. This approach suffices to obtain modestly high precision results (at least 30 digits) for $n$ up to eight or so. Beyond this level, the symbolic computing costs become too great to complete in reasonable time.
A second approach is to directly evaluate the integral in (9) using the tanh-sinh numerical quadrature scheme [9], [12, pp 312–3], where the integrand function is evaluated by either the ascending series (11) or the descending series (12), depending on whether the argument \( t \) is less than or greater than \( \lambda \). For these calculations, we found it satisfactory to take \( \lambda = D \), and to truncate the series summations when the absolute value of the term being added is less than \( 10^{-D} \) times the absolute value of the current sum.

Tanh-sinh quadrature is remarkably effective in evaluating integrals to very high precision, even in cases where the integrand function has an infinite derivative or blow-up singularity at one or both endpoints. It is well-suited for highly parallel evaluation [7], and is also amenable to computation of provable bounds on the error [8]. It is based on the transformation

\[
x = g(t) = \tanh\left[\frac{\pi/2}{2} \cdot \sinh(t)\right].
\]

In a straightforward implementation of the tanh-sinh scheme, one first calculates a set of abscissas \( x_k \) and weights \( w_k \)

\[
x_j := \tanh[\pi/2 \cdot \sinh(jh)]
\]

\[
w_j := \frac{\pi/2 \cdot \cosh(jh)}{\cosh^2[\pi/2 \cdot \sinh(jh)]},
\]

where \( h \) is the interval of integration. Then the integral of the function \( f(t) \) on \([-1, 1]\) is performed as

\[
\int_{-1}^{1} f(x) \, dx = \int_{-\infty}^{\infty} f(g(t))g'(t) \, dt \approx \sum_{-N}^{N} w_j f(x_j)
\]

where \( N \) is chosen so that the terms \( w_j f(x_j) \) are sufficiently small that they can be ignored for \(|j| > N\). Full details of a robust implementation are given in [9]. Note that in this particular application, multiple \( C_n \) can be efficiently computed for different \( n \), since the abscissas, weights and \( K_0(t) \) function values at these abscissas are independent of \( n \).

Using this approach, we have been able to evaluate \( C_n \) to very high precision (500-digit accuracy), for \( n \) as large as 1024, which is equivalent to performing a 1024-fold iterated integral in (8). Each of these runs (regardless of \( n \)) requires only about 100 s on one processor of an Apple G5 computer. Selected high-precision results are exhibited in appendix A.

12. Hypergeometric-kernel representation for \( D_n \)

Now to numerical issues for the Ising-susceptibility integrals \( D_n \). It is highly suggestive that we were able to transform the \( C_n \) integral into a one-dimensional form that admits of arbitrary-precision evaluation. For the \( D_n \), a one-dimensional form is also possible, at least formally: we do not yet know the precise convergence rate of the approach; consequently, the one-dimensional representation we achieve below may well not be practical.

A hyperbolic representation for \( D_n \) similar to (8) develops as

\[
D_n := \frac{1}{n!} \int \frac{D^2}{(\cosh x_1 + \cdots + \cosh x_n)^2} \prod_{i<j} \tanh^2 \left( \frac{x_i - x_j}{2} \right).
\]  

(38)

Knowing the identity

\[
\tanh(t - u) = \frac{\tanh t - \tanh u}{1 - \tanh t \tanh u}
\]

we fix \( n \) and ponder the formal power series

\[
\prod_{i<k} \left( \frac{t_i - t_k}{1 - t_i t_k} \right)^2 = \sum_{m_1, \ldots, m_n \geq 0} A(m_1, \ldots, m_n) t_1^{m_1} \cdots t_n^{m_n}.
\]
We intend that this define the set of $A$ coefficients. So, formally at least, we have

$$D_n = \int_0^\infty d_n(p) \, dp,$$

(39)

where the kernel $d_n$ is represented as

$$d_n(p) := \frac{2^n p}{n!} \sum_{m_k \geq 0, \text{even}} A(m_1, \ldots, m_n) \prod_{k=1}^n T_{m_k}(p),$$

where

$$T_{m}(p) := \int_0^\infty \text{tanh}^m \left( \frac{t}{2} \right) e^{-p \cosh t} \, dt,$$

a confluent hypergeometric function [1] in disguise. In fact,

$$T_{m}(p) = e^{-p} \Gamma \left( \frac{m+1}{2} \right) U \left( \frac{m+1}{2}, 1, 2p \right),$$

where $U$ is the standard confluent hypergeometric function [1]. Still formally, without regard to convergence, we claim a one-dimensional kernel for the $D_n$ as

$$d_n(p) := \frac{2^n p e^{-np}}{n!} \sum_{m_k \geq 0, \text{even}} A(m_1, \ldots, m_n) \prod_{k=1}^n \Gamma \left( \frac{m_k+1}{2} \right) U \left( \frac{m_k+1}{2}, 1, 2p \right).$$

(40)

This kernel $d_n$ is more complicated than the Bessel kernel $c_n$, which is not unexpected on the basis of the combinatorial product’s stultifying appearance in the original $D_n$ integrand. As previously intimated, we do not know the convergence rate for $d_n$, not to mention the efficiency of the integral (39), say in terms of precision versus a computational bound on the $m_k$ indices.

It is therefore admitted that this hypergeometric-kernel representation remains of theoretical interest but with as yet untapped numerical power. We do, however, posit the

**Conjecture 3.** For fixed $n$, the one-dimensional kernel $d_n(p)$ defined by (40) converges to an integrable function on $p \in (0, \infty)$, and therefore gives via (39) the correct Ising integral $D_n$.

In future research it may be useful to analyse the character of the $A$ tensor. For $n = 2$, the pattern of the $A$ coefficients is evident in the small collection:

$$\{A(2x, 2y)\}_{0 \leq x, y \leq 6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & -8 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & -12 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & -16 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 & -20 & 11 \\ 0 & 0 & 0 & 0 & 0 & 11 & -24 \end{pmatrix}. $$

Useful for calculations on the $d_n$ kernel may be the ascending and asymptotic series, respectively

$$\Gamma(a)U(a, 1, z) = \sum_{k \geq 0} \frac{(a)_k z^k}{k!} (2\psi(k+1) - \psi(k+a) - \log z),$$

(41)
Table 2. Results of qMC integration for various $D_n, E_n$. Items flagged with * are actually known (or suspected) in closed form; many of the entries are known to much higher precision than is accessible via qMC.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_n$</th>
<th>$D_{n-1}/D_n$</th>
<th>$E_n$</th>
<th>$E_{n-1}/E_n$</th>
<th>$D_n/E_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1*</td>
<td>2.00000000</td>
<td></td>
<td>2.00000000</td>
<td></td>
<td>1.00000000</td>
</tr>
<tr>
<td>2*</td>
<td>0.33333333</td>
<td>6.00000000</td>
<td>0.4582256</td>
<td>4.3973192</td>
<td>0.73288665</td>
</tr>
<tr>
<td>3*</td>
<td>0.06430739</td>
<td>5.183437</td>
<td>0.09011020</td>
<td>5.047403</td>
<td>0.713527</td>
</tr>
<tr>
<td>4*</td>
<td>0.01262502</td>
<td>5.093647</td>
<td>0.01774490</td>
<td>5.078089</td>
<td>0.714729</td>
</tr>
<tr>
<td>5</td>
<td>0.00248461</td>
<td>5.08129</td>
<td>0.00349365*</td>
<td>5.079181*</td>
<td>0.711768</td>
</tr>
<tr>
<td>6</td>
<td>4.8914 × 10^{-4}</td>
<td>5.079520</td>
<td>6.8783 × 10^{-4}</td>
<td>5.079219</td>
<td>0.711134</td>
</tr>
<tr>
<td>7</td>
<td>9.6301 × 10^{-5}</td>
<td>5.079313</td>
<td>1.3542 × 10^{-4}</td>
<td>5.07925</td>
<td>0.71112</td>
</tr>
<tr>
<td>8</td>
<td>1.8960 × 10^{-5}</td>
<td>5.07898</td>
<td>2.666 × 10^{-5}</td>
<td>5.0790</td>
<td>0.7111</td>
</tr>
</tbody>
</table>

and

$$\Gamma(a) U(a, 1, z) \sim \sum_{m \geq 0} \frac{(a)_m (-1)^m \Gamma(m + a)}{m! z^{m+a}}.$$ (42)

13. Heuristic asymptotics via quasi-Monte Carlo (qMC) methods

We have shown (theorem 3) that $D_n, E_n$ are bounded above and below by exponential decay. We also have the decay conjecture 2 that $D_n, E_n$ share the same decay constant $\Delta$. Contrast this to our proven result $C_n \rightarrow$ constant.

The quasi-Monte Carlo (qMC) integrations as shown in table 2 suggest that the decay conjecture is true and that $\Delta \approx 5$. Similar theorems and conjectures appear to be reasonable and similar for the related $E_n$, the ratios $E/D$ and so on. Yet, there are interesting open questions, such as: Is $D_{n-1}/D_n$ eventually monotonic decreasing in $n$, as table 2 suggests? Is the same true for $D_n/E_n$? The qMC algorithm we employed—a ‘spacefill-Halton hybrid’—is, for some integrands, suitable for high dimensions lying somewhat beyond the reach of the classical Halton sequences [13]. This qMC approach we employed evidently yields several good decimals even up to dimension $n = 32$. We draw this supposition from the stability of qMC for various $n$-regions, together with tests on the very much more accurately known $C_n$. (See also the recent survey on qMC [14].)

Referring to table 2: rows marked ‘*’ (and two items likewise marked) are exactly known (see closed-form evaluations for $n = 1, 2, 3, 4$ and $E_5$ in table 1) but all other entries are only numerically understood. Each table entry, for each $n$, involved $2 \cdot 10^9$ qMC points. Errors are not all rigorously known—entries here are to ‘believed’ precision, based on the qMC trends, and we admit to the usual degradation of precision with increasing dimension. Note that all of the tabulated ratios appear to approach respective constants. Though such limits are only conjectured, we have already proven that $D_n, E_n$ themselves decay at least exponentially rapidly to zero as $n \rightarrow \infty$.

There is an additional question which further computation may well address. Namely, Maillard has suggested that ratios $D_n/D_{n+2}$, meaning ratios of consecutive even or odd $D_n$ values, might converge more efficiently (or more smoothly?) based on general principles of Ising susceptibility expansions [16]. Unfortunately, the qMC values in our table 2 are evidently too imprecise to decide such an issue. Generally speaking, though, such ‘parity acceleration’ is not uncommon in other fields; for example, the pure-even, pure-odd convergents of continued
fractions are examples of split sequences that can each converge efficiently and independently to an actual common limit.

14. Quadrature for higher-dimensional $D_n$, $E_n$

Compared with the one-dimensional quadrature calculations we described earlier, multidimensional extreme-precision quadrature is very expensive indeed. Thus, to perform numerical quadrature for entities such as $D_5$, $E_5$ and beyond requires a representation in the lowest possible dimension. We have seen in section 8 that $D_n$, $E_n$ can each be reduced to an $(n-2)$-dimensional form. The details of this extra reduction can be quite intricate, so we shall summarize the explicit algebra for the elusive $D_5$, $E_5$, knowing from theorem 4 that in higher dimensions we can always have in principle follow the prescription.

For $n = 5$ let us denote variables $w, x, y, z$ and symbolically perform the interior integration over $w$ (which was $t_2$ in section 8). We use

$$A_4(x, y, z) := \left( \frac{(1 - x)(1 - xy)(1 - y)(1 - yz)(1 - z)}{(1 + x)(1 + xy)(1 + y)(1 + yz)(1 + z)} \right)^2$$

$$G_5 := 2 \left( \frac{\lambda_1 + 4}{1 + \lambda_1 w} \right) \left( \frac{\lambda_2 + 4}{1 + \lambda_2 wx} \right) \left( \frac{\lambda_3 + 4}{1 + \lambda_3 wxy} \right) \left( \frac{\lambda_4 + 4}{1 + \lambda_4 wxyz} \right)$$

$$B_5^{-1} := (1 + z + zy + zyx)(1 + w(1 + x + xy + yz)) \cdot \left( 1 + \frac{zyx}{1 + z + zy + zyx} \right).$$

Then we have, from section 8,

$$D_5 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 A_4(x, y, z) \left( \frac{\partial^4}{\partial \lambda_1 \partial \lambda_2 \partial \lambda_3 \partial \lambda_4} \right)_{\lambda_4 = 1} \int_0^1 G_5 B_5 dw \, dx \, dy \, dz$$

$$E_5 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 A_4(x, y, z) \left( \frac{\partial^4}{\partial \lambda_1 \partial \lambda_2 \partial \lambda_3 \partial \lambda_4} \right)_{\lambda_4 = 1} \int_0^1 G_5 dw \, dx \, dy \, dz.$$

The results for this procedure are two respective integrals for $D_5$, $E_5$, over the three variables $x, y, z$. As we have intimated, the details are overwhelmingly complicated, producing enormous expressions involving multivariate polynomials, rational functions and logarithms. To give but one example, we now present the stultifying triple integral we used to compute $E_5$, $E_5 = \int_0^1 \int_0^1 \int_0^1 (1 - x)^2(1 - y)^2(1 - xy)^2(1 - z)^2(1 - yz)^2(1 - xyz)^2 \times (1 - x)(1 - xy)(1 - y)(1 - yz)(1 - z)}{(1 + x)(1 + xy)(1 + y)(1 + yz)(1 + z)} \right)^2$$

$$E_5 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 A_4(x, y, z) \left( \frac{\partial^4}{\partial \lambda_1 \partial \lambda_2 \partial \lambda_3 \partial \lambda_4} \right)_{\lambda_4 = 1} \int_0^1 G_5 B_5 dw \, dx \, dy \, dz$$

$$E_5 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 A_4(x, y, z) \left( \frac{\partial^4}{\partial \lambda_1 \partial \lambda_2 \partial \lambda_3 \partial \lambda_4} \right)_{\lambda_4 = 1} \int_0^1 G_5 dw \, dx \, dy \, dz.$$
Integrals of the Ising class

\[ -42\zeta^2 - 2z + 7)x^4 - 2y(z^3(z^3 - 9z^2 - 9z + 1))y^6 + z^2(7z^4 - 14z^3 - 18z^2 - 14z + 7)y^5 + z(7z^4 + 14z^3 + 3z^2 + 14z + 7)y^4 + (z^6 - 14z^5 + 3z^4 + 84z^3 + 3z^2 - 14z + 1)y^3 - 3(3z^5 + 6z^4 - z^3 - z^2 + 6z + 3)y^2 - (9z^4 + 14z^3 - 14z^2 + 14z + 9)y + z^3 + 7z^2 + 7z + 1)x^3 + (z^2(11z^4 + 6z^2 - 6z^2 + 6z + 11))y^6 + 2z(5z^5 + 13z^4 - 2z^3 - 2z^2 + 13z + 5)y^5 + (11z^6 + 26z^5 + 44z^4 - 66z^3 + 44z^2 + 26z + 11)y^4 + (6z^5 - 4z^4 - 66z^3 - 66z^2 - 4z + 6)y^3 - 2(3z^4 + 2z^3 - 22z^2 + 2z + 33)y^2 + (6z^3 + 26z^2 + 26z + 6)y + 11z^2 + 10z + 11)x^2 - 2(z^2(5z^3 + 3z^2 + 3z + 5)y^5 + z(22z^4 + 5z^3 - 22z^2 + 5z + 22)y^4 + (5z^5 + 5z^4 - 26z^3 - 26z^2 + 5z + 5)y^3 + (3z^4 - 22z^3 - 26z^2 - 22z + 3)y^2 + (3z^3 + 5z^2 + 5z + 3)y + 5z^2 + 22z + 5)x + 15z^2 + 2z + 2(y - 1)^2(z + 1) + 2y^3(z - 1)^2(z + 1) + y^4z^2(15z^2 + 2z + 15) + y^5(15z^4 - 2z - 90z^2 - 2z + 15) + 15]/[(x - 1)^2(y - 1)^2(z - 1)^2] x(y - 1)^2(xyz - 1)^2 - [4(x + 1)(y + 1)(xyz + 1)(-z^2)y^4 + 4z(z + 1)y^3 + (z^2 + 1)y^2 - 4(z + 1)y + 4x(y^2 - 1)(y^2z^2 - 1) + x^2(z^2y^2 - 4z(z + 1)y^3 - (z^2 + 1)y^2 + 4(z + 1)(y + 1) - 1) log(x + 1)]/[x(x - 1)^3y(y - 1)^3(xy - 1)^3] - [4(y + 1)(xy + 1) x(z + 1)(x^2(z^3 - 4z - 1)y^4 + 4x(x + 1)(z^2 - 1)y^3 - (z^2 + 1)z(z - 4z - 1)y^2 - 4(x + 1)(z^2 - 1)y + z^2 - 4z - 1) log(y + 1)]/[x(y - 1)^3y(xy - 1)^3(z - 1)^3] - [4(z + 1)(yz + 1)(x^3 + 3z^2 - 3xy^2 + 4x(1 + 1)(z^2 + 1)y^3 - (z^2 + 1)z(z - 4z - 1)y^2 + 4(y + 1)x + 1)z^4 + y (y^2z^3 - 4y(y + 1)x^2 - 3(z^2y^2 + 1)x - 4(y + 1)z^3 + (5x^2y^2 + y^2 + 4x(y + 1)y + 1)z^2 + (3x + 4)y + 4z - 1) log(xy + 1)]/[x(y - 1)^3z(yz - 1)^3(xy - 1)^3)]/[x(x + 1)^2 x(y + 1)^2(z^2 + 1)^2(xyz + 1)^2] dz dy dz

There is a similar, yet more complicated integrand for \( D_5 \). The corresponding expressions for \( D_6 \) and \( E_6 \) are several times more complicated still—the computer code defining the \( D_6 \) integrand extends for over 700 lines of 60 or more characters each, even after some simplification! In appendix B we display the numerical results for \( D_5 \), \( E_5 \), \( D_6 \), \( E_6 \) obtained in this fashion.

Based on the numerical value for \( E_5 \), we applied a PSLQ integer relation detection program to recognize this constant. We succeeded in finding the experimental result

\[ E_5 \approx \frac{42 - 1984\text{Li}_4(1/2) + \frac{189\pi^4}{10} - 74\zeta(3) - 1272\zeta(3)\log 2 + 40\pi^2 \log^2 2}{3} - \frac{62\pi^2}{3} + \frac{40\pi^2 \log 2}{3} + 88 \log^4 2 + 464 \log^2 2 - 40 \log 2.\]

As before, the notation \( \approx \) is employed to emphasize that we do not yet have a formal proof for this evaluation. However, this experimental detection is quite strong—190 orders of magnitude beyond the level that could reasonably be ascribed to numerical round-off error or any other artefact.

Alas, we still have not been successful in identifying either \( C_5 \) or \( D_5 \). However, we have established, via a PSLQ computation and based on the 500-digit values given in appendix B, that:

\( \text{neither } C_5 \text{ nor } D_5 \text{ satisfies an integer linear relation} \)
with the following set of constants, where the vector of integer coefficients in the linear relation has Euclidean norm less than $4 \times 10^{12}$:

$$1, \pi, \log 2, \pi^2, \pi \log 2, \pi^2 \log 2, L_{-3}(2), \pi^3, \pi^2 \log 2, \pi \log^2 2, \log^2 2, 
\zeta(3), \pi L_{-3}(2), \log^2 2 \cdot L_{-3}(2), \pi^2 \zeta(3), \pi \log^2 2, \pi \log^4 2, G, G \pi^2, 
\pi \log 2 \cdot L_{-3}(2), \log^2 2 \cdot \zeta(3), \pi^2 L_{-3}(2), \pi^3 L_{-3}(2), 
\pi \log^2 2 \cdot L_{-3}(2), \log^2 2 \cdot L_{-3}(2), L_{-3}^2(2), \text{Im}[Li_4(e^{2 \pi i/5})], \text{Im}[Li_4(e^{3 \pi i/5})], 
\text{Im}[Li_4(i)], \text{Im}[Li_4(e^{2 \pi i/3})].$$

Here $G = \sum_{n=0}^{\infty} (-1)^n/(2n + 1)^2$ is the Catalan constant. Some constants that may appear to be ‘missing’ from this list are actually linearly redundant with this set, and thus were not included in the PSLQ search. These include

$$\text{Re}[Li_3(i)], \text{Im}[Li_3(i)], \text{Re}[Li_3(e^{2 \pi i/3})], \text{Im}[Li_3(e^{2 \pi i/3})], \text{Re}[Li_4(i)], 
\text{Re}[Li_4(e^{2 \pi i/3})], \text{Re}[Li_4(e^{2 \pi i/5})], \text{Re}[Li_4(e^{3 \pi i/5})], \text{Re}[Li_4(e^{5 \pi i/6})] \text{ and } 
\text{Im}[Li_4(e^{2 \pi i/6})].$$

In a final set of computations, we computed $D_6$ to 105-digit accuracy, and $E_6$ to 250-digit accuracy, as shown in appendix B. Unfortunately, however, we have not been able to analytically evaluate either of these constants, either experimentally or formally.

Needless to say, these computations were very demanding, both of hardware and software. Just converting the huge expressions for the integrands into working Fortran-90 code proved surprisingly tricky. For these runs, since the integrands are well-behaved at boundaries, we were able to use multi-dimensional Gaussian quadrature. We could have used tanh-sinh quadrature here, but the run times would have been somewhat longer. The computer runs themselves were performed on the Bassi system, an IBM Power5-based parallel computer system at Lawrence Berkeley Laboratory, and the Terascale Computing Facility, an Apple G5-based parallel computer system at the Virginia Institute of Technology. The computation of $D_5$ to 500 digits required 18 h on 256 CPUs; the computation of $E_6$ to 250 digits required 28 h on 256 CPUs.

We should note that computing numerical integrals sufficiently high precision to enable serious PSLQ relation searches, which typically require several hundred to several thousand digits, has only recently been achieved for a wide range of integrand functions, even for one-dimensional integrals [11, 12]. Thus our examples here of three-dimensional and four-dimensional high-precision quadrature, which require thousands of times as much computation as one-dimensional integrals, truly lie at the edge of presently available numerical techniques and computing technology. Indeed, we are not aware of any other instance of a successful three-dimensional quadrature of a nontrivial function to several-hundred-digit accuracy, much less a successful four-dimensional quadrature. In any case, our reductions to $(n-2)$ dimensions yield dramatic reductions in computational cost, compared to direct quadrature of the original $n$-dimensional integral, such as (1).

As we have noted, reasonably extensive—but far from conclusive—PSLQ experiments have failed to identify any evaluations of $C_n$, $D_n$, $E_n$ for $n > 4$, except for the experimental evaluation of $E_3$ mentioned above. The profusion of potential polylogarithmic constants of order 4 and higher, such as $Li_4(1/2)$, is one of the problems. Perhaps further study will identify the correct terms to use in these evaluations, including perhaps multi-zeta values.
15. Sum rules for susceptibility amplitudes

It is interesting that, via Painlevé differential analysis Nickel [17], using the differential theory in [24], has resolved numerical values for two infinite sums relating to the susceptibility amplitudes mentioned in the introduction, namely, recalling

\[ I_n = \frac{\pi D_n}{(2\pi)^n}, \]

\( \sum_{n=1,3,5,...} I_n = 1.000\,815\,260\,440\,212\,647\,119\,476\,363\,047\,210\,236\,9375 \ldots \) (43)

and

\( \sum_{n=2,4,6,...} I_n = 0.026\,551\,297\,359\,252\,325\,321\,072\,273\,129\,862\,563\,625\,26 \ldots . \) (44)

Our qMC values from table 2, optionally augmented by the above higher precision \( D_5, E_5, D_6, E_6 \) values, are entirely consistent with these Nickel numbers, in that we get about 20-decimal-place agreement when adding up \( D_n \) terms directly. Indeed, it would be wonderful to capture closed forms for these infinite sums.

In the same vein, for comparison we have considered \( H_n := \frac{\pi C_n}{(2\pi)^n} \). In this case we may use (9) to write

\[ \sum_{n=1,3,5,...} H_n = \pi \int_0^\infty p \sinh(K_0(p)\pi) \, dp \]

\[ = 1.010\,114\,228\,641\,994\,517\,017\,047\,968\,669\,270\,576\,602\,153\,624\,08 \ldots \] (45)

and

\[ \sum_{n=2,4,6,...} H_n = \pi \int_0^\infty p(\cosh(K_0(p)\pi) - 1) \, dp \]

\[ = 0.810\,248\,563\,808\,680\,825\,651\,910\,103\,478\,006\,142\,831\,725\,294\,803\,20 \ldots \] (46)

with the values in table 1 allowing one to confirm these values to about five places. The use of numerical values from (9) and/or estimates from (22) would allow further confirmation.

One might well ask: If the Painlevé analysis leads to high-precision values for the above sums, why does one need a closed form for say \( D_5 \) or its relatives? One answer, as posited by Maillard, is that new Ising theoretical avenues involving Fuchsian ODEs might require precise knowledge of these higher \( D_n \), starting with \( n = 5 \) [16].

16. Open problems

- We have in a sense solved what had been an open computational problem, which is to provide a workable quadrature approach for some higher susceptibility integrals \( D_{(n>4)} \). But (referring to appendix B) what is a closed form for \( D_5 \), and how far do we need to take \( D_6, E_6 \) quadrature to perform successful detection? A closed form for \( E_5 \) may well be more accessible than for \( D_5, D_6 \) based on our (conjectured) success with \( E_5 \).
- Can the two-dimensional integral (28) for \( C_5 \) be symbolically resolved? Given the historical tendency, any constants obtained would most likely shed light on those involved in the elusive \( D_5 \).
- Is there a way to calculate the hypergeometric \( D_n \)-kernel (40) efficiently, say by adroit grouping of the confluent summands? This would go a long way towards extreme-precision results for the higher \( D_n \).
- Can the methods of the exponential-decay theorem 3 be extended to find the universal decay constant \( \Delta \) in conjecture 2?
We discovered that there is a linear, rational relation $aC + bC' = c \neq 0$ between pairs $(C_{n,k}, C'_{n,k})$ with a pair being any of $(C_{1,2r}, C_{1,2r}'), (C_{2,2r}, C_{2,2r}'), (C_{1,2r+1}, C_{1,2r+1}')$, or any of $(C_{3,2r+1}, C_{3,2r+1}'), (C_{4,2r+1}, C_{4,2r+1}')$, but could find no others whatsoever. What is a conceivable, abstract-algebraic explanation for the nonexistence of such relations for certain parameter pairs? (Reference [5] has some answers.)

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Appendix A. Numerical values for $C_n$

Some 500-digit values of $C_n$ are as follows, obtained via the Bessel-kernel method (i.e., quadrature on formula (9), as in section 11). Note that $C_3, C_4$ are known in closed form, as shown in table 1. Additional data are available online [6].

\[
\begin{align*}
C_3 & : 0.7813024128964862968671874296240923563651343365428542022100062966886984 \\
& \quad 6516182180928695708322098610210423502565090305768865870552440307992607844 \\
& \quad 1998957930756967213098085932169055336438633957476728583977032551589856477 \\
& \quad 7091242889924100249818885371308788495238876782281593269542022747136358189 \\
& \quad 3707479059383768516146217899177920860361350239422760382506422626830545731 \\
& \quad 0112035525726489104581114958272539802496679979644547996026633365842227954 \\
& \quad 60055353717656228562396301698967938757682094583043 \\
C_4 & : 0.7011998560176429998165139275438458279462420038652910143788250739494056 \\
& \quad 20042015969275432592938778900585282824047235419660768699766589274854136956 \\
& \quad 482170460427514799373387126705586195085721308121642310912280637393586509 \\
& \quad 4725388655021324661906964500056599300900498070564285656606663959435880 \\
& \quad 299078826360564499252508708730415135555414121993472434832608102329416846 \\
& \quad 131914607844515860340665084683054564284935104448102000145675906901520606 \\
& \quad 312335807041636897619159644520465291911003465186463750 \\
C_5 & : 0.66575980019993742831573380830706659819749638207949765953944270353122704
\end{align*}
\]
Integrals of the Ising class

\[
\begin{align*}
C_6 : & \quad 0.64863420903100707562314984345305196088977250948162799561505088718478178 \\
& \quad 1788005579236825162345806878874630577856026938027701536062285107772881321 \\
& \quad 9046451864230224915877848383017478321796815352205732383848138693982558646 \\
& \quad 936342341276776547154769077898714018445039822718807851067223285962512604 \\
& \quad 28231725240361557398398955032766143883409972517723391720604405195636613 \\
& \quad 00113143929032927905818872272310474658497380732910871028331236398272838 \\
& \quad 82208616555375577378741536232012582576848386301001999048111 \\

C_8 : & \quad 0.63548402675916322613968489993689839348544606376327830983572450800238916 \\
& \quad 9032937027339756848064043523541244586304149729548068352148088167360413521 \\
& \quad 3106589949509550400483852455903797822155130617495416682684784964714237427 \\
& \quad 133251149187214860658155399169628214158157338079963831987798171528043525 \\
& \quad 12401360865903524067274041124570335795353976286238890061527340722256616 \\
& \quad 0112092016558520594402250356380003372739373873276161874833986524785410240 \\
& \quad 35242690609713926955186696954805468489718103770282950 \\

C_{16} : & \quad 0.6305039461732372635029526556068741948431621720810304775087911973705871 \\
& \quad 1342851877659192763501910566601982818857722862650586379030259021251047164 \\
& \quad 211123005584652503444076601171694306367509196134429529516776253103930303 \\
& \quad 076338954225894425176347983900160262575156905022542575244252372560006409374 \\
& \quad 3274793526468603824852871916766521434983765365722151925039591683519381181 \\
& \quad 431312145704351598562120835335305224258186275684402427456280670422676 \\
& \quad 72215207468384216339709665856983880586456285865788069 \\

C_{64} : & \quad 0.63047350337348367964884362028165338625355988808600159169054764761699744132 \\
& \quad 8971548840508887766760368013971973130286652582942316698018882715049609224281 \\
& \quad 36760548138258969829428890200757474414834941919486830723130043582819515980 \\
& \quad 1230323481904015476905081982491781473477053899423295297585854151547336 \\
& \quad 49367946425876887687306315849054817465842889811317033041580964887667713 \\
& \quad 70178153221634249747232867090089887482393273633450319432600881162531433 \\
& \quad 3378748354017557255302217585186907309507726430904149
Appendix B. Numerical values for $D_n$, $E_n$

The values for $D_n$, $E_n$ below all started with the respective, dimensionally reduced integrands as described in section 14. Each integral in this appendix is thus $(n-2)$-dimensional. These integrand expressions were then converted to valid Fortran-90 code, via the Mathematica FortranForm[] function, together with some offline processing to divide the full expression into ‘chunks’ of modest enough size that they could be handled by the IBM XLFortran compiler. We then prepared special three-dimensional and four-dimensional, high-precision Gaussian integration programs, which invoked parallel execution using Message Passing Interface (MPI) parallel programming constructs. The resulting programs were then run on the ‘Bassì’ system at the Lawrence Berkeley National Laboratory, which is a large cluster of IBM Power5 nodes, or the Terascale Computing Facility, which is a large cluster of Apple G5 nodes. The computation of $D_5$ to 500 digits required 18 h on 256 CPUs; the computation of $E_6$ to 250 digits required 28 h on 256 CPUs. As additional data become available, it will be made available online [6].

$D_5$:
0.00248460576234154799505915390974963506067764248751615870769216182213
785691543573792689948724512018706872110639152950115862069944997542265656264
670853284124500116682230004545703266326867938489615198247961303525258515107
1543863811369617492249285557807628042894777027871092191891116063406312541360
385984091828078640186930726810988548230378878848758305835157858552564199694
869146134911273630946052409340088718283870643464218618204509029973535634113
727612202408834546315017113540844197840922456685

$E_5$:
0.0034936537117295217406880672791842515696329449551413146836988233699924
Integrals of the Ising class

References

[22] Tracy C 2005 Private communication
[23] Tracy C 1976 Unpublished notes
