Referee report on “Experimental computation with oscillatory integrals”

This manuscript, covering integrals $I(p)/\sqrt{p}$ of powers $p$ of the sinc function $(\sin x)/x$ on $[0, \infty)$, is a good illustration of methods of experimental mathematics, even though the Conjecture (p. 2) remains unresolved. However, it suffers from a major notational inconsistency that should be corrected. Details are given below.

One will also find below a suggested complementary approach to the sinc-power-Hurwitz zeta function integrals and material related to the situation of $p = 2N$. An important early reference [medhurst] has been omitted and should be cited.

**Major points**

A. Starting in (9), the authors adopt an unusual convention for the Hurwitz zeta function $\zeta(z, a)$. The authors should write for (9) $\zeta(p, x)$ as is customary. Not only is the unusual convention potentially very confusing, but matters come to the fore in Lemma 1 (by Crandall). There, $\zeta(s, a)$ is indeed the as-usually-specified Hurwitz zeta function. This is trivial to verify: put the free parameter $\lambda = 0$ and use the relation $\Gamma(x, 0) = \Gamma(x)$, where $\Gamma(x, y)$ is the incomplete gamma function.

With the standard notation for the Hurwitz zeta function, we have the integrals (10),

$$I(p) = \pi^{1-p} \int_0^1 \sin^p(\pi t) \zeta(p, t) dt.$$

Now it is clear that asymptotic forms of $\zeta$ are of interest for large values of the first argument, and the second argument restricted to $(0, 1]$. The following is one alternative.

**Lemma.** (private communication, M. Coffey) We have uniformly for $0 < t \leq 1$,

$$\zeta(p, t) = t^{-p} + (t + 1)^{-p} + \sum_{m=0}^M \left(-p \atop m\right) [\zeta(p + m) - 1] t^m + O(t^{M+1}).$$

This formula holds for $\tilde{p} \equiv \text{Re } p > -M$.

**Proof.** This follows from

$$\zeta(p, t) = t^{-p} + (t + 1)^{-p} + \sum_{j=2}^{\infty} j^{-p}(1 + t/j)^{-p}$$

$$= t^{-p} + (t + 1)^{-p} + \sum_{m=0}^{\infty} \left(-p \atop m\right) [\zeta(p + m) - 1] t^m.$$
Therefore, we have, for example,
\[
\lim_{p \to \infty} I(p) = \lim_{p \to \infty} \sqrt{p^{1-p}} \int_0^1 \frac{\sin^p(\pi t)}{t^p} dt.
\]

B. Put
\[
I_n^{MR}(0) = \frac{2}{\pi} \int_0^\infty \left( \frac{\sin x}{x} \right)^n dx.
\]
Then the following result is contained in [medhurst] for \( n \) an even integer,
\[
I(n) = \frac{\pi}{2} \sqrt{n} I_n^{MR}(0)
\]
\[
= \sqrt{\frac{3}{2\pi}} \left[ 1 - \frac{13}{20} \frac{1}{n^2} + \frac{1}{1120} \frac{27}{n^3} + \frac{1}{3200} \frac{52791}{n^4} + \frac{1}{6656000} \frac{482427}{n^5} \right. \\
- \left. \frac{1}{10035200000} \frac{124996631}{n^6} + O \left( \frac{1}{n^7} \right) \right].
\]
Obviously the reference [medhurst] should be cited. It would seem desirable that it also be cited in the similar reference [8].

C. The expression (29) or (31) for \( p = 2N \) even can be rewritten by recognizing the Eulerian numbers \( \left\langle \frac{n}{k} \right\rangle \). We then have
\[
\frac{I(2N)}{\sqrt{2N}} = \frac{\pi}{2(2N-1)!} \left\langle \frac{2N-1}{N-1} \right\rangle.
\]
Here,
\[
(-1)^N \sum_{m=0}^{2N} (-1)^m \binom{2N}{N-m} m^{2N-1} = \sum_{m=0}^{N} (-1)^m \binom{2N}{m} (N-m)^{2N-1} = A(2N-1, N) = \left\langle \frac{2N-1}{N-1} \right\rangle,
\]
in the \( A \) notation of [L. Comtet’s book Advanced Combinatorics, 1974] (p. 243). This identification is very useful because we may then apply known asymptotic results for \( A \). From [giladi] (6.16) we have for \( n \gg 1 \),
\[
A \left( n, \frac{n-1}{2} \right) \sim \frac{2\sqrt{3}}{e} \left( \frac{n+1}{e} \right)^n.
\]
Together with Stirling’s formula for $\Gamma(2N)$, we obtain the expected result, $I(2N) \sim \sqrt{3\pi/2}$ as $N \to \infty$.

**Minor points**

Two lines above (14), the sum should read $\sum_{j > 1000}$.

The numerical value for $I(3)$ given in (27) requires a minus sign.

As indicated below, the classical formula of Kummer should be cited from a standard textbook and/or the original 1847 paper by E. Kummer.

Equation (25) is another instance of the potential ambiguity in the notation for $\zeta(s, a)$. Presumably the $'$ applies to the second argument in the present context.

**Remark.** The observation concerning the function $\sin^p(\pi t)\zeta(p, t)$ at $p = 3.5$ may be extended. Namely, suppose that $p = k + \delta$ where $k = \lfloor p \rfloor$ and $\delta = \{p\} \neq 0$, the fractional part of $p$. Then the $(k+1)$-th and higher order derivatives of this integrand function display discontinuous behavior at 0 and 1.

**References**


[medhurst] R. G. Medhurst and J. H. Roberts, Evaluation of the integral $I_n(b) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos(bx) dx$, Math. Comp. 19, 113-117 (1965).

For formatting consistency, the volume numbers in [5], [20], and [23] should be given in boldface, in the style of the other references.

[3] June; supply publication details

[7] There is a surprising change of title for this reference. The first word is “Surprising” instead of “Some”.

