The Landen Transformation

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1. The generating equation.

The starting point is the generating equation

$$\sqrt{1-l^2\sin^2 u} = a\sqrt{1-k^2\sin^2 t} + b\cos t$$

which we differentiate and get

$$-\frac{l^2\sin 2u\, du}{2\sqrt{1-l^2\sin^2 u}} = -\left[\frac{a\, k^2\sin t\cos t}{\sqrt{1-k^2\sin^2 t}} + b\sin t\right]dt \Rightarrow$$

$$\frac{du}{\sqrt{1-l^2\sin^2 u}} = \frac{2\left(b\sin t\sqrt{1-k^2\sin^2 t} + a\, k^2\sin t\cos t\right)}{l^2\sin 2u\sqrt{1-k^2\sin^2 t}}dt.$$

The dividend on the right side resembles

$$\sin(t+v) = \sin t\cos v + \cos t\sin v = \sin t\sqrt{1-k^2\sin^2 t} + \cos t\,(k\sin t)$$

where $\sin v = k\sin t$, so we set $2u = t+v$ and $b = ak$ and deduce

$$\frac{du}{\sqrt{1-l^2\sin^2 u}} = \frac{2akdt}{l^2\sqrt{1-k^2\sin^2 t}}.$$

Let's square the generating equation

$$1-\frac{l^2}{2}+\frac{l^2}{2}\cos 2u = a^2\left(1+2k\cos t\sqrt{1-k^2\sin^2 t}+k^2\cos 2t\right)$$

and substitute the expression

$$\cos 2u = \cos(t+v) = \cos t\sqrt{1-k^2\sin^2 t} - k\sin^2 t = \cos t\sqrt{1-k^2\sin^2 t} - k\left(\frac{1-\cos 2t}{2}\right).$$

The left side becomes

$$1-\frac{l^2}{2}-\frac{kl^2}{4}+\frac{l^2}{2}\cos t\sqrt{1-k^2\sin^2 t}+\frac{kl^2}{4}\cos 2t$$

so we can determine l and a from

$$\frac{l^2/2}{1-l^2/2-kl^2/4}=\frac{2l^2}{4-2l^2-kl^2}=2k\Rightarrow l^2\left(1+2k+k^2\right)=4k\Rightarrow l^2=\frac{4k}{(1+k)^2}$$

$$\Rightarrow a^2=1-l^2/2-kl^2/4=l^2/4k=(1+k)^{-2}\Rightarrow a=\frac{1}{1+k}.$$

If we examine the following equations

$$\begin{aligned}\cos 2u&=\cos t\sqrt{1-k^2\sin^2 t}-k\sin^2 t\\ \sin 2u&=\sin t\sqrt{1-k^2\sin^2 t}+k\cos t\sin t\end{aligned}$$

we notice that

$$\begin{aligned}1+k\cos 2u&=\sqrt{1-k^2\sin^2 t}\left(\sqrt{1-k^2\sin^2 t}+k\cos t\right)\\ k\sin 2u&=k\sin t\left(\sqrt{1-k^2\sin^2 t}+k\cos t\right)\end{aligned}$$

or

$$\sqrt{1-k^2\sin^2 t}=\frac{1+k\cos 2u}{(1+k)\sqrt{1-l^2\sin^2 u}},\, k\sin t=\frac{k\sin 2u}{(1+k)\sqrt{1-l^2\sin^2 u}}$$

and especially

$$v\ =\ 2u\ -\ t=\operatorname{arc\,sin}(k\sin t)=\operatorname{arc\,tan}\left(\frac{k\sin t}{\sqrt{1-k^2\sin^2 t}}\right)=\operatorname{arc\,tan}\left(\frac{k\sin 2u}{1+k\cos 2u}\right)$$

which gives the correct quadrant for the angle t when $u>\pi/2$.

Hence, the ascending Landen transformation can be written as

$$\int_0^{u_0}\frac{du}{\sqrt{1-l^2\sin^2 u}}=\frac{1+k}{2}\int_0^{t_0}\frac{dt}{\sqrt{1-k^2\sin^2 t}},\, u=\frac{1}{2}\left(t+\operatorname{arc\,tan}\left(\frac{k\sin t}{\sqrt{1-k^2\sin^2 t}}\right)\right)$$

and descending

$$\int_0^{t_0}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\left(1+\sqrt{1-l^2}\right)\int_0^{u_0}\frac{du}{\sqrt{1-l^2\sin^2 u}},\, t=2u-\operatorname{arc\,tan}\left(\frac{k\sin 2u}{1+k\cos 2u}\right).$$

Here

$$l=\frac{2\sqrt{k}}{1+k}\iff k=\frac{1-\sqrt{1-l^2}}{1+\sqrt{1-l^2}}=\frac{l^2}{\left(1+\sqrt{1-l^2}\right)^2}.$$

2. Elliptic integral of the second kind.

Let's apply the ascending Landen transformation to the integral

$$E(l,u)=\int_0^{u_0}\sqrt{1-l^2\sin^2 u}\,du.$$

By using the equations

$$\frac{du}{\sqrt{1-l^2\sin^2 u}}=\frac{1+k}{2}\frac{dt}{\sqrt{1-k^2\sin^2 t}}$$

$$\cos 2u=\cos t\sqrt{1-k^2\sin^2 t}-k\sin^2 t$$

we get

$$1-l^2\sin^2 u=1-\frac{4k}{(1+k)^2}\frac{1-\cos t\sqrt{1-k^2\sin^2 t}+k\sin^2 t}{2}=$$

$$\frac{1+2k+k^2-2k-2k^2\sin^2 t+2k\cos t\sqrt{1-k^2\sin^2 t}}{(1+k)^2}=$$

$$\frac{2\left(1-k^2\sin^2 t\right)-(1-k^2)+2k\cos t\sqrt{1-k^2\sin^2 t}}{(1+k)^2}$$

and further

$$E(l,u)=\int_0^{t_0}\frac{2\left(1-k^2\sin^2 t\right)-(1-k^2)+2k\cos t\sqrt{1-k^2\sin^2 t}}{2\,(1+k)\sqrt{1-k^2\sin^2 t}}\,dt$$

$$=\frac{1}{1+k}\int_0^{t_0}\left[\sqrt{1-k^2\sin^2 t}-\frac{1-k^2}{2\sqrt{1-k^2\sin^2 t}}+k\cos t\right]dt$$

$$=\frac{1}{1+k}\left(E(k,t)+\frac{1-k^2}{2}F(k,t)+k\sin t\right)=\frac{E(k,t)+k\sin t}{1+k}-\frac{1-k}{2}F(k,t).$$

Let's generate an algorithm for calculating this integral

$$E_0=\frac{E_1+k_1S_1}{1+k_1}-\frac{1-k_1}{2}F_1=$$

$$\frac{1}{1+k_1}\left[\frac{E_2+k_2S_2}{1+k_2}-\frac{1-k_2}{2}F_2+k_2S_1\right]-\frac{1-k_1}{2}\frac{1+k_2}{2}F_2=$$

$$\frac{1}{1+k_1}\left[\frac{1}{1+k_2}\left[\frac{E_3+k_3S_3}{1+k_3}-\frac{1-k_3}{2}F_3+k_2S_2\right]-\frac{1-k_2}{2}\frac{1+k_3}{2}F_3+k_1S_1\right]$$

$$-\frac{1-k_1}{2}\frac{1+k_2}{2}\frac{1+k_3}{2}F_3 = E_3/A - BF_3 + C$$

where

$$A = (1+k_1)(1+k_2)(1+k_3)$$

$$B = \frac{(1-k_3)}{2(1+k_1)(1+k_2)} + \frac{(1-k_2)(1+k_3)}{2^2(1+k_1)} + \frac{(1-k_1)(1+k_2)(1+k_3)}{2^3}$$

$$C = \frac{k_3 \sin t_3}{(1+k_1)(1+k_2)(1+k_3)} + \frac{k_2 \sin t_2}{(1+k_1)(1+k_2)} + \frac{k_1 \sin t_1}{1+k_1}.$$

From this we get the algorithm (we DO NOT use the AGM-scheme)

$$A = 1, B = C = 0$$
$$\text{WHILE}(k \ > \text{errorbound})$$

$$k = \left(\frac{k}{1+\sqrt{1-k^2}}\right)^2$$

$$t = 2t - \arctan\left(\frac{k \sin 2t}{1 + k \cos 2t}\right)$$

$$B = B\left(\frac{1+k}{2}\right)$$

$$B = B + \frac{1-k}{2A}$$

$$A = A(1+k)$$

$$C = C + k \sin t \,/A$$

$$\text{WHILE} - \text{END}$$

$$E = (1/A - B)\, t + C.$$

3. Elliptic integral of the third kind.

The integral to be transformed is

$$\Pi(n; l, u) = \int_0^{u_0} \frac{\mathrm{du}}{(1 - n \sin^2 u)\sqrt{1 - l^2 \sin^2 u}}.$$

We again use the same equations as before

$$1 - n\sin^2 u = 1 - n\left(\frac{1 - \cos t\sqrt{1 - k^2\sin^2 t} + k\sin^2 t}{2}\right)$$

$$= \frac{2 - n - kn\sin^2 t + n\cos t\sqrt{1 - k^2\sin^2 t}}{2}$$

$$\Pi(n; l, u) = \frac{1+k}{2}\int_0^{t_0} \frac{2}{2 - n - kn\sin^2 t + n\cos t\sqrt{1 - k^2\sin^2 t}} \frac{dt}{\sqrt{1 - k^2\sin^2 t}} =$$

$$(1+k)\int_0^{t_0} \frac{2 - n - kn\sin^2 t - n\cos t\sqrt{1 - k^2\sin^2 t}}{\left(2 - n - kn\sin^2 t\right)^2 - n^2\left(1 - \sin^2 t\right)\left(1 - k^2\sin^2 t\right)} \frac{dt}{\sqrt{1 - k^2\sin^2 t}} =$$

$$(1+k)\int_0^{t_0} \frac{2 - n - kn\sin^2 t - n\cos t\sqrt{1 - k^2x^2}}{4(1-n) - n\left(4k - n\,(1+k)^2\right)\sin^2 t} \frac{dt}{\sqrt{1 - k^2\sin^2 t}} =$$

$$\frac{1+k}{4(1-n)}\int_0^{t_0} \frac{2 - n - kn\sin^2 t - n\cos t\sqrt{1 - k^2\sin^2 t}}{\left(1 - m\ \sin^2 t\right)\sqrt{1 - k^2\sin^2 t}} dt.$$

Here

$$m = \frac{n\left(4k\ - n(1+k)^2\right)}{4(1-n)} = \frac{n(l^2 - n)(1+k)^2}{4(1-n)}.$$

Note that $m = 0$ when $n = l^2$. This is a special case which will be considered later.

We can now apply same kind of method we used in the previous elliptic integral of the second kind:

$$\Pi(n; l, u) = \frac{1+k}{4(1-n)}\int_0^{t_0} \frac{2(1-n) + n - kn/m + kn/m - kn\sin^2 t - n\cos t\sqrt{1 - k^2\sin^2 t}}{\left(1 - m\ \sin^2 t\right)\sqrt{1 - k^2\sin^2 t}} dt$$

$$= \int_0^{t_0} \left(\frac{a}{\left(1 - m\sin^2 t\right)\sqrt{1 - k^2\sin^2 t}} + \frac{b}{\sqrt{1 - k^2\sin^2 t}} - \frac{c\cos t}{1 - m\sin^2 t}\right) dt$$

$$= a\,\Pi(m; k, t) + b\,F(k, t) - c\int_0^{t_0} \frac{\cos t}{1 - m\sin^2 t} dt$$

where

$$c = \frac{1+k}{4}\frac{n}{1-n}, m = \frac{n\left(4k\ - n(1+k)^2\right)}{4(1-n)} = \frac{n(l^2-n)\,(1+k)^2}{4(1-n)}$$

$$b = ck/m = \frac{1+k}{4}\frac{4k}{(l^2-n)\,(1+k)^2} = \frac{1+k}{4}\frac{l^2}{l^2-n}, a\ = \frac{1+k}{2} - b + c.$$

The last elementary integral has the solutions

$$\int_0^{t_0} \frac{\cos t}{1 - m\sin^2 t} dt = \frac{\operatorname{arc\,tanh}(\sqrt{m}\sin t)}{\sqrt{m}}, m > 0$$

$$\int_0^{t_0} \frac{\cos t}{1 - \sin^2 t} dt = \sin t, m = 0$$

$$\int_0^{t_0} \frac{\cos t}{1 - m\sin^2 t} dt = \frac{\operatorname{arc\,tan}\left(\sqrt{-m}\sin t\right)}{\sqrt{-m}}, m < 0$$

and the values of the degenerate integrals $\Pi(m;0,t)$ and $\Pi(m;1,t)$ are

$$\Pi(m;0,t) = \int_0^{t_0} \frac{dt}{1 - m\sin^2 t} = \int_0^{t_0} \frac{1+\tan^2 t}{1+(1-m)\tan^2 t} dt = \int_0^{t_0} \frac{\tan^{(1)} t}{1+(1-m)\tan^2 t} dt$$

$$= \frac{\operatorname{arc\,tan}\left(\sqrt{1-m}\tan t\right)}{\sqrt{1-m}}, m\ <\ 1$$

$$= \tan t, m = 1$$

$$= \frac{\operatorname{arc\,tanh}\left(\sqrt{m-1}\tan t\right)}{\sqrt{m-1}}, m > 1, \sin t\ < m^{-1/2}.$$

The first integral has to be modified in order to take care of the periodic trigonometric functions

$$\tan\left(\Pi\sqrt{1-m}\right) = \sqrt{1-m}\tan t \Rightarrow \tan\left(t - \Pi\sqrt{1-m}\right) = \frac{\left(1-\sqrt{1-m}\right)\tan t}{1+\sqrt{1-m}\tan^2 t}$$

$$= \frac{\left(1-\sqrt{1-m}\right)\cos t\sin t}{\cos^2 t + \sqrt{1-m}\sin^2 t} = \frac{\left(1-\sqrt{1-m}\right)\sin 2t}{1+\cos 2t\ + \sqrt{1-m}(1-\cos 2t)} = \frac{n\sin 2t}{1+n\cos 2t}$$

$$\Rightarrow \Pi\ = \frac{1}{\sqrt{1-m}}\left(t - \operatorname{arc\,tan}\left(\frac{n\sin 2t}{1+n\cos 2t}\right)\right),\ n\ = \frac{1-\sqrt{1-m}}{1+\sqrt{1-m}} = \frac{m}{\left(1+\sqrt{1-m}\right)^2}.$$

The value of the second degenerate integral $\Pi(m;1,t)$ is

$$\Pi(m;1,t)=\int_0^{t_0}\frac{dt}{\left(1-m\sin^2 t\right)\cos t}=\int_0^{t_0}\frac{\cos t\,dt}{\left(1-m\sin^2 t\right)\left(1-\sin^2 t\right)}=$$

$$\frac{1}{1-m}\int_0^{t_0}\left(\frac{\cos t}{1-\sin^2 t}-\frac{m\cos t}{1-m\sin^2 t}\right)dt$$

$$=\frac{\operatorname{arc\,tanh}(\sin t)-\sqrt{m}\operatorname{arc\,tanh}(\sqrt{m}\sin t)}{1-m},m\geqq 0,m\neq 1$$

$$=\frac{\operatorname{arc\,tanh}(\sin t)+\sqrt{-m}\operatorname{arc\,tan}(\sqrt{-m}\sin t)}{1-m},m<0.$$

We can determine the value of $\Pi(1;1,t)$ by means of a transformation formula for $\Pi(1;k,t)$:

$$\Pi(1;k,t)=\int\frac{dt}{\left(1-\sin^2 t\right)\sqrt{1-k^2\sin^2 t}}=\int\frac{dt}{\cos^2 t\sqrt{1-k^2\sin^2 t}}=$$

$$\frac{\tan t}{\sqrt{1-k^2\sin^2 t}}-\int\frac{k^2\sin t\cos t\tan t\,dt}{\left(1-k^2\sin^2 t\right)^{3/2}}=\frac{\tan t}{\sqrt{1-k^2\sin^2 t}}+\int\frac{1-k^2\sin^2 t-1}{\left(1-k^2\sin^2 t\right)^{3/2}}dt$$

$$=\frac{\tan t}{\sqrt{1-k^2\sin^2 t}}+\int\frac{dt}{\sqrt{1-k^2\sin^2 t}}-\int\frac{dt}{\left(1-k^2\sin^2 t\right)\sqrt{1-k^2\sin^2 t}}$$

$$=\frac{\tan t}{\sqrt{1-k^2\sin^2 t}}+F(k,t)-\Pi(k^2;k,t)\Rightarrow$$

$$\Pi(1;1,t)=\frac{\tan t}{\cos t}+\int\frac{\cos t\,dt}{1-\sin^2 t}-\Pi(1;1,t)=\frac{\tan t}{\cos t}+\operatorname{arc\,tanh}(\sin t)-\Pi(1;1,t)$$

$$\Rightarrow\Pi(1;1,t)=\frac{1}{2}\left(\operatorname{arc\,tanh}(\sin t)+\frac{\tan t}{\cos t}\right).$$

An algorithm almost identical to that of the integral of the second kind is now easy to construct

$$\Pi_0=a_1\Pi_1+b_1F_1-c_1T_1=a_1(a_2\Pi_2+b_2F_2-c_2T_2)+b_1\frac{1+k_2}{2}F_2-c_1T_1$$

$$=a_1\left(a_2(a_3\Pi_3+b_3F_3-c_3T_3)+b_2\frac{1+k_3}{2}F_3-c_2T_2\right)$$

$$+b_1\frac{1+k_2}{2}\frac{1+k_3}{2}F_3-c_1T_1$$

and we get

$$A = a_1 a_2 a_3$$

$$B = a_1 a_2 b_3 + a_1 b_2 \frac{1+k_3}{2} + b_1 \frac{1+k_2}{2} \frac{1+k_3}{2}$$

$$C = a_1 a_2 c_3 T_3 + a_1 c_2 T_2 + c_1 T_1.$$

The algorithm (when $m \neq k^2$ and $k < 1$)

$$A = 1, B = C = 0$$

$$\text{WHILE}(k > \text{errorbound})$$

$$c = \frac{m}{1-m}$$

$$b = \frac{k^2}{k^2 - m}$$

$$a = 2 - b + c$$

$$m = \frac{m(k^2 - m)}{1-m}$$

$$k = \left(\frac{k}{1 + \sqrt{1-k^2}} \right)^2$$

$$t = 2t - \arctan\left(\frac{k \sin 2t}{1 + k \cos 2t} \right)$$

$$a = a\frac{1+k}{4}, b = b\frac{1+k}{4}, c = c\frac{1+k}{4}, m = m\left(\frac{1+k}{2}\right)^2$$

$$T = \arctan(h)\left(\sqrt{|m|}\sin t\right) / \sqrt{|m|}$$

$$B = \frac{1+k}{2} B$$

$$B = B + Ab$$

$$C = C + AcT$$

$$A = Aa$$

$$\text{WHILE} - \text{END}$$

$$\text{IF}(m < 1)$$

$$u = \frac{1}{\sqrt{1-m}}$$

$$m = \frac{m}{\left(1+\sqrt{1-m}\right)^2}$$

$$u = u\left(t - \arctan\left(\frac{m\sin 2t}{1+m\cos 2t}\right)\right)$$

$$\text{ELSE}$$

$$u = \frac{1}{\sqrt{m-1}}\operatorname{arc\,tanh}(\sqrt{m-1}\tan t)$$

$$\Pi = Au + Bt - C.$$

In the case of $l^2 = n$ we get the transformation

$$\Pi(l^2; l, u) = \frac{1+k}{4(1-n)}\int_0^{t_0} \frac{2 - n - kn\sin^2 t - n\cos t\sqrt{1-k^2\sin^2 t}}{\sqrt{1-k^2\sin^2 t}}dt =$$

$$\frac{1+k}{4(1-n)}\int_0^{t_0} \frac{2 - n - n/k + n/k - kn\sin^2 t - n\cos t\sqrt{1-k^2\sin^2 t}}{\sqrt{1-k^2\sin^2 t}}dt =$$

$$\int_0^{t_0}\left(a\sqrt{1-k^2\sin^2 t} + \frac{b}{\sqrt{1-k^2\sin^2 t}} - c\cos t\right)dt \Rightarrow$$

$$\Pi(l^2; l, u) = a\,E(k,t) + b\,F(k,t) - c\sin t$$

where the coefficients are

$$n = \frac{4k}{(1+k)^2} \Rightarrow 1 - n = \left(\frac{1-k}{1+k}\right)^2 \Rightarrow \frac{n}{1-n} = \frac{4k}{(1-k)^2} \Rightarrow$$

$$c = \frac{1+k}{4}\frac{4k}{(1-k)^2} = \frac{k(1+k)}{(1-k)^2} \Rightarrow a = c/k = \frac{1+k}{(1-k)^2} \Rightarrow$$

$$b = \frac{1+k}{2} - a + c = \frac{1+k}{2}\left(1 - \frac{2(1-k)}{(1-k)^2}\right) = -\frac{(1+k)^2}{1-k}.$$

So

$$\Pi(l^2;l,u) = \frac{1+k}{(1-k)^2}\left(E(k,t) - \frac{1-k^2}{2}F(k,t) - k\sin t\right).$$

On the other hand

$$E(l,u) = \frac{1}{1+k}\left(E(k,t) - \frac{1-k^2}{2}F(k,t) + k\sin t\right) \Rightarrow$$

$$\Pi(l^2;l,u) = \left(\frac{1+k}{1-k}\right)^2\left(E(l,u) - \frac{2k}{1+k}\sin t\right) = \frac{1}{1-l^2}\left(E(l,u) - \frac{2k\sin t}{1+k}\right)$$

$$= \frac{E(l,u) - \left(1-\sqrt{1-l^2}\right)\sin t}{1-l^2}.$$

Because

$$\sin t = \frac{\sin 2u}{(1+k)\sqrt{1-l^2\sin^2 u}} = \frac{\left(1+\sqrt{1-l^2}\right)\sin u\cos u}{\sqrt{1-l^2\sin^2 u}}$$

the result becomes

$$\Pi(l^2;l,u) = \frac{E(l,u)}{1-l^2} - \frac{l^2}{1-l^2}\frac{\sin u\cos u}{\sqrt{1-l^2\sin^2 u}}.$$

We can now use this to calculate $\Pi(1;l,u)$:

$$\Pi(1;l,u) = \frac{\tan u}{\sqrt{1-l^2\sin^2 u}} + F(l,u) - \Pi(l^2;l,u) =$$

$$\frac{\tan u}{\sqrt{1-l^2\sin^2 u}} + \frac{l^2}{1-l^2}\frac{\sin u\cos u}{\sqrt{1-l^2\sin^2 u}} + F(l,u) - \frac{E(l,u)}{1-l^2} =$$

$$\frac{\tan u}{\sqrt{1-l^2\sin^2 u}}\left(\frac{1-l^2+l^2\cos^2 u}{1-l^2}\right) + F(l,u) - \frac{E(l,u)}{1-l^2} =$$

$$\frac{\tan u\sqrt{1-l^2\sin^2 u} - E(l,u)}{1-l^2} + F(l,u).$$

Here is a complete set of formulae to construct an algorithm to calculate the values of $\Pi(m;k,t)$ by descending Landen transformation when $0 \le m \le 1$. (I have done this and checked the results from the tables in the book Milton Abramovitz/Irene Stegun: Handbook of mathematical functions).

I would be extremely grateful to receive comments on the readability of this script (you must take into consideration that I am only a layperson from Finland fascinated by the beauty of the realm of the elliptic integrals).