Uniformly smooth renorming of Banach spaces with modulus of convexity of power type 2.

(English summary)


The moduli of convexity and smoothness of a Banach space $X$ are the functions

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon \right\}$$

and

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\},$$

defined for $\varepsilon \in [0, 2]$ and $\tau \geq 0$, respectively. They provide a quantitative measure of the properties that $X$ is uniformly convex and uniformly smooth respectively. $X$ is $p$-uniformly convex if $\delta_X(\varepsilon) \geq c_1 \varepsilon^p$ and $X$ is $q$-uniformly smooth if $\rho_X(\tau) \leq c_2 \tau^q$. The main result of the paper establishes a relation between these two notions, which improves an earlier estimate by S. A. Rakov [Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 135 (1984), 120–134; MR0741702 (85f:46036)]:

Theorem: There is an absolute constant $k_1$ such that if $X$ satisfies

$$\liminf_{\varepsilon \to 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} \geq \frac{1}{8(1 + b)}$$

for some $b \geq 0$, then $X$ is $q$-uniformly smooth renormable for $q = 1 + \frac{1}{1+k_1 b}$.

Along with the proof, the authors obtain the following inequality of independent interest:

Theorem: There exists an absolute positive constant $k_2$ such that for every Banach space $X$ and any $x, y \in X$ with $\|x\| = \|y\| = 1$,

$$\limsup_{\tau \to 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq 1 + k_2 \left( 2 \limsup_{\tau \to 0} \frac{\rho_X(\tau)}{\tau^2} - 1 \right).$$

Reviewed by Antonio Avilés

References

13. O. Hanner, On the uniform convexity of $L^p$ and $l^p$, Ark. Mat. 3 (1956) 239–244. MR0077087 (17,987d)
Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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