Subdifferential Properties of Quasiconvex and Pseudoconvex Functions: Unified Approach

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Abstract. In this paper, we are mainly concerned with the characterization of quasiconvex or pseudoconvex nondifferentiable functions and the relationship between those two concepts. In particular, we characterize the quasiconvexity and pseudoconvexity of a function by mixed properties combining properties of the function and properties of its subdifferential. We also prove that a lower semicontinuous and radially continuous function is pseudoconvex if it is quasiconvex and satisfies the following optimality condition: \(0 \in \partial f(x)\) \(\Rightarrow f\) has a global minimum at \(x\). The results are proved using the abstract subdifferential introduced in Ref. 1, a concept which allows one to recover almost all the subdifferentials used in nonsmooth analysis.

Key Words. Nonsmooth analysis, abstract subdifferential, quasiconvexity, pseudoconvexity, mixed property.

1. Preliminaries

Throughout this paper, we use the concept of abstract subdifferential introduced in Ausel et al. (Ref. 1). This definition allows one to recover almost all classical notions of nonsmooth analysis. The aim of such an approach is to show that a lot of results concerning the subdifferential properties of quasiconvex and pseudoconvex functions can be proved in a unified way, and hence for a large class of subdifferentials.

We are interested here in two aspects of convex analysis: the characterization of quasiconvex and pseudoconvex lower semicontinuous functions and the relationship between these two concepts.

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Throughout this paper, \(
\tilde{\partial}
\) stands for the following abstract notion of subdifferential.

**Definition 1.1.** See Ref. 1. We call subdifferential, denoted by \(\tilde{\partial}\), any operator which associates a subset \(\tilde{\partial}f(x)\) of \(X^*\) to any lower semicontinuous \(f: X \to \mathbb{R} \cup \{+\infty\}\) and any \(x \in X\), and satisfies the following properties:

1. **(P1)** \(\tilde{\partial}f(x) = \{x^* \in X^* | \langle x^*, y-x \rangle + f(x) \leq f(y), \forall y \in X\}\), whenever \(f\) is convex;
2. **(P2)** \(0 \in \tilde{\partial}f(x)\), whenever \(x \in \text{dom } f\) is a local minimum of \(f\);
3. **(P3)** \(\tilde{\partial}(f+g)(x) \subseteq \tilde{\partial}f(x) + \tilde{\partial}g(x)\), whenever \(g\) is a real-valued convex continuous function which is \(\tilde{\partial}\)-differentiable at \(x\).

Here, \(g\) is \(\tilde{\partial}\)-differentiable at \(x\) means that both \(\tilde{\partial}g(x)\) and \(\tilde{\partial}(-g)(x)\) are nonempty. We say that a function \(f\) is \(\tilde{\partial}\)-subdifferentiable at \(x\) when \(\tilde{\partial}f(x)\) is nonempty.

This abstract subdifferential allows one to recover, by a unique definition, a large class of subdifferentials. Among this class, let us mention some examples (for precise definitions, see Ref. 1 and references therein):

- the Clarke–Rockafellar subdifferential \(\partial^{CR}f\);
- the lower and upper Dini subdifferentials \(\partial^{-}f\) and \(\partial^{+}f\);
the lower Hadamard subdifferential $\partial^{H-}f$, also called contingent subdifferential;
the Fréchet subdifferential $\partial^F f$;
the Lipschitz smooth subdifferential $\partial^{LS}f$, also called proximal subdifferential $\partial^p f$ whenever $X$ is a Hilbert space.

The verification that these subdifferentials satisfy properties (P1) to (P3) is straightforward (see Ref. 4) and does not require any assumption on the Banach space. The use of the word $\partial$-differentiable is justified by the following observation:

a function is $\partial^{D^+}$-differentiable
[resp. $\partial^{H-}$-differentiable, $\partial^F$-differentiable]
at $x$ if and only if it is Gâteaux [resp. Hadamard, Fréchet]
differentiable at $x$.

In all the sequel, we will focus our attention on the class of subdifferentials which satisfy properties (P1) to (P3) and one of the following inclusions:

$\partial \subseteq \partial^{D^+}$ or $\partial \subseteq \partial^{CR}$.

This assumption is not needed in all the proofs and in any case is not really restrictive, since the Clarke-Rockafellar and the upper Dini subdifferentials are the biggest (in the sense of inclusion) among the classical subdifferentials. In particular, one has

$$\partial^{LS} \subseteq \partial^F \subseteq \partial^{H-} \subseteq \partial^{CR}$$

$$\cap$$

$$\partial^{D^+} \subseteq \partial^{D^+}.$$ 

Let us recall the definition of the Clarke-Rockafellar subdifferential and the definition of the upper Dini subdifferential which will be helpful for the sequel,

$$\partial^{CR} f(x) = \{x^* \in X^* | \langle x^*, v \rangle \leq f^1(x, v), \forall v \in X\},$$

with $f^1(x, v) = \sup_{\varepsilon > 0} \int_0^\varepsilon \sup_{\zeta > 0} \int_0^\zeta \inf_{a B_\varepsilon(x)} \inf_{d B_\varepsilon(v)} [f(u + t\zeta) - \alpha]/t,$

(one can take $\alpha = f(u)$ whenever $f$ is l.s.c.);

$$\partial^{D^+} f(x) = \{x^* \in X^* | \langle x^*, v \rangle \leq f^{D^+}(x, v), \forall v \in X\},$$

with $f^{D^+}(x, v) = \limsup_{\iota \searrow 0} [f(x + \iota v) - f(x)]/\iota.$
Definition 1.2. See Ref. 1. A norm $\| \cdot \|$ on $X$ is said to be $d$-smooth if all the (real-valued, convex, continuous) functions of the following form are $d$-differentiable:

(i) $d_{[a,b]}^2(x) := \min_{c \in [a,b]} \| x - c \|^2,$

where $[a, b]$ is a closed segment in $X$,

(ii) $A_2(x) := \sum_n \mu_n \| x - v_n \|^2,$

where $\sum_n \mu_n = 1, \mu_n \geq 0,$ and $(v_n)$ converges in $X$.

We say that a Banach space admits a $d$-smooth renorm if it admits an equivalent norm which is $d$-smooth. Let us give some interesting examples of such $d$-smooth norms:

(a) a norm is $d^D$-smooth iff it is $d^D$-differentiable on $X \backslash \{0\},$ that is, according to a previous remark, iff it is Gateaux-differentiable on $X \backslash \{0\}$ (elementary proof);

(b) any norm is $d^C^R$-smooth since the functions $d^C_{[a,b]}$ and $A_2$ are locally Lipschitz.

Concerning (a), we note that the same equivalence, with respect to Hadamard or Frechet differentiability, is true for $d = d^H$ or $d = d^F.$ So, any separable [resp. reflexive] Banach space admits a $d^D$-smooth [resp. $d^F$-smooth] renorm; see, e.g., Ref. 6.

Finally, we need the following result which is a special case of the mean-value inequality stated in Ref. 1.

Proposition 1.1. Let $X$ be a Banach space with a $d$-smooth renorm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. For any $a \in \text{dom } f$ and $b \in X$ such that $f(a) < f(b),$ there exist $c \in [a, b] \text{ and sequences } (x_n) \text{ converging to } c$ and $(x^*_n), x^*_n \in \partial f(x_n),$ such that $\langle x^*_n, d - x_n \rangle > 0, \quad \forall n,$

for every $d = c + t(b - a)$ with $t > 0.$

2. Two Characterizations of Quasiconvexity

This section is devoted to the study of quasiconvex functions. More precisely, we extract from the proof of the main result of Aussel et al. (Ref. 7, Theorem 1) a mixed property ($Q_1$) combining properties of the function

\[ Q_1 \quad (x_n') \leq c + t(b - a) \quad \forall t > 0. \]
and properties of the subdifferential. Using the same arguments as in Ref. 7, Theorem 1, we show that this mixed property characterizes the quasi-convexity of l.s.c. functions. This characterization will play a central role in the sequel.

Let us recall that a function is quasiconvex if, for any \( x, y \in \mathcal{X} \) and any \( z \in [x, y] \), one has

\[
f(z) \leq \max\{f(x), f(y)\}.
\]

It is well known that, in the differentiable case, quasiconvex functions satisfy

\[
\langle \nabla f(x), y - x \rangle > 0 \Rightarrow f(x) \leq f(y).
\]

In the nondifferentiable case, this property becomes

\[(Q) \quad \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0 \Rightarrow f(x) \leq f(y).
\]

Our first result asserts that the following slightly stronger mixed property actually characterizes the quasiconvexity of l.s.c. functions:

\[(Q_r) \quad \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0 \Rightarrow f(z) \leq f(y), \quad \forall z \in [x, y].
\]

**Theorem 2.1.** Let \( X \) be a Banach space with a \( \partial \)-smooth renorm, and let \( f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) be a l.s.c. function. Then, the following assertions are equivalent:

1. \( f \) is quasiconvex;
2. \( \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0 \Rightarrow f(z) \leq f(y), \quad \forall z \in [x, y].\)

**Proof.** (i) \( \Rightarrow \) (ii) Case \( \partial f \subset \partial^{CR} f \). Let \( x, y \in \text{dom} \partial f \) and \( x^* \in \partial f(x) \) such that

\[
f^*(x, y - x) \geq \langle x^*, y - x \rangle > 0.
\]

Hence, there exists \( \epsilon > 0 \) such that, for every \( n \in \mathbb{N}^+ \), one can find \( x_n \in B_{\epsilon/n}(x) \) [and then \( y - x_n \in B_n(y - x) \)] and \( t_n \in [0, 1/n] \) which satisfy

\[
f(x_n + t_n(y - x_n)) > f(x_n).
\]

Since \( f \) is quasiconvex, the previous inequality implies that, for every \( t \in [0, 1] \), we have

\[
f(x_n + t(y - x_n)) \leq f(y),
\]

and hence, by lower semicontinuity of \( f \),

\[
f(z) \leq f(y), \quad \forall z \in [x, y].
\]

(i) \( \Rightarrow \) (ii) Case \( \partial f \subset \partial^{BD} f \). The proof is very simple. Indeed, if \( x, y \) and \( x^* \in \partial^{BD} f(x) \) satisfy \( f^{BD}(x, y - x) > 0 \), then \( f(\tilde{z}) > f(x) \) for some
It satisfies property \((Q)\) for \(d = d^{CR}\) but is not quasiconvex.

In previous works, Hassouni (Ref. 8, locally Lipschitz function) and Penot and Quang (Ref. 9, continuous function) characterize quasiconvexity using property \((Q)\). But property \((Q_s)\) is proved to be the good expression in the more general setting of l.s.c. functions.

In the following result, we show that, for a continuous function or radially continuous function (that is, continuous on each segment), both properties are equivalent.

**Proposition 2.1.** Let \(X\) be a Banach space with a \(\delta\)-smooth renorm. Every l.s.c. radially continuous function which satisfies \((Q)\) is quasiconvex.

**Proof.** Let \(x, y \in X\), and let \(z \in [x,y]\) be such that \(f(z) > \max[f(x), f(y)]\). Applying Proposition 1.1 to the points \(x\) and \(z\), we obtain two sequences \((a_n)\) and \((a^*_n)\), with \(a_n \to z \in [x, y]\), \(a^*_n \in \partial f(a_n)\), and \(\langle a^*_n, y - x_n \rangle > 0\), \(\forall n \in \mathbb{N}\). Hypothesis (ii) implies that, for every \(n \in \mathbb{N}\), the point \(z_n\) defined on \([x_n, y]\) by \(z_n = \lambda x_n + (1 - \lambda)y\) satisfies \(f(z_n) \leq f(y)\), and hence by lower semicontinuity \(f(z) \leq f(y)\).

Obviously, every l.s.c. quasiconvex function also satisfies property \((Q)\). But the converse is false, in general, for l.s.c. functions. Consider for example the function \(f\) defined by

\[
\begin{align*}
f(x) &= 0, \quad \text{if } x = 0 \text{ or } 1, \\
f(x) &= 1, \quad \text{otherwise.}
\end{align*}
\]

It satisfies property \((Q)\) for \(\delta = \delta^{CR}\) but is not quasiconvex.

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\[
f(a) = f(b) = \min_{c \in [x,y]} f(c).
\]
The equivalence between the quasiconvexity of a function and the quasimonotonicity of its subdifferential has been studied in Auselle et al. (Refs. 7, 10) and in Luc (Ref. 10) for \( d = d_{CR} \).

The aim of the next two results is to show that the implication

\[ f \text{ quasiconvex} \implies \partial f \text{ quasimonotone} \]

can be stated without any regularity assumption on \( f \) and that the reverse implication can be seen as a consequence of Theorem 2.1.

**Proposition 2.2.** Let \( X \) be a Banach space. Then, the Clarke–Rockafellar subdifferential and the upper Dini subdifferential of any quasiconvex function \( f: X \to \mathbb{R} \cup \{+\infty\} \) are quasimonotone.

**Proof.** The case of the Clarke–Rockafellar subdifferential consists of a slight refinement of the second part of the proof of Ref. 7, Theorem 1. We include the proof for completeness. Suppose that \( f \) is quasiconvex, and let \( x, y \in \text{dom } \partial_{CR} f \) and \( x^* \in \partial_{CR} f(x) \) be such that \( \langle x^*, y-x \rangle > 0 \). It is sufficient to prove that \( f(\cdot, y-x) \leq 0 \). For every \( \varepsilon > 0 \), there exists \( \gamma \in ]0, \varepsilon[ \) such that \( \langle x^*, v-x \rangle > 0 \), \( \forall v \in B_{\varepsilon}(y) \). Let fix \( \delta \in B_{\varepsilon}(y) \). Since \( f(\cdot, \delta-x) \) is strictly positive, there exist \( \varepsilon' \in ]0, \varepsilon[ \), \( u_0 \in B_{\varepsilon'}(x) \), \( \alpha \in B_{\varepsilon'}(f(x)) \), and \( \tau \in ]0, 1[ \) which satisfy

\[ \delta - u_0 \in B_{\varepsilon'}(\delta-x) \quad \text{and} \quad f(u_0 + \tau(\delta - u_0)) > \alpha > f(u_0). \]

These inequalities, according to the quasiconvexity assumption, yield

\[ f(\delta + t(u_0 - \delta)) \leq f(\delta), \quad \text{for all } \tau \in ]0, 1[. \]

Moreover, from the choice of \( \gamma \) and \( \varepsilon' \), the direction \( u_0 - \delta \) is an element of \( B_{\varepsilon'}(x-y) \). Thus, summing up the previous steps, for any \( \varepsilon > 0 \), there exists \( \gamma > 0 \) such that, for any \( v \in B_{\varepsilon}(y) \), any \( \beta \in B_{\varepsilon}(f(y)) \), \( f(v) \leq \beta \), and any...
Consequently, if \( x^* \in \partial f(y,x-y) \) is such that \( x^* > 0 \), we immediately obtain that \( f(x) < f(y) \), and hence that \( f^D+(y,x-y) < 0 \), thus showing that \( dD+ f \) is a quasimonotone point-to-set map.

**Theorem 2.2.** Let \( X \) be a Banach space with a \( \delta \)-smooth renorm, and let \( f: X \to \mathbb{R} \cup \{ +\infty \} \) be a l.s.c. function. Then, \( f \) is quasiconvex iff \( \partial f \) is quasimonotone.

**Proof.** Since the abstract subdifferential \( \partial f \) is assumed to be included in \( \partial^{CR} f \), the “only if” part follows from Proposition 2.2.

To prove the “if” part, we follow the lines of the second part of the proof of Ref. 1, Theorem 5.4. Let us assume that \( \partial f \) is quasimonotone. According to Theorem 2.1, we just have to show that the l.s.c. function \( f \) satisfies property \((Q_s)\). Let \( x \in \text{dom} \partial f, y \in \text{dom} f (x \neq y) \), and \( z \in [x,y] \) such that \( f(z) > f(y) \). Applying Proposition 1.1 to \( y \) and \( z \), we get a sequence \( (y_n) \) converging to \( y \in [z,y] \) and a sequence \( (y_n^*) \) such that \( y_n^* \in \partial f(y_n) \) and \( \langle y_n^*, y - y_n \rangle > 0 \), \( \forall n \).

By quasimonotonocity of \( \partial f \), we have
\[
\langle x^*, x - y_n \rangle \geq 0,
\]
for all \( n \) and all \( x^* \in \partial f(x) \). Then,
\[
\langle x^*, y - x \rangle = \| y - x \| \langle y - y_n^*, y - x \rangle \leq 0.
\]
Thus, the function \( f \) satisfies property \((Q_s)\), and the proof is complete. \( \square \)

**3. Quasiconcave and Quasiaffine Functions**

In this section, we consider the case of quasiconcave and quasiaffine functions for which analogous mixed characterizations are proved using appropriate mixed properties.
A function $f$ is said to be quasiconcave if $-f$ is quasiconvex and it is said to be quasiaffine if $f$ and $-f$ are quasiconvex.

Let us consider the following mixed property

$$(Q^-) \quad \exists x^* \in \partial f(x) : \langle x^*, y-x \rangle < 0 \Rightarrow f(z) \geq f(y), \quad \forall z \in [x, y].$$

The characterization of the quasiconcavity of a function $f$ by the property $(Q^-)$ cannot be deduced in general from Theorem 2.1. Indeed, considering the function $(-f)$ in place of a function $f$ in Theorem 2.1 yields a characterization of the quasiconcavity of $f$ in terms of $-\partial(-f)$, which in general is different from $\partial f$. The function $f(x) = \sqrt{|x|}$ is a standard counterexample for $d_{C, R}$, since $d_{C, R} f(0) = \mathbb{R}$ and $d_{C, R}(-f)(0) = \emptyset$; however, the subdifferential equality $\partial^{CR} f = -\partial^{CR}(-f)$ is true if $f$ is assumed to be locally Lipschitz (see, e.g., Ref. 11).

**Proposition 3.1.** Let $X$ be a Banach space with a $\partial$-smooth renorm, and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a continuous function. Then, $f$ is quasiconcave if and only if, for any $x, y$ of $X$, the function $f$ satisfies

$$(Q^-) \quad \exists x^* \in \partial f(x) : \langle x^*, y-x \rangle < 0 \Rightarrow f(z) \geq f(y), \quad \forall z \in [x, y].$$

**Proof.** It follows the same line as the proof of Theorem 2.1. Let us assume that $f$ satisfies property $(Q^-)$ and that $x, y \in X$ and $z \in [x, y]$ are such that $f(z) < f(y)$. From Proposition 1.1, there exist a sequence $(a_n)$ converging to $ae[z, y]$ and a sequence $(a^*_n)$, $a^*_n \in \partial f(a_n)$, which verify

$$\langle a^*_n, y-a_n \rangle > 0, \quad \forall n \in \mathbb{N}.$$ 

Let $t_1$ and $t_2$ be some positive numbers, with $0 \leq t_1 \leq t_2$, such that

$$z = a + t_1 (a-y), \quad x = a + t_2 (a-y),$$

and define two sequences $(x_n)$ and $(z_n)$ by

$$z_n = a + t_1 (a_n - y), \quad x_n = a + t_2 (a_n - y).$$

For $n$ large enough, we have $\langle a^*_n, x_n - a_n \rangle < 0$, and hence according to property $(Q^-)$, $f(z_n) \geq f(x_n)$. Finally, since the function $f$ is continuous, we get $f(z) \geq f(x)$.

Conversely, assume that $f$ is quasiconcave. Let $x \in \text{dom} \, \partial f$, $y \in X$, $z \in [x, y]$, and $x^* \in \partial f(x)$ such that $\langle x^*, y-x \rangle < 0$. If $\partial f \subset \partial^{CR} f$, then the points $x$ and $y$ satisfy $f'(x, x-y) > 0$, and hence there exists $\epsilon > 0$ such that, for any $n \in \mathbb{N}$, one can find $x_n \in B_{\epsilon/n}(x)$ and $t_n \in [0, 1/n]$ satisfying

$$f(x_n + t_n (x_n - y)) > f(x_n).$$
For any \( n \), the two points \( x_n \) and \( z_n = \lambda x_n + (1 - \lambda) y \) [with \( \lambda \) defined by \( z = \lambda x + (1 - \lambda) y \)] are on the segment line \( ]x_n + t_n (x_n - y), y[ \). Using the quasiconcavity of \( f \), we have

\[
f(\lambda x_n + (1 - \lambda) y) \geq f(y),
\]

and hence, since \( f \) is upper semicontinuous, \( f(z) \geq f(y) \).

If \( \partial f = \partial f^+ \), then we have \( f^+(x, x - y) > 0 \), and hence for every \( n \), there exists \( t_n \in [0, 1/n] \) verifying \( f(x + t_n (x - y)) > f(x) \). But \( f \) is quasiconcave and \( x \in ]x + t_n (x - y), y[ \), for all \( n \). Consequently \( f(z) \geq f(y) \) and \( f \) satisfies property \((Q^-)\). \( \square \)

Let us mention that one can easily find counterexamples which show that, for the "if" part of the proposition, the continuity hypothesis cannot be replaced by a lower or upper semicontinuity hypothesis. Consider for example the indicator function of the set \( \{0, 1\} \) or the indicator function of the set \( \mathbb{R} \setminus \{0\} \).

As an immediate consequence of Theorem 2.1 and Proposition 3.1, we obtain the following characterization of quasiaffine functions.

**Corollary 3.1.** Let \( X \) be a Banach space with a \( \partial \)-smooth renorm, and let \( f: X \to \mathbb{R} \cup \{+\infty\} \) be a continuous function. Then, the following assertions are equivalent:

(i) \( f \) is quasiaffine;

(ii) \( \exists x^* \in \partial f(x): \langle x^*, d \rangle > 0 \Rightarrow f(x + td) \leq f(x + t_2d), \forall t_1 \leq t_2. \)

Indeed, the conjunction of the mixed properties \((Q_i)\) and \((Q^-i)\) is equivalent to

\[ \exists x^* \in \partial f(x): \langle x^*, d \rangle > 0 \Rightarrow f_{x,d}: t \mapsto f(x + td) \text{ is nondecreasing on } \mathbb{R}, \]

which is exactly assertion (ii). Another equivalent way to describe assertion (ii) of the corollary is

\[ \exists x \in [x, y], \exists x^* \in \partial f(z): \langle x^*, y - x \rangle > 0 \Rightarrow f(z) \leq f(y). \]

Using the corresponding expression in the differentiable case, Martos (Ref. 12) characterized quasiaffine functions in finite-dimensional spaces.

4. Pseudoconvexity

The original definition of pseudoconvexity by Mangasarian in the differentiable setting was extended by many authors in the following way
A function $f$ is said to be pseudoconvex if, for any $x, y \in X$, one has
\[ \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle \geq 0 \Rightarrow f(x) \leq f(y). \]

As in the differentiable case, every pseudoconvex function $f$ satisfies the following fundamental properties:

(a) every local minimizer of $f$ is a global minimizer;
(b) $0 \in \partial f(x) \Rightarrow f$ has a global minimum at $x$.

Unfortunately, the relation between quasiconvexity and pseudoconvexity is not simple. For example, the function $f(x) = x^3$ is quasiconvex and not pseudoconvex, whereas the function defined by
\[ f(x) = \begin{cases} 1, & \text{if } x = 0, \\ x^2, & \text{elsewhere}, \end{cases} \]
is pseudoconvex, with e.g. $\partial = \partial^{CR}$, but not quasiconvex.

Extending a result of Crouzeix and Ferland (Ref. 2) stated for differentiable functions, the following theorem gives the precise relation between pseudoconvexity and quasiconvexity for l.s.c. radially continuous functions.

**Theorem 4.1.** Let $X$ be a Banach space with a $\partial$-smooth renorm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. and radially continuous function. Then, the following assertions are equivalent:

(i) $f$ is pseudoconvex;
(ii) $f$ is quasiconvex and $0 \in \partial f(x) \Rightarrow f$ has a global minimum at $x$.

**Proof.** (i) $\Rightarrow$ (ii). From the definition of pseudoconvexity, it is obvious that, if $0 \in \partial f(x)$, then
\[ f(x) \leq f(y), \quad \text{for every } y \in X. \]

So $x$ is a global minimum of the function $f$. On the other hand, the function $f$ is l.s.c., radially continuous and satisfies property $(Q)$ of Section 2, since every pseudoconvex function satisfies property $(Q)$. Then, from Proposition 2.1, $f$ is quasiconvex.

(ii) $\Rightarrow$ (i). Let $x \in \text{dom } \partial f$, $y \in X$, and $x^* \in \partial f(x)$ such that $\langle x^*, y - x \rangle \geq 0$. If $0 \in \partial f(x)$, then $x$ is a global minimum of $f$ and in particular one has $f(x) \leq f(y)$. Otherwise $[0 \notin \partial f(x)]$, there exists $d \in X$ such that $\langle x^*, d \rangle > 0$. Now let us define the sequence $(y_n)$ by
\[ y_n = y + \frac{1}{2n} \|d\| d. \]
Theorem 4.2. Let $X$ be a Banach space with a $d$-smooth renorm, and let $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. and radially continuous function. Then, the following assertions are equivalent:

We claim that $0$ is not an element of $\partial f(y)$.

Indeed, since $f'(x, y-x) > 0$, there exist $\epsilon > 0$, $x' \in B_\epsilon(x)$, and $\tau \in [0, 1[$ such that $f(x' + \tau(y-x')) > f(x')$. 

In finite dimensions, for $d = d_{\text{dim}}$ and for an upper semicontinuous function, the implication $(ii) \Rightarrow (i)$ is given in Komlosi (Ref. 13).

Now using the relation between quasiconvexity and pseudoconvexity (Theorem 4.1) and the characterization of quasiconvexity by the quasimonotonicity of its subdifferential (Theorem 2.2), it is possible to give two characterizations of l.s.c. radially continuous pseudoconvex functions.

Let us recall that a point-to-set map $A: X \rightarrow X^*$ is said to be pseudomonotone if, for any $x, y \in X$, one has

$$\exists x^* \in A(x): \langle x^*, y-x \rangle > 0 \Rightarrow \forall y^* \in A(y), \langle y^*, y-x \rangle > 0.$$
From Theorem 4.1, the function $f$ is quasiconvex and then we have $f(x') < f(y)$.

Since $f$ is pseudoconvex, this last inequality implies that, for every $a \in \partial f(y)$, the real $\langle a, y - x' \rangle$ is strictly positive, which proves the claim.

Now, let us observe that there exists $\eta > 0$ such that $\langle x^*, u - x \rangle > 0$, for every $u \in B_\eta(y)$. Then, the pseudoconvexity of $f$ implies that, for every $u \in B_\eta(y), f(u) \geq f(x)$. Since $\langle x^*, x - y \rangle \geq 0$, we also obtain that $f(x) \geq f(y)$. Consequently, $y$ is a local minimizer of $f$ and from property (P2), $0$ is an element of $\partial f(y)$, which contradicts the claim.

(i) $\Rightarrow$ (iii) Case $\partial f \subseteq \partial^0 f$. We use a direct proof. Let $x \in \text{dom} \partial f, y \in X$, and $x^* \in \partial f(x)$ with $\langle x^*, y - x \rangle > 0$. Then, there exists $\tau \in ]0, 1[$ such that $f(x + \tau(y - x)) > f(x)$. Since $f$ is quasiconvex (by Theorem 4.1), we obtain $f(y) > f(x)$. Now, the pseudoconvexity of the function implies that, for every $y^* \in \partial f(y)$, we have $\langle y^*, x - y \rangle < 0$, thus proving that $\partial f$ is pseudomonotone.

(iii) $\Rightarrow$ (i). Using Theorem 4.1 again, we will prove that $f$ is pseudo-convex. Indeed, the point-to-set map $\partial f$ is pseudomonotone, and hence quasimonotone. According to Theorem 2.2, the function $f$ is then quasiconvex. On the other hand, if $x$ is not a minimizer of $f$, there exists $y \in X$ such that $f(y) < f(x)$. Applying Proposition 1.1, one can find $a \in \text{dom} \partial f$ and $a^* \in \partial f(a)$ such that $\langle a^*, x - a \rangle > 0$ and then, by the pseudomonotonicity of $\partial f, \langle x^*, x - a \rangle > 0$ for every $x^* \in \partial f(x)$. Hence, $0$ is not an element of $\partial f(x)$. Consequently, $f$ satisfies

$$0 \in \partial f(x) \Rightarrow x \text{ global minimum of } f,$$

and the proof is complete.

The idea of the proof of (i) $\Rightarrow$ (iii) for $\partial^{cR}$ is due to Penot and Quang (Ref. 9).

5. Radially Nonconstant Functions

Following Komlosi (Ref. 14), we say that a function $f$ is radially nonconstant (r.n.c. for short) if one cannot find any line segment on which $f$ is constant; i.e.,

$$\forall x, y \in X, \exists z \in ]x, y[, \text{ with } f(x) \neq f(z).$$

The purpose of this section is to give a different light to some of the preceding results by proving them under the r.n.c. assumption, instead of the continuity (or radial continuity) assumption.
**Proposition 5.1.** Let $X$ be a Banach space with a $\delta$-smooth renorm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. and radially nonconstant function. If $f$ is pseudoconvex, then $\partial f$ is pseudomonotone.

Let us remark that Proposition 5.1 does not allow one to recover the implication (i) $\Rightarrow$ (iii) of Theorem 4.2, since there exists continuous (and convex) functions which are not radially nonconstant.

**Proof.** Assume, for a contradiction, that $f$ is pseudoconvex and that $\partial f$ is not pseudomonotone. So, there exist $x, y \in \text{dom } \partial f$, $x^* \in \partial f(x)$, and $y^* \in \partial f(y)$ such that $\langle x^*, y - x \rangle > 0$ and $\langle y^*, y - x \rangle \leq 0$. Since $f$ is pseudoconvex, these inequalities imply that $f(y) \geq f(x)$ and $f(x) \geq f(y)$. Moreover, for any $z \in [x, y]$, we have

$$\langle x^*, z - x \rangle = (\|z - x\|/\|y - x\|) \langle x^*, y - x \rangle > 0,$$

and hence by pseudoconvexity $f(x) = f(y) \leq f(z)$. The function $f$ being r.n.c., there exists $\zeta \in [x, y]$ such that $f(\zeta) = f(x) = f(y)$. Let $\lambda \in [f(x), f(\zeta)]$. By semicontinuity of $f$, we have $f(u) > \lambda$ for all $u$ in a neighborhood $V$ of $\zeta$. Let $\tilde{x} \in [x, \zeta]$ be such that $\|\tilde{x} - \zeta\|/\|x - \zeta\| \in V \cap [x, \zeta]$. Since $f$ is r.n.c., there exists $a \in [\tilde{x}, \zeta]$ satisfying $f(a) = f(\tilde{x})$. Now, let us suppose that $f(a) < f(\tilde{x})$. According to Proposition 1.1, there exist a sequence $(b_n)$ converging to $b \in [a, \tilde{x}]$ and a sequence $(b_n^*)$, $b_n^* \in \partial f(b_n)$ for which

$$\langle b_n^*, y - b_n \rangle > 0, \quad \forall n.$$

Since $f$ is pseudoconvex and l.s.c., it follows that $f(y) \geq f(b)$. But $b \in [\tilde{x}, \zeta] \subset V$, and then $f(b) > \lambda > f(y)$. Consequently, we have $f(a) > f(\tilde{x})$. Using the same arguments one can prove that this is impossible too, thus supplying the desired contradiction.

As an immediate consequence of Proposition 5.1, we obtain the following relation between pseudoconvexity and quasiconvexity.

**Corollary 5.1.** Let $X$ be a Banach space with a $\delta$-smooth renorm. Every l.s.c., radially nonconstant, and pseudoconvex function is quasiconvex.

**Proof.** If $f$ is a l.s.c., r.n.c., and pseudoconvex function, then according to Proposition 5.1, its subdifferential is pseudomonotone and hence quasimonotone. By Theorem 2.2, the function is then quasiconvex.
We say that a function \( f: X \to \mathbb{R} \cup \{+\infty\} \) is:

(a) strictly quasiconvex iff, for any \( x, y \in X \) and any \( z \in ]x, y[ \), one has \( f(z) < \max\{f(x), f(y)\} \);

(b) strictly pseudoconvex iff, for any \( x, y \in X \), the following holds:

\[ \exists x^* \in \partial f(x): \langle x^*, y-x \rangle \geq 0 \Rightarrow f(x) < f(y). \]

Obviously every strictly quasiconvex [resp. strictly pseudoconvex] function is quasiconvex [resp. pseudoconvex]. Diewert (Ref. 3) observed that, conversely, every quasiconvex r.n.c. function is strictly quasiconvex.

A consequence of the next result is that the same holds true for pseudoconvex functions.

**Proposition 5.2.** Let \( X \) be a Banach space with a \( \delta \)-smooth renorm, and let \( f: X \to \mathbb{R} \cup \{+\infty\} \) be a l.s.c. function. Then, the following assertions are equivalent:

(i) \( f \) is pseudoconvex and radially nonconstant;

(ii) \( f \) is strictly quasiconvex and strictly pseudoconvex.

**Proof.** (i) \( \Rightarrow \) (ii). According to Corollary 5.1, the radially nonconstant function \( f \) is quasiconvex and hence strictly quasiconvex. To prove that \( f \) is strictly pseudoconvex, let us assume, for a contradiction, that there exist \( x \in \text{dom } \partial f, y \in \text{dom } f, \) and \( x^* \in \partial f(x) \) such that \( \langle x^*, y-x \rangle \geq 0 \) and \( f(x) \geq f(y) \). Consequently, for any \( z \in ]x, y[ \), we have \( \langle x^*, z-x \rangle \geq 0 \), and hence \( f(z) \geq f(x) = f(y) \). But this contradicts the strict quasiconvexity of \( f \).

(ii) \( \Rightarrow \) (i). This is obvious, since every strictly quasiconvex function is radially nonconstant. \( \square \)

**6. Final Remarks**

(i) Throughout this paper, two different assumptions are made. The first one is the inclusion of the subdifferential operator in the Clarke–Rockafellar or upper Dini subdifferential [Assumption (H)]. The other one, is the existence of a \( \delta \)-smooth renorm.

A swift reading is sufficient to mention that these hypotheses are not necessary in all the proofs. They are, in a sense, complementary. Indeed, in most equivalence proofs, one hypothesis allows to show an implication and the other is used to prove the converse.
To be more precise, a direct or indirect use of Proposition 1.1 requires the existence of a $\partial$-smooth renorm, whereas Assumption (H) is necessary for proofs based on the definition of $\partial^{D^+}$ or $\partial^{CR}$.

(ii) Another way to characterize generalized convexity and to describe the relationships between different kinds of generalized convexity is to use criteria and properties based on the notion of generalized derivative instead of the subdifferential. In recent works [Komlosi (Refs. 14, 15), Penot (Ref. 16), etc.], some abstract approach with generalized derivatives are developed. Since some subdifferentials covered by our abstract definition are not defined by a generalized derivative (the Fréchet and proximal subdifferentials, for example), these two approaches have to be considered as complementary points of view.

References


