The First Derivative of Ramanujans Qubic Continued Fraction

Nikos Bagis
Department of Informatics
Aristotele University of Thessaloniki Greece
bagkis@hotmail.com

Abstract

We give the complete evaluation of the first derivative of the Ramanujans qubic continued fraction using Elliptic functions. The Elliptic functions are easy to handle and give the results in terms of Gamma functions and radicals from tables. The results appear to be new.

keywords Jacobian Elliptic Functions; Continued Fractions; Ramanujan; Qubic Fraction; Derivative

1 Introduction

The Ramanujan's Cubic Continued Fraction is (see [3], [7], [8], [9], [11]).

$$V(q) := \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\ldots \tag{1}$$

Our main result is the evaluation of the first derivative of Ramanuajan's qubic fraction. For this, we follow a different way from previous works and use the theory of Elliptic functions, which is more easy to handle. Our method consists to find the complete polynomial equation of the Qubic fraction in terms only of the inverse elliptic nome k_r, which is a solvable, in radicals, quartic equation. Using the derivative of k_r which we evaluate in Section 2 of this article, we find the desired formula.
We give some definitions first
Let

$$(a;q)_k = \prod_{n=0}^{k-1}(1-aq^n) \tag{2}$$

Then we define

$$f(-q) = (q;q)_\infty \tag{3}$$

and

$$\Phi(-q) = (-q;q)_\infty \tag{4}$$

Also let

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1-x^2\sin^2(t)}}dt \tag{5}$$

be the elliptic integral of the first kind.
We denote

$$\theta_4(u,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui} \tag{6}$$

the Elliptic Theta function of the 4th-kind.

$$\prod_{n=1}^{\infty} (1-q^{2n})^6 = \frac{2kk'K(k)^3}{\pi^3 q^{1/2}} \tag{7}$$

and from

$$q^{1/3} \prod_{n=1}^{\infty} (1+q^n)^8 = 2^{-4/3} \left(\frac{k}{1-k^2}\right)^{2/3} \tag{8}$$

we get

$$f(-q)^8 = \prod_{n=1}^{\infty} (1-q^n)^8 = \frac{2^{8/3}}{\pi^4} q^{-1/3} k^{2/3} (k')^{8/3} K(k)^4 \tag{9}$$

The variable k is defined from the equation

$$\frac{K(k')}{K(k)} = \sqrt{r} \tag{10}$$

where r is positive , $q = e^{-\pi\sqrt{r}}$ and $k' = \sqrt{1-k^2}$. Note also that whenever r is positive rational, the k are algebraic numbers.
For the above one can see and [15].

2 The Derivative $\{r, k\}$

Lemma 1. If $|t| < \pi a/2$ and $q = e^{-\pi a}$ then

$$\sum_{n=1}^{\infty} \frac{\cosh(2tn)}{n \sinh(\pi a n)} = \log(f(-q^2)) - \log\left(\theta_4(it, e^{-a\pi})\right) \tag{11}$$

Proof. From the Jacobi Triple Product Identity (see [4]) we have

$$\theta_4(z,q) = \prod_{n=0}^{\infty} (1-q^{2n+2})(1-q^{2n-1}e^{2iz})(1-q^{2n-1}e^{-2iz}) \tag{12}$$

By taking the logarithm of both sides and expanding the logarithm of the individual terms in a power series it is simple to show (17) from (18).

Lemma 2.
Let $q = e^{-\pi\sqrt{r}}$

$$\phi(x) = 2\frac{d}{dx}\left(\frac{\partial}{\partial t}\log\left(\vartheta_4\left(\frac{it\pi}{2}, e^{-2\pi x}\right)\right)_{t=x}\right) \tag{13}$$

then

$$\frac{d(\sqrt{r})}{dk}=\frac{K^{(1)}(k)}{\phi\left(\frac{K(\sqrt{1-k^2})}{K(k)}\right)}=\frac{K^{(1)}(k)}{\phi\left(\frac{K(k')}{K(k)}\right)} \tag{14}$$

Where $K^{(1)}(k)$ is the first derivative of K.

Proof. From Lemma 1 we have

$$2\frac{\partial}{\partial t}\log\left(\vartheta_4\left(\frac{it\pi}{2},e^{-2\pi x}\right)\right)_{t=x}=-\pi\sum_{n=1}^{\infty}\frac{1}{\cosh(n\pi x)}=\frac{\pi}{2}-K(k_x)$$

then

$$\sqrt{x(k_2)}-\sqrt{x(k_1)}=-\int_{k_1}^{k_2}\frac{K^{(1)}(k)}{\phi\left(\frac{K\left(\sqrt{1-k^2}\right)}{K(k)}\right)}dk$$

Differantiating the above relation with respect to k we get the result.

Lemma 3. Set $q=e^{-\pi\sqrt{r}}$ and

$$\{r,k\}:=\frac{dr}{dk}=2\frac{K(k')K^{(1)}(k)}{K(k)\phi\left(\frac{K(k')}{K(k)}\right)}$$

Then

$$\{r,k\}=\frac{\pi K'}{K^2}\frac{\frac{1-5k^2}{k-k^3}+\frac{6K^{(1)}}{K}}{(k^2-5)K+6E} \tag{15}$$

Proof. From (9) taking the logarithmic derivative with respect to k and using Lemma 2 we get:

$$\pi\{r,k\}\left(1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)=\left(\frac{1-5k^2}{(k-k^3)}+\frac{6K^{(1)}}{K}\right)\frac{4K'}{K} \tag{16}$$

But it is known that

$$\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=\frac{1}{24}+\frac{K}{6\pi^2}((5-k^2)K-6E) \tag{17}$$

From the above relations we get the result.

Note.

1) The first derivative of K is

$$K^{(1)}=\frac{E}{k\cdot k'^2}-\frac{K}{k_r}$$

where $k=k_r$ and $k'=k'_r=\sqrt{1-k_r^2}$.

2) In the same way we can find form the relation

$$k_{4r}=\frac{1-k'_r}{1+k'_r} \tag{18}$$

the modular equation of 2-degree derivative.
Noting first that (the proof is easy):

$$\{r, k'_r\} = \frac{k'_r}{k_r} \{r, k_r\} \tag{19}$$

we have

$$\{r, k_{4r}\} = \frac{k'_r(1 + k'_r)^2}{2k_r} \{r, k_r\} \tag{20}$$

3 The Ramanujan's Cubic Continued Fraction

Let

$$V(q) := \frac{q^{1/3}}{1+} \frac{q + q^2}{1+} \frac{q^2 + q^4}{1+} \frac{q^3 + q^6}{1+} \dots \tag{21}$$

is the Ramanujan's cubic continued fraction, then

Lemma 4.

$$V(q) = \frac{2^{-1/3}(k_{9r})^{1/4}(k'_r)^{1/6}}{(k_r)^{1/12}(k'_{9r})^{1/2}} \tag{22}$$

where the k_{9r} are given by (see [7]):

$$\sqrt{k_r k_{9r}} + \sqrt{k'_r k'_{9r}} = 1 \tag{23}$$

Proof. It is known (see and [12] pg. 596) that

$$V(q) = q^{1/3} \frac{(q; q^2)_\infty}{(q^3; q^6)^3_\infty}$$

But

$$\Phi(-q) = (-q, q)_\infty = \frac{1}{(q, q^2)_\infty}$$

thus

$$V(q) = q^{1/3} \frac{\Phi(-q^3)^3}{\Phi(-q)}$$

and equation (19) follows from (8).

Lemma 5.
If

$$G(x) = \frac{x}{\sqrt{2\sqrt{x} - 3x + 2x^{3/2} - 2\sqrt{x}\sqrt{1 - 3\sqrt{x} + 4x - 3x^{3/2} + x^2}}}$$

and

$$k_r = G(w) \tag{24}$$

then

$$k_{9r} = \frac{w}{k_r}$$

and

$$k'_{9r} = \frac{(1-\sqrt{w})^2}{k'_r}$$

Proof. Set the values of k_r and k_{9r} in (20).
If we set

$$W = 2 - 3\sqrt{w} + 2w - 2(1-\sqrt{w})\sqrt{1-\sqrt{w}+w} \tag{25}$$

then

$$V(q) = \frac{(k'_r)^{2/3} w^{1/4}}{2^{1/3}(k_r)^{1/3}(1-\sqrt{w})} = \frac{(W - w^{3/2})^{1/3} W^{-1/6}}{2^{1/3}(1-\sqrt{w})} \tag{26}$$

after solving (22) with respect to w and making the simplifications we arrive at

$$2V^3(q) = \frac{\sqrt{W}}{(1+\sqrt{W})^2} \tag{27}$$

and

$$(k_r)^2 = \sqrt{W}\left(\frac{2+\sqrt{W}}{1+2\sqrt{W}}\right)^3 \tag{28}$$

Hence we get the following equation

$$(k_r)^{2/3} = Z^2 \frac{\sqrt{2}V(q)^{3/2} + Z^3}{-\sqrt{2}V(q)^{3/2} + 2Z^3} \tag{29}$$

Where $Z = \sqrt[12]{W(q)}$. Reducing the above equation in polynomial form we have

$$s k_r^{2/3} + sZ^2 - 2k_r^{2/3} Z^3 + Z^5 = 0 \tag{30}$$

and

$$s^2 = 2V^3(q) = \frac{Z^6}{(1+Z^6)^2} \tag{31}$$

From these two equations we arrive to

Theorem 1. Set $T = \sqrt{1-8V^3(q)}$ then

$$(k_r)^2 = \frac{(1-T)(3+T)^3}{(1+T)(3-T)^3} \tag{32}$$

Equation (29) is a solvable quartic equation with respect to T.
An example of evaluation is

$$V(e^{-\pi}) = \frac{1}{2}\left(-2-\sqrt{3}+\sqrt{3(3+2\sqrt{3})}\right) \tag{33}$$

Main Theorem.

$$V'(q) = -\frac{2}{3}\frac{V(q)+V^4(q)}{k_r\{r,k\}\sqrt{1-8V^3(q)}} \tag{34}$$

The derivative $\{r,k\}$ is given form (15).
Proof. Derivate (32) with respect to r then

$$\sqrt{\frac{2k_r}{\{k,r\}}} = \frac{4T(3+T)}{(3-T)^2(1+T)}\sqrt{\frac{dT}{dr}} \tag{35}$$

or

$$T_r = \frac{dT}{dr} = \frac{1}{8k_r\{r,k\}}\frac{(9-T^2)(1-T^2)}{T^2} \tag{36}$$

Using the relation $T = \sqrt{1-8V(q)^3}$, we get the result.

We use oftenly the notations $V[r] := V(e^{-\pi\sqrt{r}})$, $T[r] := T(e^{-\pi\sqrt{r}})$.

Proposition.

$$V[4r] = \frac{1-T[r]}{4V[r]} \tag{37}$$

Proof. From (18) and Theorem 1 we get the result using simplifications with Mathematica. For a more completed proof see [9].

Proposition.

$$\frac{T'[4r]}{T'[r]} = \frac{(1-T[r])(3+T[r])}{8\sqrt{T[r]}(1+T[r])^{5/3}(3-T[r])^{1/2}} \tag{38}$$

Proof. From (18), (19), (20) and (34) we get

$$V'[4r] = \frac{-2\{k,r\}}{3\frac{1-k'}{1+k'}\frac{k'(1+k')^2}{2k}}\frac{V[4r]+V[4r]^4}{T[r]} \tag{39}$$

If we use the duplication formula (37) we get the result.

Note.
1) We can calculate now easy the values of $V'(q)$ from (31) using (29) and (15). An example of evaluation is

$$k_1 = \frac{1}{\sqrt{2}}$$

,

$$E(k_1)=\frac{4\pi^{3/2}}{\Gamma(-1/4)^2}+\frac{\Gamma(3/4)^2}{2\sqrt{\pi}}$$

and

$$K(k_1)=\frac{8\pi^{3/2}}{\Gamma(-1/4)^2}$$

When $r=1$ we get

$$\{r,k\}=\frac{8\sqrt{2}\Gamma(3/4)^4}{\pi^2}$$

Hence

$$V'(e^{-\pi})=-\frac{\sqrt{6(12+7\sqrt{3})}-6-3\sqrt{3}}{24\Gamma(3/4)^4}\pi^2 \tag{40}$$

2) It is

$$T_1=T(e^{-\pi\sqrt{3}})=-39+22\sqrt{3}-\frac{2\cdot 6^{2/3}(-123+71\sqrt{3})}{\left(-4725+2728\sqrt{3}-\sqrt{4053-2340\sqrt{3}}\right)^{1/3}}+$$

$$+2\cdot 6^{1/3}\left(-4725+2728\sqrt{3}-\sqrt{4053-2340\sqrt{3}}\right)^{1/3}$$

and $V_1=V(e^{-\pi\sqrt{3}})=\frac{1}{2}\sqrt[3]{1-T_1^2}$.
From tables it is:

$$\{3,k_3\}=\frac{96\cdot 2^{1/3}\pi\Gamma(2/3)^2\left(2^{1/3}\sqrt{3}\Gamma(1/3)^4-16(3+2\sqrt{3})\pi\Gamma(2/3)^2\right)}{\sqrt{2-\sqrt{3}}\left((3+2\sqrt{3})\Gamma(1/3)^8-96\cdot 2^{2/3}(12+7\sqrt{3})\pi^3\Gamma(4/3)^2\right)}$$

We find the value of $V'(e^{-\pi\sqrt{3}})$ in terms of Gamma function and algebraic numbers.

$$V'(e^{-\pi\sqrt{3}})=-\frac{2}{3}\frac{V_1+V_1^4}{k_3\,\{3,k_3\}\,\sqrt{1-8V_1^3}}$$

3) Solving (36) with respect to T we get

$$T=\sqrt{5-4cT_r+\frac{1}{2}\sqrt{-36+(-10+8cT_r)^2}} \tag{41}$$

Where $c=\{r,k\}\,k_r$

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Some Notes On a Continued Fraction of Ramanujan

Nikos Bagis
Department of Informatics
Aristotele University of Thessaloniki Greece
bagkis@hotmail.com

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Abstract

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1 Introduction

Let

$$(a;q)_k = \prod_{n=0}^{k-1}(1-aq^n) \quad (1)$$

Then we define

$$f(-q) = (q;q)_\infty \quad (2)$$

and

$$\Phi(-q) = (-q;q)_\infty \quad (3)$$

Also let

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1-x^2\sin^2(t)}}dt \quad (4)$$

be the elliptic integral of the first kind
We denote

$$\theta_4(u,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2nui} \quad (5)$$

the Elliptic Theta function of the 4th-kind.

$$\prod_{n=1}^{\infty}(1-q^{2n})^6 = \frac{2kk'K(k)^3}{\pi^3 q^{1/2}} \quad (6)$$

and from

$$q^{1/3}\prod_{n=1}^{\infty}(1+q^n)^8 = 2^{-4/3}\left(\frac{k}{1-k^2}\right)^{2/3} \quad (7)$$

we get

$$f(-q)^8 = \prod_{n=1}^{\infty}(1-q^n)^8 = \frac{2^{8/3}}{\pi^4}q^{-1/3}k^{2/3}(k')^{8/3}K(k)^4 \quad (8)$$

The variable k is defined from the equation

$$\frac{K(k')}{K(k)} = \sqrt{r} \tag{9}$$

where r is positive , $q = e^{-\pi\sqrt{r}}$ and $k' = \sqrt{1-k^2}$. Note also that whenever r is positive rational, the k are algebraic numbers.

In Berndt's book: Ramanujan's Notebook Part III, ([B3] pg.21), one can find the following expansion

Theorem 1. Suppose that either q, a and b are complex numbers with $|q| < 1$, or q, a, and b are complex numbers with $a = bq^m$ for some integer m. Then

$$U = U(a,b;q) = \frac{(-a;q)_\infty(b;q)_\infty - (a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty + (a;q)_\infty(-b;q)_\infty} =$$

$$\frac{a-b}{1-q+}\,\frac{(a-bq)(aq-b)}{1-q^3+}\,\frac{q(a-bq^2)(aq^2-b)}{1-q^5+}\,\frac{q^2(a-bq^3)(aq^3-b)}{1-q^7+}\ldots \tag{10}$$

Suppose now

$$X = \frac{(-a;q)_\infty(b;q)_\infty}{(a;q)_\infty(-b;q)_\infty} \tag{11}$$

Then holds

$$\frac{X-1}{X+1} = U \tag{12}$$

Corollary. Set

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{13}$$

then

$$\frac{\phi(q)-1}{\phi(q)+1} = \frac{q}{1+q}\,\frac{-q^3}{1+q^3+}\,\frac{-q^5}{1+q^5+}\,\frac{-q^7}{1+q^7}\ldots \tag{14}$$

Proof. Take $q \to q^2$ in (12) and then set $a \to q$ and $b \to q^2$.

Corollary.

$$\frac{\Phi(-q)-f(-q)}{\Phi(-q)+f(-q)} = \frac{q}{1-q+}\,\frac{q^3}{1-q^3+}\,\frac{q^5}{1-q^5+}\,\frac{q^7}{1-q^7}\ldots \tag{15}$$

Proof. Set $b = 0$ in (12) and then $a = q$.

Corollary.

$$\frac{\Phi(-q)-f(-q)}{\Phi(-q)+f(-q)}=-\frac{\phi(-q)-1}{\phi(-q)+1} \tag{16}$$

Proof. It follows from Corollary's 1, 2

Proposition 1.

$$\sum_{n=0}^{\infty}\frac{q^n}{1-a^2q^{2n}}=\frac{1}{1-q+}\frac{-a^2(1-q)^2}{1-q^3+}\frac{-qa^2(1-q^2)^2}{1-q^5+}\frac{-q^2a^2(1-q^3)^2}{1-q^7+}\ldots \tag{17}$$

Proof. Divide relation (12) by $a-b$ and then take the limit $b \to a$.

Corollary.

$$\frac{K(k_r)}{2\pi}+\frac{1}{4}=\frac{1}{1-q+}\frac{(1-q)^2}{1-q^3+}\frac{q(1-q^2)^2}{1-q^5+}\frac{q^2(1-q^3)^2}{1-q^7+}\ldots \tag{18}$$

Proof. Set in (19) $a=i$, and $q=e^{-\pi\sqrt{r}}$.

Now set

$$u(a,q)=\frac{2a}{1-q+}\frac{a^2(1+q)^2}{1-q^3+}\frac{a^2q(1+q^2)^2}{1-q^5+}\frac{a^2q^2(1+q^3)^2}{1-q^7+}\ldots \tag{19}$$

and

$$P=\left(\frac{(-a;q)_\infty}{(a;q)_\infty}\right)^2 \tag{20}$$

Then

$$\frac{P-1}{P+1}=u(a,q) \tag{21}$$

or

Proposition 2.

$$\left(\frac{(-a;q)_\infty}{(a;q)_\infty}\right)^2=-1+\frac{2}{1-}\frac{2a}{1-q+}\frac{a^2(1+q)^2}{1-q^3+}\frac{a^2q(1+q^2)^2}{1-q^5+}\frac{a^2q^2(1+q^3)^2}{1-q^7+}\ldots \tag{22}$$

Taking the logarithms in both sides of (24), we get easily as in Lemma 1 :

Proposition 3.

$$4\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)(1-q^{2n+1})}=\log\left(-1+\frac{2}{1-u(a,q)}\right) \tag{23}$$

Here we must mention that hods the more general formula (the proof is as in Proposition 2)

$$2\sum_{n=0}^{\infty}\frac{a^{2n+1}-b^{2n+1}}{(2n+1)(1-q^{2n+1})}=\log\left(-1+\frac{2}{1-U(a,b;q)}\right) \tag{24}$$

Thus

$$\left(-1+\frac{2}{1-U(a,b;q)}\right)^2=\frac{\left(-1+\frac{2}{1-u(a,q)}\right)}{\left(-1+\frac{2}{1-u(b,q)}\right)} \tag{25}$$

and for to study U we have to study only u.

Corollary.

$$\int_0^q \frac{1}{1-x+}\frac{-x(1-x)^2}{1-x^3+}\frac{-x^2(1-x^2)^2}{1-x^5+}\frac{x^3(1-x^3)^2}{1-x^7+}\ldots dx=$$

$$=\frac{1}{4}\log\left(-1+\frac{2}{1-}\frac{2\sqrt{q}}{1-q+}\frac{q(1+q)^2}{1-q^3+}\frac{q^2(1+q^2)^2}{1-q^5+}\ldots\right) \tag{26}$$

Proof. Differentiate with respect to a the relation (25) and use (19). After that integrate to get the desired result.
In some cases the u fraction can calculated in terms of elliptic functions. For example:

$$-1+\frac{2}{1-u(q,q)}=\frac{\pi}{2k_r'K(k_r)} \tag{27}$$

In general holds

$$4\sum_{n=0}^{\infty}\frac{q^{\nu(2n+1)}}{(2n+1)(1-q^{2n+1})}=-4\sum_{j=1}^{\nu-1}\operatorname{arctanh}(q^j)-\log\left(\frac{2k_r'K(k_r)}{\pi}\right)$$

from which we lead to the following:

Proposition 4.

$$-1+\frac{2}{1-u(q^\nu,q)}=-1+\frac{2}{1-}\frac{2q^\nu}{1-q+}\frac{q^{2\nu}(1+q)^2}{1-q^3+}\frac{q^{2\nu+1}(1+q^2)^2}{1-q^5+}\frac{q^{2\nu+2}(1+q^3)^2}{1-q^7+}\ldots=$$

$$= \frac{\pi}{2k_r' K(k_r)} \exp\left(-4\sum_{j=1}^{\nu-1} \operatorname{arctanh}(q^j)\right) \tag{28}$$

Proposition 5.

$$-1 + \frac{2}{1 - U(q^{\nu_1}, q^{\nu_2}, q)} = \exp\left[-2\left(\sum_{j_1=1}^{\nu_1-1} \operatorname{arctanh}(q^{j_1}) - \sum_{j_2=1}^{\nu_2-1} \operatorname{arctanh}(q^{j_2})\right)\right] \tag{29}$$

Proof. The proof follows easily from (25), (26) and (27) and (30).

Another formula related with u continued fraction is

$$-1 + \frac{2}{1 - u(q^{\nu+1/2}, q)} = \exp\left(-4\sum_{n=0}^{\infty} \frac{q^{(2n+1)(\nu+1/2)}}{(2n+1)(1-q^{2n+1})}\right)$$

$$= \exp\left(-4\sum_{j=0}^{\nu-1} \operatorname{arctanh}(q^{j+1/2}) + \operatorname{arctanh}(k_r)\right)$$

Hence also

$$k_r = \tanh\left(4\sum_{j=0}^{\nu-1} \operatorname{arctanh}(q^{j+1/2}) + \log\left(-1 + \frac{2}{1 - u(q^{\nu+1/2}, q)}\right)\right) \tag{30}$$

For every ν positive integer.
Hence we obtain a continued fraction for k_r

$$\frac{k_r'}{1 - k_r} = -1 + \frac{2}{1 - u(q^{1/2}, q)} \tag{31}$$

Inspired from the above relations and Propositions we have

Proposition 6.(Unproved)
If c is positive real and ν_1, ν_2 positive integers then:

$$-1 + \frac{2}{1 - U(q^{\nu_1+c}, q^{\nu_2+c}, q)} \stackrel{?}{=}$$

$$= \exp\left[-2\left(\sum_{j_1=1}^{\nu_1-1} \operatorname{arctanh}(q^{j_1+c}) - \sum_{j_2=1}^{\nu_2-1} \operatorname{arctanh}(q^{j_2+c})\right)\right] \tag{32}$$

Also we observe that holds and

$$\left(-1+\frac{2}{1-U(a,-b;q)}\right)^2 \stackrel{?}{=} \left(-1+\frac{2}{1-u(a,q)}\right)\left(-1+\frac{2}{1-u(b,q)}\right) \tag{33}$$

This relation is similarly to (27). We also get the following unproved

Corollary.(Unproved)
Let $w \in \mathrm{Im}(\mathbf{C})$, then

$$\left|-1+\frac{2}{1-U(q^{\nu_1+c},-wq^{\nu_2+c},q)}\right| \stackrel{?}{=} \left(-1+\frac{2}{1-u(q^{\nu_1+c},q)}\right) \tag{34}$$

Seting $c=0$ in (35) and using Proposition 3 we get

Corollary. When $w, z \in \mathrm{Im}(\mathbf{C})$ and $q = e^{-\pi\sqrt{r}}$, $r > 0$, then
(i)

$$\left|-1+\frac{2}{1-U(q^{\nu_1},-wq^{\nu_2},q)}\right| = \frac{\pi}{2k_r' K(k_r)}\exp\left(-4\sum_{j=1}^{\nu_1-1}\operatorname{arctanh}(q^j)\right)$$

(ii)

$$\left|-1+\frac{2}{1-U(-zq^{\nu_1+c},-wq^{\nu_2+c},q)}\right| = 1$$

We write (26) as

$$\log\left(-1+\frac{2}{1-U(a,b,q)}\right) = 2\sum_{n=1}^{\infty}\frac{a^n-b^n}{n(1-q^n)} - \sum_{n=1}^{\infty}\frac{a^{2n}-b^{2n}}{n(1-q^{2n})} =$$

$$= 2\sum_{n=1}^{\infty}\frac{(aq^{-1/2})^n-(bq^{-1/2})^n}{n(q^{-n/2}-q^{n/2})} - \sum_{n=1}^{\infty}\frac{(aq^{-1/2})^{2n}-(bq^{-1/2})^{2n}}{n(q^{-n}-q^n)}$$

Set now $b = e^{-v-2t}$, $q = e^{-2t}$, $a = e^v$ then

$$\log\left(-1+\frac{2}{1-U(a,b,q)}\right) = \log\left(-1+\frac{2}{1-U(e^v,e^{-v-2t},e^{-2t})}\right) =$$

$$= 2\sum_{n=1}^{\infty}\frac{e^{(v+t)n}-e^{-(v+t)n}}{n(e^{tn}-e^{-tn})} - \sum_{n=1}^{\infty}\frac{e^{2(v+t)n}-e^{-2(v+t)n}}{n(e^{2tn}-e^{-2tn})} =$$

$$= 2\sum_{n=1}^{\infty}\frac{\sinh((v+t)n)}{n\sinh(tn)} - \sum_{n=1}^{\infty}\frac{\sinh(2(v+t)n)}{n\sinh(2tn)} \tag{a}$$

Set

$$U_1(t) := \lim_{h\to 0}\left(-1+\frac{2}{1-U(e^{-t+h},e^{-t+h},e^{-2t+h})}\right) \tag{35}$$

Hence we get the following

Proposition 7.

$$\frac{d^{2\nu+1}}{dt^{2\nu+1}}\log(U_1(t)) = 2\sum_{n=1}^{\infty}\frac{n^{2\nu+1}}{n\sinh(tn)} - 2\sum_{n=1}^{\infty}\frac{(2n)^{2\nu+1}}{n\sinh(2tn)} \tag{36}$$

Proof. Use Lemma and proceed derivating relation (a). Everytime take the limit $v \to -t$.

Corollary.

$$-\frac{d}{dt}\log\left(U_1(t)\right) = -\frac{\log(U_1(t+dh))}{dh} = 2\sum_{n=0}^{\infty}\frac{1}{\sinh(t(2n+1))} \tag{37}$$

Proof. Set $\nu = 0$ in Proposition.

Lemma

$$\frac{U_1(t+dh)^{\nu} - U_1(t+dh)^{-\nu}}{2\nu dh} \stackrel{?}{=} \frac{\log(U_1(t+dh))}{dh} \tag{38}$$

Lemma

$$\frac{F(U_1(t+dh)) - F\left(U_1(t+dh)^{-1}\right)}{2dh} \stackrel{?}{=} F'(1)\frac{\log(U_1(t+dh))}{dh} \tag{39}$$

Proposition 8. For arbitrary non costant analytic functions F_j define $T_{F_j}(x) = \log(F_j(x))$, $j = 1, 2, \ldots$ then

$$\frac{T_{F_m}\circ\underbrace{\ldots}_{m}\circ T_{F_1}(\log(U_1(x+dh)))}{dh} = 2\sum_{n=0}^{\infty}\frac{1}{\sinh((2n+1)x)} \tag{40}$$

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Infinite Series and Divisor Sums

Nikos Bagis

Department of Informatics

Aristotle University of Thessaloniki

Thessaloniki Greece

bagkis@hotmail.com

SUBMITTED

Abstract

In this work we present and prove formulas having infinite and finite parts. The finite parts are divisor sums which already known from the ancient Greeks. These sums can lead us to very interesting formulas when attached to infinite expressions.

§1.General Theorems on Series

Proposition 1. If a, b are positive real numbers and f is analytic in (-1,1) with $f(0)=0$, then

$$e^{\int_{\infty}^{x} f(e^{-t})dt} = \prod_{n=1}^{\infty}(1-e^{-nx})^{\frac{1}{n}\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)} \quad : (1)$$

Where μ is the Moebius function i.e.:

$$\mu(k) = \begin{cases} 1 & \text{if } k=1 \\ (-1)^r, & \text{if } k \text{ is the product of r discrete primes} \\ 0 & \text{else} \end{cases}$$

For the Moebius Function one can see [Ap].

Proof. Because $f(0) = 0$ and f analytic in (-1,1), the integral $\int_{\infty}^{x} f(e^{-t})dt$ exists for every $x > 0$. We assume that exists sequence $X(n)$ such that:

$$e^{\int_{\infty}^{x} f(e^{-t})dt} = \prod_{n=1}^{\infty}(1-e^{-nx})^{X(n)}$$

$$\int_{\infty}^{x} f(e^{-t})dt = \sum_{n=1}^{\infty} X(n)\log\left(1-e^{-nx}\right) = -\sum_{n=1}^{\infty} X(n)\sum_{k=1}^{\infty}\frac{e^{-knx}}{k} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} X(n)n\frac{e^{-knx}}{kn}$$

$$= -\sum_{n=1}^{\infty}\frac{e^{-nx}}{n}\sum_{d|n} X(d)d \text{ . (A)}$$

Thus taking the derivatives in (A)

$$f(x) = \sum_{n=1}^{\infty} e^{-nx}\sum_{d|n} X(d)d \text{ : (2)}$$

But $f(x) = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \Rightarrow f(e^{-x}) = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}e^{-nx}$. From (2) we have

$\frac{f^{(n)}(0)}{n!} = \sum_{d|n} X(d)d$. Thus from the Moebius Inversion Theorem [Ap] we have

$\frac{f^{(n)}(0)}{n!} = \sum_{d|n} X(d)d \Leftrightarrow X(n) = \frac{1}{n}\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)$ and there holds the relation

$e^{\int_{\infty}^{x} f(e^{-t})dt} = \prod_{n=1}^{\infty}(1-e^{-nx})^{\frac{1}{n}\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)}$. Thus the proof is complete.

Examples in Proposition 1.

1) Let $f(x) = x$, then we have $\frac{f^{(n)}(0)}{n!} = \begin{cases} 1, n=1 \\ 0, n=2,3,4,... \end{cases} = 1[n]$

thus $X(n) = \frac{1}{n}\sum_{d|n} 1[d]\mu\left(\frac{n}{d}\right) = 1/n\mu(n)$

then

$$\prod_{n=1}^{\infty}\left(1-q^n\right)^{\frac{\mu(n)}{n}} = e^q \text{ : (2)}$$

2) Let $\frac{f^{(n)}(0)}{n!} = n,\ n = 1,2,...$ then $f(x) = \frac{x}{(x-1)^2}$ and

$X(n) = \frac{1}{n}\sum_{d|n} d\mu\left(\frac{n}{d}\right) = 1/n\varphi(n)$

$$\prod_{n=1}^{\infty}\left(1-q^n\right)^{\frac{\varphi(n)}{n}} = e^{\frac{q}{q-1}} \text{ : (3)}$$

3) Let $\dfrac{f^{(n)}(0)}{n!}=\dfrac{\mu(n)}{n^{v}}$, $n=1,2,...$ then $f(x)=\sum_{n=1}^{\infty}\dfrac{\mu(n)x^{n}}{n^{v}}$ and

$$X(n)=\frac{1}{n}\sum_{d|n}d^{-v}\mu(d)\mu\left(\frac{n}{d}\right)=1/n\sigma_{-v}^{(-1)}(n)$$

$$\lim_{q\to1^{-}}\prod_{n=1}^{\infty}\left(1-q^{n}\right)^{\frac{\sigma_{-v}^{(-1)}(n)}{n}}=e^{-1/\zeta(1+v)}\ :(4)$$

Theorem 1. When *a*, *b* >0 and f has Taylor series in [-1, 1] then

$$\prod_{n=1}^{\infty}\left(\frac{1-e^{-nb}}{1-e^{-na}}\right)^{\frac{1}{n}\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)}=e^{\int_{a}^{b}f(e^{-t})dt}\ :(5)$$

Proof.
It follows easily from Proposition 1.

Proposition 2. If *a* is a positive real number

$$\sum_{n=1}^{\infty}\frac{\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)}{e^{na}-1}=f(e^{-a})\ :(6)$$

Proof.
Set $x=a>0$ in (1), take the logarithms in the two sides and then derivate, after some simplifications we get the desired result.

Proposition 3. If *x* is a positive real number

$$\sum_{n=1}^{\infty}\frac{\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)}{e^{nx}+1}=-2f(e^{-2x})+f(e^{-x})\ :(7)$$

Proof.
Set $x=a$ and $x=2a$ in (1) to take two relations, divide them. Take the logarithms and derivate. After a few simplifications we get (5)

Proposition 4. If $A(n)$ is arbitrary sequence of numbers we have for $x>0$

$$\frac{d^{v}}{dx^{v}}\left(\sum_{n=1}^{\infty}\frac{\sum_{d|n}A(d)\mu\left(\frac{n}{d}\right)}{e^{nx}-1}\right)=\sum_{n=1}^{\infty}\frac{\sum_{d|n}A(d)(-d)^{v}\mu\left(\frac{n}{d}\right)}{e^{nx}-1}\ :(8)$$

Proof. From Proposition 1 we have

$$\frac{d^v}{dx^v}\left(\sum_{n=1}^{\infty}\frac{\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)}{e^{nx}-1}\right)=\frac{d^v}{dx^v}f(e^{-x})=\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}\frac{d^v}{dx^v}\left(e^{-nx}\right)=\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}(-n)^v e^{-nx}$$

Using again Proposition 1 we get the result.

Lemma.

$$\sum_{n=1}^{\infty}\frac{X(n)}{e^{nx}-1}=\sum_{n=1}^{\infty}\sum_{d|n}X(d)e^{-nx} \ : (9)$$

Proof. Set $\sum_{d|n}\frac{f^{(d)}(0)}{d!}\mu\left(\frac{n}{d}\right)=X(n)$, then from Moebius Inversion Theorem we have $\frac{f^{(n)}(0)}{n!}=\sum_{d|n}X(d)$, using Proposition 2 we have the result.

Proposition 5. Let $\sum_{d|n}X(d)=\frac{g^{(n)}(0)}{n!}$, then for every f we have

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sum_{d|n}X(d)f\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty}g(q^n)f(n) \ : (10)$$

Proof. Let $\sum_{d|n}X(d)=\frac{g^{(n)}(0)}{n!}$, from Lemma 1 we have

$$\sum_{n=1}^{\infty}\frac{X(n)f(m)}{e^{nmx}-1}=\sum_{n=1}^{\infty}\frac{g^{(n)}(0)}{n!}f(m)e^{-nmx}.$$

Summing with respect to m we have $\sum_{n=1}^{\infty}\frac{\sum_{d|n}X(d)f\left(\frac{n}{d}\right)}{e^{nx}-1}=\sum_{n=1}^{\infty}f(n)g(e^{-nx})$ and the result follows.

Proposition 6. Let $\sum_{d|n}X(d)=\frac{g^{(n)}(0)}{n!}$, then for every f we have

$$\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sum_{d|n}X(d)f\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty}(g(q^n)-2g(q^{2n}))f(n) \ : (11)$$

Theorem 2. When $|q|<1$

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sum_{d|n}\frac{f(d)}{H(q^d)}\varphi_H\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty}f(n) \ : (12)$$

where $\varphi_H(n)=\sum_{d|n} h_d\mu\left(\frac{n}{d}\right)$ and $H(x)=\sum_{k=1}^{\infty} h_k x^k$.

§2. Some general applications

From $\sum_{n=1}^{\infty}\frac{X(n)}{e^{nx}-1}=\sum_{n=1}^{\infty}\sum_{d|n} X(d)e^{-nx}$ we have setting $X(n)=n^v$:

$$\sum_{n=1}^{\infty}\frac{n^v}{e^{nx}-1}=\sum_{n=1}^{\infty}\sigma_v(n)e^{-nx} \ : (1)$$

where $\sum_{d|n} d^v=\sigma_v(n)$, also

$$\sum_{n=1}^{\infty}\frac{\sum_{d|n} d^{-v}\mu(n/d)}{e^{nx}-1}=Li_v(e^{-x}) \, , x>0$$

or

$$\sum_{n=1}^{\infty}\frac{q^n\sum_{d|n} d^{-v}\mu(n/d)}{1-q^n}=Li_v(q) \ : (2)$$

The next known formula exist $\sum_{k=1}^{n} f(\gcd(n,k))=\sum_{k=1}^{n} f((n,k))=\sum_{d|n} f(d)\varphi(n/d)$,

where φ is Euler phi arithmetic function

Let $X(n)=\varphi(n)$ is Euler-phi function, then $\sum_{d|n}\varphi(d)=n$. Thus from Lemma we have

$\sum_{n=1}^{\infty}\frac{\varphi(n)}{e^{nx}+1}=\frac{1}{2}\frac{\cosh(x)}{\sinh^2(x)}$ and form Proposition 5 we have

$$\sum_{n=1}^{\infty}\frac{\sum_{d|n}\varphi(d)f\left(\frac{n}{d}\right)}{e^{nx}+1}=\frac{1}{2}\sum_{n=1}^{\infty} f(n)\frac{\cosh(nx)}{\sinh^2(nx)}$$

or better

$$2\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n} f((n,k))}{e^{nx}+1}=\sum_{n=1}^{\infty} f(n)\frac{\cosh(nx)}{\sinh^2(nx)} \ : (3)$$

$(n,k)=\gcd(n,k)$ for every f such that the two sums are convergent.

Integrating (3) we get

$$\prod_{n=1}^{\infty}\left(1+q^n\right)^{\frac{1}{n}\sum_{k=1}^{n} f((n,k))}=\exp\left(\sum_{n=1}^{\infty}\frac{f(n)q^n}{n(1-q^{2n})}\right) \ : (4)$$

Note that f may also depend by x. Identity (4) is very curious. We can write a)

$$2\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}\frac{\sinh((n,k)x)^2}{(n,k)^s\cosh((n,k)x)}}{e^{nx}+1}=\zeta(s)\ ,\ \forall x>0\ \ :(5)$$

Also in general

$$4\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f((n,k))\frac{\sinh((n,k)x)^2}{\cosh((n,k)x)}}{e^{nx}+1}=\sum_{n=1}^{\infty}f(n)\ :(6)$$

Thus if $ab=2\pi$ (see [T] pg 60):

$$\sqrt{a}\left(\frac{f(0)}{2}+2\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f(a(n,k))\frac{\sinh((n,k)x)^2}{\cosh((n,k)x)}}{e^{nx}+1}\right)=\sqrt{b}\left(\frac{f_c\wedge(0)}{2}+2\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f_c\wedge(b(n,k))\frac{\sinh((n,k)x)^2}{\cosh((n,k)x)}}{e^{nx}+1}\right)$$

: (7)

for x real positive.

b) From the Jacobi Triple Identity (see [An] pg. 169-170) we have that:

$$\sum_{n=1}^{\infty}\frac{\cosh(tn)}{n\sinh(\pi an)}=\log\left(f(-e^{-2n\pi a})\right)-\log\left(\vartheta_4(it/2,e^{-a\pi})\right),\ |t|<a\pi\ :(J)$$

using (4) we get

$$\prod_{n=1}^{\infty}\left(1+q^n\right)^{\frac{1}{n}\sum_{k=1}^{n}\cos(2t(n,k))}=\left(\frac{f(-q^2)}{\vartheta_4(t,q)}\right)^{1/2}\ :(8)$$

and

$$\prod_{n=1}^{\infty}\left(1+q^n\right)^{\frac{1}{n}\sum_{k=1}^{n}(-1)^{(n,k)}}=\left(\frac{f(-q^2)}{\vartheta_4(\frac{\pi}{2},q)}\right)^{1/2}\ :(9)$$

where $|q|<1$ and $f(-q)=\prod_{k=1}(1-q^k)$.

Now set $|q|<1$, $q=e^{-\pi\sqrt{r}}$, $r>0$ and

$$\psi(v,z,q)=\sum_{n=1}^{\infty}\frac{n^v z^n}{n(1-q^n)}\ :(10)$$

then from (J) we have the following evaluation

$$\psi(2v,q,q^2)=\sum_{n=1}^{\infty}\frac{n^{2v}q^n}{n(1-q^{2n})}=-\frac{1}{2(-4)^v}\frac{\partial^{2v}}{\partial t^{2v}}\log(\vartheta_4(t,q))\bigg|_{t=0}\ ,\ v=1,2,\ldots\ :(11)$$

$$\frac{1}{8}\frac{\partial^2}{\partial t^2}\log(\vartheta_4(t,e^{-\pi\sqrt{r}}))\Bigg|_{t=0} = \psi(2,e^{-\pi},e^{-2\pi}) =$$

$$=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} = \frac{K(k_r)}{2\pi^2}\left(K(k_r)-E(k_r)\right).$$

But

$$-4\sum_{n=1}^{\infty}\frac{\log\left(1+e^{-n\pi}\right)}{n}\sum_{k=1}^{n}(n,k)^2 = -\frac{1}{2}\frac{\partial^2}{\partial t^2}\log(\vartheta_4(t,e^{-\pi}))\Bigg|_{t=0}$$

Thus we get an evaluation

$$-8\sum_{n=1}^{\infty}\frac{\log\left(1+e^{-n\pi}\right)}{n}\sum_{k=1}^{n}(n,k)^2 = -\frac{1}{\pi}+\frac{8\pi}{\Gamma\left(-\frac{1}{4}\right)^2\Gamma\left(\frac{3}{4}\right)^2}$$

K, E are the elliptic Integrals of the first and second type and k is the inverse elliptic nome. (see [W,W], [G,R]).

For to evaluate other values one can use tables stored in the Web.

Continuing we have as with (4)

$$2\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f((n,k))}{e^{nx}-1} = \sum_{n=1}^{\infty}f(n)\frac{1}{\cosh(nx)-1} \quad : (12)$$

or

$$4\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f((n,k))\sinh((n,k)x)^2}{e^{2nx}-1} = \sum_{n=1}^{\infty}f(n) \quad : (13)$$

Thus from [T]:

3) When $ab=2\pi$:

$$\sqrt{a}\left(\frac{f(0)}{2}+4\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f(a(n,k))\sinh((n,k)x)^2}{e^{2nx}-1}\right) = \sqrt{b}\left(\frac{f_c\wedge(0)}{2}+4\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}f_c\wedge(b(n,k))\sinh((n,k)x)^2}{e^{2nx}-1}\right)$$

: (14)

For every x in $\mathbb{R}$.

Taking the Mellin Transform (see [T]):

$$(Mf)(s)=\int_0^{\infty}f(t)t^{s-1}dt$$

in both sides of (12) we have

$$2\Gamma(s)\zeta(s-1)\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\sum_{n=1}^{\infty}f(n)\int_0^{\infty}\frac{x^{s-1}}{\cosh(nx)-1}dx$$

or

$$4\Gamma(s)\zeta(s-1)\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\int_0^{\infty}\left(\sum_{n=1}^{\infty}\frac{f(n)}{\sinh(nx/2)^2}\right)x^{s-1}dx=M\left(\sum_{n=1}^{\infty}\frac{f(n)}{\sinh(nx/2)^2}\right)(s)$$

$$M\left(\sum_{n=1}^{\infty}\frac{X(n)}{\sinh(nx)^2}\right)(s)=4\cdot 2^{-s}\Gamma(s)\zeta(s-1)L(X,s)\ :(15)$$

Identity (15) is very interesting some Examples and Applications are given:

Examples of relation (15)

1)

$$M\left(\sum_{n=1}^{\infty}\frac{1}{\sinh(nx)^2}\right)(s)=4\cdot 2^{-s}\Gamma(s)\zeta(s-1)\zeta(s):(16)$$

2) Let $X_k(n)$ be any periodic sequence with period *k*, then (see [Ap])

$$M\left(\sum_{n=1}^{\infty}\frac{X_k(n)}{\sinh(nx)^2}\right)(s)=4\cdot 2^{-s}\Gamma(s)\zeta(s-1)k^{-s}\sum_{r=1}^{k}X_k(r)\zeta\left(s,\frac{r}{k}\right)\ :(17)$$

3) For the Rogers Ramanujan continued fraction

$$R(q)=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}...\ :(18)$$

(see [B3] and [B,G,2]) we have

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{n^2}{\sinh(nx/2)^2}=-4\frac{d^2}{dx^2}\log\left(R(e^{-x})\right)\ :(19)$$

where $\left(\frac{n}{5}\right)$ is the Legendre symbol

This example can generalized as follows (see [B,G,2])

$$\sum_{n=1}^{\infty}\frac{X_2(n)n^2}{\sinh(nx/2)^2}=-4\frac{d^2}{dx^2}\log\left(\frac{\vartheta_4((p-2a)ix/4,e^{-px/2})}{\vartheta_4((p-2b)ix/4,e^{-px/2})}\right)\ :\ (20)$$

where

$$X_2(n)=\begin{cases} 1, n\equiv (p-a)\bmod p \\ -1, n\equiv (p-b)\bmod p \\ 1, n\equiv a \bmod p \\ -1, n\equiv b \bmod p \\ 0, p\mid n \end{cases} \quad : (21)$$

Also we get

$$M\left(\frac{d^2}{dx^2}\log\left(\frac{\vartheta_4((p-2a)ix/4,e^{-px/2})}{\vartheta_4((p-2b)ix/4,e^{-px/2})}\right)\right)(s)+2^{-s}\Gamma(s)\zeta(s-1)\sum_{n=1}^{\infty}\frac{X_2(n)n^2}{n^s}=0$$

Or using the periodicity of $X_2(n)$ we can find in a closed form (in terms of Hurwitz Zeta function), the Mellin transform :

4)

$$\int_0^{\infty}\log\left(\frac{\vartheta_4((p-2a)ix/4,e^{-px/2})}{\vartheta_4((p-2b)ix/4,e^{-px/2})}\right)x^{s-1}dx=-\frac{\Gamma(s)\zeta(s+1)}{4p^s}\sum_{r=1}^{p}X_2(r)\zeta\left(s,\frac{r}{p}\right) \; : (22)$$

where

$$\vartheta_4(z,q)=\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}e^{2ikz} \;,\; |q|<1 \; : (23)$$

is the elliptic theta function of the 4-th kind and $p,a,b\in\mathbb{N},\ p>2a,\ p>2b$.

5) Also

$$M\left(1-\vartheta_2(e^{-\pi x})^4-\vartheta_3(e^{-\pi x})^4\right)(s)=24\pi^{-s}\Gamma(s)\zeta(s-1)(2^{1-s}-1)\zeta(s) \; : (24)$$

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The complete evaluation of Rogers Ramanujan and other continued fractions with elliptic functions

Nikos Bagis
Department of Informatics, Aristotle University
Thessaloniki, Greece
bagkis@hotmail.com

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Abstract

In this article we present evaluations of continued fractions studied by Ramanujan. More precisely we give the complete polynomial equations of Rogers-Ramanujan and other continued fractions, using tools from the elementary theory of the Elliptic functions. We see that all these fractions are roots of polynomials with coeficients depending only on the inverse elliptic nome-q and in some cases the Elliptic Integral-K. In most of simplifications of formulas we use Mathematica.

1 Introductory definitions and formulas

For $|q| < 1$, the Rogers Ramanujan continued fraction (RRCF) is defined as

$$R(q) := \frac{q^{1/5}}{1+}\frac{q^1}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \quad (1)$$

We also define

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \quad (2)$$

$$f(-q) := \prod_{n=1}^{\infty}(1-q^n) = (q;q)_\infty \quad (3)$$

$$\Phi(-q) := \prod_{n=1}^{\infty}(1+q^n) = (-q;q)_\infty \quad (4)$$

and also hold the following relations of Ramanujan

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad (5)$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} \quad (6)$$

From the Theory of Elliptic Functions we have:
Let

$$K(x) = \int_0^{\pi/2}\frac{1}{\sqrt{1-x^2\sin(t)^2}}dt \quad (7)$$

It is known that the inverse elliptic nome $k = k_r$, $k_r'^2 = 1 - k_r^2$ is the solution of

$$\frac{K(k_r')}{K(k)} = \sqrt{r} \quad (8)$$

In what it follows we assume that $r \in \mathbf{R}_+^*$. When r is rational then k_r is algebraic.

$$k_r = \frac{8q^{1/2}\Phi(-q)^{12}}{1+\sqrt{1+64q\Phi(-q)^{24}}} \quad (9)$$

We can write the functions f and Φ using elliptic functions. It holds

$$\Phi(-q) = 2\frac{2^{-1/6}q^{-1/24}(k_r)^{1/12}}{(k_r')^{1/6}} \quad (10)$$

$$f(-q)^8 = \frac{2^{8/3}}{\pi^4}q^{-1/3}(k_r)^{2/3}(k_r')^{8/3}K(k_r)^4 \quad (11)$$

also holds

$$f(-q^2)^6 = \frac{2k_r k_r' K(k_r)^3}{\pi^3 q^{1/2}} \quad (12)$$

From [B,G] it is known that

$$R'(q) = 1/5q^{-5/6} f(-q)^4 R(q) \sqrt[6]{R(q)^{-5} - 11 - R(q)^5} \quad (13)$$

2 Evaluations of Rogers Ramanujan Continued Fraction

Theorem 2.1
If $q = e^{-\pi\sqrt{r}}$ and

$$a = a_r = \left(\frac{k_r'}{k_{25r}'}\right)^2 \sqrt{\frac{k_r}{k_{25r}}} M_5(r)^{-3} \quad (14)$$

Then

$$R(q) = \left(-\frac{11}{2} - \frac{a_r}{2} + \frac{1}{2}\sqrt{125 + 22a_r + a_r^2}\right)^{1/5} \quad (15)$$

Where $M_5(r)$ is root of: $(5x-1)^5(1-x) = 256(k_r)^2(k_r')^2 x$.
Proof.
Suppose that $N = n^2\mu$, where n is positive integer and μ is positive real then it holds that

$$K[n^2\mu] = M_n(\mu)K[\mu] \quad (16)$$

Where $K[\mu] = K(k_\mu)$
The following formula for $M_5(r)$ is known

$$(5M_5(r) - 1)^5(1 - M_5(r)) = 256(k_r)^2(k_r')^2 M_5(r) \quad (17)$$

Thus if we use (5) and (10) and the above consequence of the Theory of Elliptic Functions, we get:

$$R^{-5}(q) - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} = a = a_r$$

Solving with respect to R we get the result.

The relation between k_{25r} and k_r is

$$k_r k_{25r} + k_r' k_{25r}' + 2 \cdot 4^{1/3}(k_r k_{25r} k_r' k_{25r}')^{1/3} = 1 \quad (18)$$

We will try to evaluate k_{25r}. For this we set

$$k_{25r} k_r = w^2 \quad (19)$$

then setting directly to (17) the following parametrization of w (see also [B3] pg.280):

$$w=\sqrt{\frac{L(18+L)}{6(64+3L)}} \tag{20}$$

we get

$$\frac{(k_{25r})^{1/2}}{w^{1/2}}=\frac{w^{1/2}}{(k_r)^{1/2}}=\frac{1}{2}\sqrt{4+\frac{2}{3}\left(\frac{L^{1/6}}{M^{1/6}}-4\frac{M^{1/6}}{L^{1/6}}\right)^2}+\frac{1}{2}\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}}-4\frac{M^{1/6}}{L^{1/6}}\right) \tag{21}$$

where

$$M=\frac{18+L}{64+3L}$$

From the above relations we get also

$$-\frac{k_r-w}{\sqrt{k_r w}}=\frac{k_{25r}-w}{\sqrt{k_{25r}w}}=\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}}-4\frac{M^{1/6}}{L^{1/6}}\right) \tag{22}$$

We can consider the above equations as follows: Taking an arbitrary number L we construct an w. Now for this w we calculate the two numbers k_{25r} and k_r. Thus when we know the w, the k_r and k_{25r} are given by (20) and (21). The result is: We can set a number L and from this calculate the two inverse elliptic nome‘s, or equivalently, find easy solutions of (17). But we don't know the r. One can see (from the definition of k_r) that the r can be evaluated from equation

$$r=r_{k_1}=r[L]=\frac{K^2(k_r')}{K^2(k_r)} \tag{23}$$

or

$$r=r_{k_{25}}=r[25L]=\frac{1}{5}\frac{K^2(k_{25r}')}{K^2(k_{25r})} \tag{24}$$

However there is no way to evaluate the r in a closed form, such as roots of polynomials, or else. Some numerical evaluations as we will see, indicate as that even k_r are algebraic numbers, the r are not rational.

Theorem 2.2

Set

$$A_L=a\left(\frac{K^2(k_L')}{K^2(k_L)}\right)=\frac{(k_L)^3(1-(k_L)^2)M_5(L)^{-3}}{(k_L)^2w-w^5} \tag{25}$$

then

$$R\left(e^{-\pi\sqrt{r[L]}}\right)=\left(-\frac{11}{2}-\frac{A_L}{2}+\frac{1}{2}\sqrt{125+22A_L+A_L^2}\right)^{1/5} \tag{26}$$

where the k_L and w are given by (19) and (20).

Example.

Set $L=1/3$ then

$$w=\frac{1}{3}\sqrt{\frac{11}{78}}$$

and

$$k_{25r} = \frac{1}{3}\sqrt{\frac{11}{78}}\left(\frac{-4(\frac{11}{13})^{1/6}+(\frac{13}{11})^{1/6}}{\sqrt{6}}+\frac{1}{2}\sqrt{4+\frac{2}{3}\left(-4\left(\frac{11}{13}\right)^{1/6}+\left(\frac{13}{11}\right)^{1/6}\right)^2}\right)^2$$

and

$$k_r = \frac{\frac{1}{3}\sqrt{\frac{11}{78}}}{\left(\frac{-4(\frac{11}{13})^{1/6}+(\frac{13}{11})^{1/6}}{\sqrt{6}}+\frac{1}{2}\sqrt{4+\frac{2}{3}\left(-4\left(\frac{11}{13}\right)^{1/6}+\left(\frac{13}{11}\right)^{1/6}\right)^2}\right)^2}$$

where the r is given by

$$r = \frac{K^2\left(\sqrt{1-k_r^2}\right)}{K^2\left(k_r\right)}$$

Now we can see (The results are known in the Theory of Elliptic Functions) how we can found evaluations of $R(q)$ when r is given and k_r is known:
From (19) it is

$$L = -9+9w^2+\sqrt{3}\sqrt{27+74w^2+27w^4} \tag{27}$$

from the relation between M and L we get

$$M = \frac{1}{64}\left(9-9w^2+\sqrt{81+222w^2+81w^4}\right) \tag{28}$$

Hence from (20)

$$t = \frac{w-k_r}{\sqrt{k_r w}} \tag{29}$$

also

$$t = \sqrt{\frac{2}{3}}\left(\frac{1}{y^{1/6}}-4y^{1/6}\right) \tag{30}$$

where $y = M/L$. Hence ($k = k_r$):

$$\frac{M}{L} = \left(\frac{\sqrt{3}(k-w)+\sqrt{3k^2+26kw+3w^2}}{8\sqrt{2kw}}\right)^6 \tag{31}$$

or

$$\frac{1}{64}\left(\frac{9-9w^2+\sqrt{81+222w^2+81w^4}}{-9+9w^2+\sqrt{81+222w^2+81w^4}}\right) = \left(\frac{\sqrt{3}(k-w)+\sqrt{3k^2+26kw+3w^2}}{8\sqrt{2kw}}\right)^6$$

or

$$\frac{\sqrt{6}w}{-9+9w^2+\sqrt{81+222w^2+81w^4}}=\left(\frac{\sqrt{3}(k-w)+\sqrt{3k^2+26kw+3w^2}}{8\sqrt{2kw}}\right)^3 \tag{32}$$

setting now

$$k_r^*=\left(\frac{-1+4p^2+\sqrt{1-2p^2+16p^4}}{\sqrt{6}p}\right)^2 \tag{33}$$

and

$$w=\left(\frac{6^{3/4}p^{3/2}}{\sqrt{-1+64p^6+\sqrt{1+88p^6+4096p^{12}}}}\right)^2 \tag{34}$$

$$W=-1+4p^2+\sqrt{1-2p^2+16p^4}$$

and

$$T=-1+64p^6+\sqrt{1+88p^6+4096p^{12}}$$

we have

$$k_r^*w=k_r$$

Also

$$p=\left(\frac{T(2+T)}{216+128T}\right)^{1/6}=\left(\frac{W(2+W)}{6+8W}\right)^{1/2} \tag{35}$$

But

$$w=6k_r\left(\frac{W+2}{(6+8W)W}\right) \tag{34a}$$

$$T=\frac{\sqrt{6}W^2}{k_r}\sqrt{\frac{W(2+W)}{6+8W}}$$

where the equation for finding W from k_r is

$$-108k_r^2\left(\frac{W(W+2)}{8W+6}\right)^{5/2}+\sqrt{6}k_rW^2\left(1-64\left(\frac{W(W+2)}{8W+6}\right)^3\right)+$$

$$+3W^4\left(\frac{W(W+2)}{8W+6}\right)^{1/2}=0 \tag{36}$$

We give the complete polynomial equation of p arising from (35):

$$k_r^2+2\sqrt{6}k_rk_r'^2p-24k_r^2p^2-10\sqrt{6}k_rk_r'^2p^3+240k_r^2p^4+32\sqrt{6}k_rk_r'^2p^5+$$

$$+(54-1388k_r^2+54k_r^4)p^6-128\sqrt{6}k_rk_r'^2p^7+3840k_r^2p^8+640\sqrt{6}k_rk_r'^2p^9-$$

$$-6144k_r^2p^{10}-2048\sqrt{6}k_rk_r'^2p^{11}+4096k_r^2p^{12}=0 \tag{37}$$

It is evident that the Rogers Ramanujan Continued Fraction is a polynomial equation with coeficients depending by k_r.

From (32) we have

$$\sqrt{6k_r^*} = \frac{-1+4p^2+\sqrt{1-2p^2+16p^4}}{p}$$

where p is root of (36).
Using Mathematica we get the following simplification formula for

$$x = \sqrt{k_r^*} = 1/\sqrt{k_{25r}}$$

$$k_r^2 + 4k_r'^2 k_r x - 6k_r^2 x^2 + 20k_r'^2 k_r x^3 + 15x^4 - 16k_r'^2 k_r x^5 + (16 - 52k_r^2 + 16k_r^4)x^6 +$$
$$+16k_r'^2 k_r x^7 + 15k_r^2 x^8 - 20k_r'^2 k_r x^9 - 6k_r^2 x^{10} - 4k_r'^2 k_r x^{11} + k_r^2 x^{12} = 0 \qquad (38)$$

set now

$$c_r = \frac{k_r'^2 (k_r^*)^5}{(k_r^*)^4 - k_r^2} \qquad (39)$$

and

$$G(q) = (R^{-5}(q) - 11 - R^5(q))^{1/3}$$

then
Theorem 2.3
i)

$$3125c_r^2 - 6250c_r^{5/3}G(q) + 4375c_r^{4/3}G^2(q) - 1500c_r G^3(q) + 275c_r^{2/3}G^4(q) +$$
$$+2c_r^{1/3}(-13 + 128k_r'^2 k_r^2)G^5(q) + G^6(q) = 0 \qquad (39a)$$

Also
ii)

$$k_r^6 + k_r^3(-16+10k_r^2)w + 15k_r^4 w^2 - 20k_r^3 w^3 + 15k_r^2 w^4 + k_r(10-16k_r^2)w^5 + w^6 = 0 \qquad (39b)$$

Once we know k_r we can calculate w from the above equation and hence the k_{25r}. Hence the problem reduces to solve 6-th degree equations. The first is (16) and the second is (39b).
Proof.
i)
We have

$$R(q)^{-5} - 11 - R^5(q) = a_r = \frac{k_r^3(1-k_r^2)}{w(k_r^2 - w^4)} M_5(r)^{-3}$$
$$= \frac{k_r'^2 (k_r^*)^5}{(k_r^*)^4 - k_r^2} M_5(r)^{-3}$$

and $M_5(r)$ satisfies $(5x-1)^5(1-x) = 256(k_r)^2(k_r')^2 x$.
After elementary algebraic calculations we get the result.
ii)
From (35) we get:

$$-\sqrt{6}t - 3U + 108t^2U^5 + 64\sqrt{6}tU^6 = 0$$

and

$$t=\frac{k_r}{W^2}=\frac{w}{6U^2}$$

and

$$-32\sqrt{6}wU^6+(9-54w^2)U^3+3\sqrt{6}w=0 \qquad (a)$$

$$U=\left(\frac{W(W+2)}{8W+6}\right)^{1/2} \qquad (b)$$

and also

$$\left(\frac{W(W+2)}{8W+6}\right)^{1/2}=\sqrt{\frac{w}{6k_r}}W \qquad (c)$$

Hence solving the system we obtain the 6-th degree equation.

Corollary

The solution of (39b) with respect to k_r when we know w is

$$\frac{w^{1/2}}{(k_r)^{1/2}}=\frac{1}{2}\sqrt{4+\frac{2}{3}\left(\frac{L^{1/6}}{M^{1/6}}-4\frac{M^{1/6}}{L^{1/6}}\right)^2}+\frac{1}{2}\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}}-4\frac{M^{1/6}}{L^{1/6}}\right) \qquad (40)$$

Where

$$w=\sqrt{\frac{L(18+L)}{6(64+3L)}}$$

$$M=\frac{18+L}{64+3L}$$

Theorem 2.3

$$R'(q)=\frac{2^{4/3}(k_r)^{5/12}(k_r')^{5/3}}{5(k_{25r})^{1/12}(k_{25r}')^{1/3}\sqrt{M_5(r)}}\times$$

$$\times\left(-\frac{11}{2}-\frac{a_r}{2}+\frac{1}{2}\sqrt{125+22a_r+a_r^2}\right)^{1/5}\frac{K^2(k_r)}{\pi^2 q} \qquad (41)$$

Proof.

Combining (11) and (10) and Theorem 2.1 we get the proof.

Evaluations.

$$R(e^{-2\pi})=\frac{-1}{2}-\frac{\sqrt{5}}{2}+\sqrt{\frac{5+\sqrt{5}}{2}}$$

$$R'(e^{-2\pi})=8\sqrt{\frac{2}{5}\left(9+5\sqrt{5}-2\sqrt{50+22\sqrt{5}}\right)}\frac{e^{2\pi}}{\pi^3}\Gamma\left(\frac{5}{4}\right)^4$$

Sumarizing our results we can say that:

1) Theorem 2.1 is quite usefull for evaluating $R(q)$ when we know k_r and k_{25r}. But this it whas known allrady to Ramanujan by using the function $X(-q)=(-q;q^2)_\infty$, (see [8]).

2) Theorem 2.2 is more kind of a Lemma rather a Theorem and it might help

for further research.
3) Theorem 2.3 is a proof that the Rogers Ramanujan continued fraction is a root of a polynomial equation with coeficients the k_r where r positive real.
4) Theorem 2.4 is a consequence of a Ramanujan integral first proved by Andrews (see [5]) and it is usefull for evaluations of $R'(e^{-\pi\sqrt{r}})$, $r \in \mathbf{Q}$.
The above theorems can used to derive also modular equations of $R(q)$, from the modular equations of k_r. More precise we can guess an equality with the help of a methematical pacage (for example in Mathematica there exist the command 'recognize'), and then proceed to proof, using the Theorems which we present in this article. We follow this prosedure with other fractions (the Rogers Ramanujan is litle dificult) such as the qubic or Ramanujan-Gollnitz-Gordon.
These two last continued fractions are more easy to handle. The elliptic function theory and the sigular moduli k_r will exctrac and give us several proofs of modular indenties.

3 The H-Continued Fraction

Heng Huat Chan and Sen-Shan Huang [11] studied the Ramanujan Gollnitz-Gordon continued fraction

$$H(q) := \frac{q^{1/2}}{1+q+}\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^6}{1+q^7+}\ldots \tag{42}$$

where $|q| < 1$.
In a paper of C. Adiga and T. Kim [2] one can find the next identity for this fraction

$$H(q)^{-1} - H(q) = \frac{M^2(q^2)}{M^2(q^4)} \tag{43}$$

where

$$M(q) = \frac{q^{1/8}}{1+}\frac{-q^1}{1+q^1}\frac{-q^2}{1+q^2+}\frac{-q^3}{1+q^3+}\ldots = q^{1/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{44}$$

Next we will use some properties of the inverse Elliptic Nome and show how this can help us to evaluate the H-fraction. For to complete our purpose we need the relation between k_r and k_{4r}. There holds the following
Lemma 3.1

$$k_{4r} = \frac{1-k_r'}{1+k_r'} \tag{45}$$

and

$$K[4r] = \frac{1+k_r'}{2}K[r] \tag{46}$$

Proof.
For (43) see ([B3], pg. 102, 215). The identity for $K[4r]$ is known from the theory of elliptic functions
Theorem 3.1

$$H(q) = -P + \sqrt{P^2+1} \tag{47}$$

where

$$P=\frac{k_r}{(1-k_r')}$$

or

$$k_r=\frac{4(H-H^3)}{(1+H^2)^2} \tag{48}$$

Proof.
It is known that, under some conditions in the sequence b_n (see [16]) it holds

$$\frac{1}{1+}\frac{-b_1}{1+b_1}\frac{-b_2}{1+b_2+}\ldots=1+\sum_{n=1}^{\infty}\prod_{k=1}^{n}b_k \tag{49}$$

Hence if we set $b_n=q^n$, $|q|<1$, then

$$M(q)=\theta_2(q^{1/2})=q^{-1/8}\sqrt{\frac{k_{r/4}K(k_{r/4})}{2\pi}} \tag{50}$$

where

$$\theta_2(q)=\sum_{n=-\infty}^{\infty}q^{(n+1/2)^2}$$

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}=1/2q^{-1/8}\theta_2(q^{1/2})$$

Using Lemma 3.1 and identity (41) we get the proof.
Theorem 3.2
If $ab=\pi^2$, then

$$\left(H(e^{-a})+2-\frac{1}{H(e^{-a})}\right)\left(H(e^{-b})+2-\frac{1}{H(e^{-b})}\right)=8 \tag{51}$$

Proof.
Set

$$\psi(q)=\sum_{n=0}^{\infty}q^{(n+1)n/2} \tag{52}$$

and

$$\phi(q)=\sum_{n=-\infty}^{\infty}q^{n^2} \tag{53}$$

Identity (49) becomes.
If $ab=4\pi^2$

$$\left(2-\frac{\psi(e^{-a})^2}{e^{-a/4}\psi(e^{-2a})^2}\right)\left(2-\frac{\psi(e^{-b})^2}{e^{-b/4}\psi(e^{-2b})^2}\right)=8 \tag{54}$$

From [B3] pg.43 we have if $ab=2\pi$, then

$$\psi(e^{-a^2})=\frac{\sqrt{b}}{2\sqrt{a}}e^{a^2/8}\phi(-e^{-b^2/2}) \tag{55}$$

$$\psi(e^{-2a^2}) = \frac{\sqrt{b/2}}{2\sqrt{a}} e^{a^2/4} \phi(-e^{-b^2/4}) \tag{56}$$

Hence if $ab = \pi^2/4$, ([B3] pg.98)

$$\left(1 - \frac{\phi(e^{-a})}{\phi(-e^{-a})}\right)\left(1 - \frac{\phi(e^{-b})}{\phi(-e^{-b})}\right) = 2 \tag{57}$$

But this is equivalent to

$$k'_{1/(4r)} = \frac{1 - k'_r}{1 + k'_r} \tag{58}$$

(For details [B3] pg.98, 102 and 215). Which is equivalent to

$$k_r = k'_{1/r}. \tag{59}$$

But this is true from the definition of the modulus-k (see relation (7)).
This completes the proof.

Corollary.
If $ab = \pi^2$

$$(1 + \sqrt{2} + H(e^{-a}))(1 + \sqrt{2} + H(e^{-b})) = 2(2 + \sqrt{2}) \tag{60}$$

Proof.
This follows from Theorem 3.2 and as in [B3] pg. 84

Evaluations.

$$H\left(e^{-\pi/2}\right) = \sqrt{1 + 2\sqrt{2} - 2\sqrt{2 + \sqrt{2}}}$$

$$H\left(e^{-\pi\frac{\sqrt{2}}{2}}\right) = \sqrt{3 + 2\sqrt{2} - 2\sqrt{4 + 3\sqrt{2}}}$$

Now it is easy to see how we can construct modular equations of these continued fractions from the modular equations of the inverse elliptic nome. For example for the H continued fraction we give the second degree modular equation:

Theorem 3.3

$$H^2(q) = \frac{H(q^2) - H^2(q^2)}{1 + H(q^2)} \tag{61}$$

Proof.
If $ab = 4$ and $k_r = k(e^{-\pi\sqrt{r}})$, $q_r = e^{-\pi\sqrt{r}}$ then

$$(1 + k_a)(1 + k_b) = 2$$

which can be written as

$$(1 + k_{4a})(1 + k'_a) = 2$$

But from Theorem 3.1

$$k_a = \frac{4H^2(q^{1/2})}{(H^2(q^{1/2}) - 1)^2}$$

and the result follows after elementary algebraic computations.

Also from (47) and (60) one can get

$$\sqrt{k'_r} = \frac{H(q^2) + 2H(q^2) - 1}{H(q^2) - 2H(q^2) - 1} \tag{62}$$

For to proceed we must mention that the relation between k_{9r} and k_r is given by

$$\sqrt{k_r k_{9r}} + \sqrt{k'_r k'_{9r}} = 1 \tag{63}$$

4 The Ramanujan's Cubic Continued Fraction

Let

$$V(q) := \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\ldots \tag{64}$$

is the Ramanujan's cubic continued fraction, then
Lemma 4.1

$$V(q) = \frac{2^{-1/3}(k_{9r})^{1/4}(k'_r)^{1/6}}{(k_r)^{1/12}(k'_{9r})^{1/2}} \tag{65}$$

where the k_{9r} are given by (61)
Proof.
It is known (see and [14] pg. 596) that

$$V(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty}$$

But

$$\Phi(-q) = (-q,q)_\infty = \frac{1}{(q,q^2)_\infty}$$

thus

$$V(q) = q^{1/3}\frac{\Phi(-q^3)^3}{\Phi(-q)}$$

and equation (63) follows from (8).
Lemma 4.2
If

$$G(x) = \frac{x}{\sqrt{2\sqrt{x} - 3x + 2x^{3/2} - 2\sqrt{x}\sqrt{1 - 3\sqrt{x} + 4x - 3x^{3/2} + x^2}}}$$

and

$$k_{9r} = \frac{w}{k_r}$$

and

$$k'_{9r} = \frac{(1-\sqrt{w})^2}{k'_r}$$

then

$$k_r = G(w) \tag{66}$$

Proof.
Set the values of k_r and k_{9r} in (61).
Also holds

$$\frac{1}{(V(q)V(q^3))^{12}} = 256w\frac{\left(1-\frac{w^2}{G(w)^2}\right)^2}{1-G(w)^2}\frac{\left(1-\frac{G(w)^2G^{-1}(w/G(w))^2}{w^2}\right)^3}{G^{-1}(w/G(w))^3}$$

If we set

$$W = 2-3\sqrt{w}+2w-2(1-\sqrt{w})\sqrt{1-\sqrt{w}+w} \tag{67}$$

then

$$V(q) = \frac{(k_r')^{2/3}w^{1/4}}{2^{1/3}(k_r)^{1/3}(1-\sqrt{w})} = \frac{(W-w^{3/2})^{1/3}W^{-1/6}}{2^{1/3}(1-\sqrt{w})} \tag{68}$$

after solving (65) with respect to w and making the simplifications we arrive at

$$2V^3(q) = \frac{\sqrt{W}}{(1+\sqrt{W})^2} \tag{69}$$

and

$$(k_r)^2 = \sqrt{W}\left(\frac{2+\sqrt{W}}{1+2\sqrt{W}}\right)^3 \tag{70}$$

Hence we get the following equation

$$(k_r)^{2/3} = Z^2\frac{\sqrt{2}V(q)^{3/2}+Z^3}{-\sqrt{2}V(q)^{3/2}+2Z^3} \tag{71}$$

Where $Z = \sqrt[12]{W(q)}$. Reducing the above equation in polynomial form we have

$$sk_r^{2/3}+sZ^2-2k_r^{2/3}Z^3+Z^5 = 0 \tag{72}$$

and

$$s^2 = 2V^3(q) = \frac{Z^6}{(1+Z^6)^2} \tag{73}$$

From these two equations we arrive to
Theorem 4.1
Set $T = \sqrt{1-8V(q)^3}$ then holds the next equation

$$(k_r)^2 = \frac{(1-T)(3+T)^3}{(1+T)(3-T)^3} \tag{74}$$

Corollary 4.1
If $X = \sqrt{W(q)} = \frac{1-T}{1+T}$ and $Y = \sqrt{W(q^2)}$, then

$$X^{1/2}\left(\frac{2+X}{1+2X}\right)^{3/2} = 2\frac{Y^{1/4}}{\left(\frac{1+2Y}{2+Y}\right)^{3/4}+Y^{1/2}\left(\frac{2+Y}{1+2Y}\right)^{3/4}} \tag{75}$$

The dublication formula is
Proposition 4.1
Set $u = T(q^2)$, $v = T(q)$, then

$$\frac{\sqrt{(1-u)}(3+u)^{3/2}}{\sqrt{(1+u)}(3-u)^{3/2}} = \frac{(3-v)^{3/2}\sqrt{1+v}-4v^{3/2}}{(3-v)^{3/2}\sqrt{1+v}+4v^{3/2}} \tag{76}$$

We can simplify the problem of finding modular equations of degree 3 using the Cubic continued fraction. As someone can see with direct algebraic calculations and with definitions of W, $V(q)$ and Lemma 4.2 there holds:
Proposition 4.2
If

$$k_{9r} = \frac{w}{k_r}$$

then

$$w = \left(\frac{1-4V(q)^3-8V(q)^6-\sqrt{1-8V(q)^3}}{4V(q)^3\left(1-2V(q)^3-\sqrt{1-8V(q)^3}\right)}\right)^2 \tag{77}$$

The Ramanujan's modular equation which relates $V(q)$ and $V(q^3)$ is

$$V(q)^3 = V(q^3)\frac{1-V(q^3)+V(q^3)^2}{1+2V(q^3)+4V(q^3)^2} \tag{78}$$

(see [10]) one can get from the above formula and Proposition 4.2 the folowing:
Proposition 4.3

$$k_{81r} = \left(\frac{1+2V(q^3)^2-\sqrt{1-8V(q^3)^3}}{1+2V(q^3)^2+\sqrt{1-8V(q^3)^3}}\right)^2 k_r \tag{79}$$

Corollary 4.2
Set $u = H(q)$, $v = H(q^6)$ and

$$t = \frac{4T(q)}{(1+T(q))(3-T(q))}$$

, then

$$\frac{\sqrt{u^4-6u^2+1}(v^2+2v-1)}{(u^2+1)(v^2-2v-1)} = t \tag{80}$$

Proof.
Set $q \to q^3$ in (60) and then use Proposition 4.2 .
Evaluations
a)

$$V(e^{-\pi}) = \frac{1}{2^{2/3}}\left(-67-39\sqrt{3}+(9+6\sqrt{3})\sqrt{2(12+7\sqrt{3})}\right)$$

b)

$$\frac{(3-T(e^{-\pi\sqrt{2}}))^3(3+T(e^{-\pi\sqrt{2}}))^3}{(1-T(e^{-\pi\sqrt{2}}))(1+T(e^{-\pi\sqrt{2}}))} = 5832$$

Using the tables of k_r we can find a wide number of evaluations for the cubic continued fraction.
c) From [13] we have

$$b_{M,N} = \frac{Ne^{-\frac{(N-1)\pi}{4}\sqrt{\frac{M}{N}}}\psi^2(-e^{-\pi\sqrt{MN}})\phi^2(-e^{-2\pi\sqrt{MN}})}{\psi^2(-e^{-\pi\sqrt{\frac{M}{N}}})\phi^2(-e^{-2\pi\sqrt{\frac{M}{N}}})}$$

Then from Theorem 4.1

$$b_{M,3}^2 = \frac{9(1-T^2)}{T^2(9-T^2)}$$

where

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

$|q| < 1$.

5 Other Continued Fractions

Section 1.

Another continued fraction is

$$S(q) = q^{1/8}\frac{1}{1+}\frac{q}{1+}\frac{q^2+q}{1+}\frac{q^3}{1+}\frac{q^4+q^2}{1+}\frac{q^5}{1+}\dots \tag{81}$$

for which it is known that

$$S(q) = q^{1/8}\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} \tag{82}$$

after using Euler's Theorem: $(-q,q)_\infty = 1/(q;q^2)_\infty$ and making some simplifications and rearrangements in the products we find

$$S(q) = q^{1/8}\frac{\Phi(-q^2)^2}{\Phi(-q)}$$

Now making use of (8) we get

$$S(q) = \frac{2^{-1/6}(k_{4r})^{1/6}(k_r')^{1/6}}{(k_r)^{1/12}(k_{4r}')^{1/3}} \tag{83}$$

Using the relation between k_{4r} and k_r form Lemma 3.1, we get
Theorem 5.1

$$S(q) = \frac{(k_r)^{1/4}}{\sqrt{2}} \tag{84}$$

Hence the fraction S is the inverse elliptic nome and as someone can see there holds a very large number of modular equations, but since it is trivial we not mention here.

Section 2.

The continued fraction

$$Q(q)=\frac{q^{1/2}}{1-q+}\frac{q(1-q)^2}{(1-q)(q^2+1)+}\frac{q(1-q^3)^2}{(1-q)(q^4+1)+}\frac{q(1-q^5)^2}{(1-q)(q^6+1)+}\cdots \quad (85)$$

which is known that

$$Q(q)=q^{1/2}\frac{(q^4;q^4)^2_\infty}{(q^2;q^4)^2_\infty}=M(q^2)^2 \quad (86)$$

it becomes

$$Q(q)=...=q^{1/2}f(-q^4)^2\Phi(-q^2)^2=M(q^2)^2 \quad (87)$$

or

Theorem 5.2

$$Q(q)=\frac{1}{\pi}K(k_{4r})\sqrt{k_{4r}}=\frac{1}{2\pi}K(k_r)k_r \quad (88)$$

Proof.

It follows from the relation between Q and M.

Evaluation

$$Q(e^{-\pi\sqrt{2}})=\frac{(\sqrt{2}-1)}{\sqrt{2\pi}}\frac{\Gamma(9/8)}{\Gamma(5/8)}$$

Theorem 5.3

If

$$u=\frac{Q(q)}{Q(q^2)},v=\frac{Q(q^3)}{Q(q^6)}$$

then

$$v^4+u^4-v^3u^3+6v^2u^2-16vu=0 \quad (89)$$

Proof.

From relation (59), we get

$$4\sqrt{vu}+\sqrt{(v^2-4)(u^2-4)}=\sqrt{(v^2+4)(u^2+4)}$$

after some simplification we get the result

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Dear Prof. I cut and translate some parts of my PhD

Stirling Numbers of the first and second kind

a) Stirling 1. ${}_1S_n^{(m)}$

$$x(x-1)...(x-n+1) = \sum_{m=0}^{n} {}_1S_n^{(m)} x^m,\ x \in \mathbb{R},\ n \in \mathbb{N}.$$

b) Stirling 2. ${}_2S_n^{(m)}$

$$x^n = \sum_{m=0}^{n} {}_2S_n^{(m)} x(x-1)...(x-m+1)$$

$${}_2S_n^{(m)} = \frac{1}{m!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n.$$

See [AS] pg. 822-825.

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\cdot\frac{(-x)_n}{n!},\ \text{where}\ a_n(f) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k),$$

See paper with Prof. A. Melas

a) Let $f(x) = \dfrac{x\cos(x)}{x+1}$ (analytic $\mathbb{R}$ -{-1}), with Mathematica

$$\texttt{a[n]} = \frac{e^{-i}\left(-1 + (1 - e^{i})^n + (-1 + (1 - e^{-i})^n)\, e^{2i} + ((1 - e^{-i})^n + (1 - e^{i})^n)\, e^{i}\, n\right)}{2\,(1 + n)}.$$

Evaluating the series $\sum_{n=0}^{\infty} a_n(f)\cdot\frac{(-x)_n}{n!}$ with Mathematica we get $\dfrac{x\cos(x)}{x+1}$.

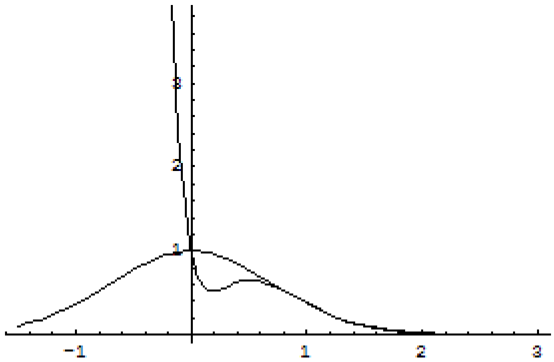
b) Let $f(x) = e^{x/4}x^3$

$$\texttt{a[n]} = -(1 - e^{1/4})^{-3+n}\, e^{1/4}\, n\,(1 + e^{1/4}(1 + n(-3 + e^{1/4} n))).$$

$$\sum_{n=0}^{\infty} a_n(f)\cdot\frac{(-x)_n}{n!} = e^{x/4}x^3.$$

c) $f(x) = e^{-x^2}$, then $T_M f(x) = \sum_{n=0}^{M} a_n(f)\cdot\frac{(-x)_n}{n!}$ για *M*=100, *x = 0.9*, τότε:

$$|f(0.9) - T_{100}f(0.9)| = 0.001.$$



Σχήμα 1: γραφική παράσταση της *f(x)* και $T_{100}f(x)$

d) Let $f(x) = J_0(x/4)$, is J_0 Bessel of the first kind, for $x = \pi$ we have:

$$\left| f(\pi) - \sum_{n=0}^{100} a_n(f) \frac{(-\pi)_n}{n!} \right| = 1.1919x10^{-71}.$$

In general we have good behavior with all

$$f(x) = {}_nF_m \left[\begin{matrix} a_1,\dots,a_n \\ b_1,\dots,b_m \end{matrix} ;x \right], \text{ με } n < m$$

e) If $f(x) = \dfrac{1}{\Gamma(x+1)}$, then:

$$a_k(f) = \sum_{m=0}^{k} (-1)^m \binom{m}{k} \frac{1}{m!} = L_k(1),$$

$L_k(x)$ -Laguerre polynomials, (see [S], chapter 8). Thus

$$\frac{1}{\Gamma(x+1)} = \sum_{k=0}^{\infty} \frac{L_k(1)}{k!} (-x)_k .$$

And knowing that $\Gamma'(1) = -\gamma$, $\left. \dfrac{d(-x)_k}{dx} \right|_{x=0} = -(k-1)!$ it is

$$\gamma = -\sum_{k=1}^{\infty} \frac{L_k(1)}{k}$$

f) Let $a_n(f) = \dfrac{a^n}{n^2+1}$, then

$$f(x) = \mathrm{Re}\left(ia^{-i} Beta(a, i, 1+x) \right)$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathrm{Re}(ia^{-i} Beta(a, i, 1+k)) = \frac{a^n}{n^2+1},$$

where: $Beta(z,a,b) := \int_0^z t^{a-1}(1-t)^{b-1}\,dt$, $Beta(1,a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

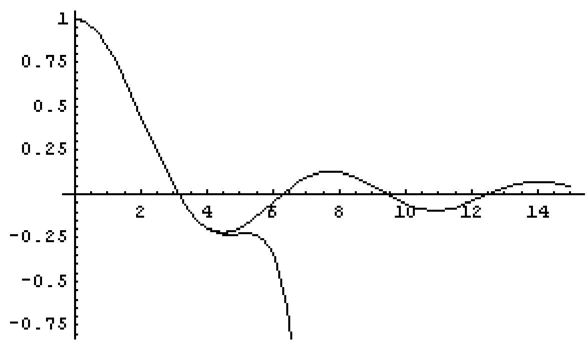
g) For $x >> 1$ in $\sum_{n=0}^{M} a_n(f)\cdot\frac{(-x)_n}{n!}$, $M >> 1$, and $f(x) = \frac{e^x}{e^x+1}$ in Mathematica

```
f[t_] := Exp[t] / (Exp[t] + 1)

a[n_] := Sum[(-1)^k Binomial[n, k] f[k], {k, 0, n}]

N[f[x] /. x → 1/2, 20] =

0.62245933120185456464

N[Sum[a[n] Pochhammer[-x, n] / n!, {n, 0, 100}] /. x → 1/2, 20] =

0.62229762475954152025.
```

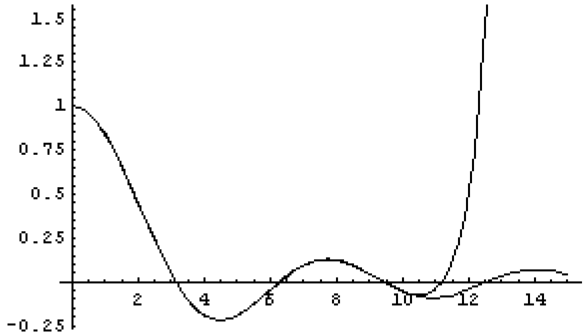
For x>>1

```
N[f[x] /. x → 5 E, 20] =

0.99999874880366142748

N[Sum[a[n] Pochhammer[-x, n] / n!, {n, 0, 100}] /. x → 5 E, 20] =

0.99999874880366142749
```

h) $f(x) = \frac{\sin(x)}{x}, x \in \mathbb{R}$. For the first M=10 (Taylor Series) around $x_0 = 0$.



With Pochhammer: M=10



Proposition.

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}f(k)g(k)=\sum_{k=0}^{\infty}\frac{a_k(f)}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}g(k+l)$$

1) If $f(t)=\dfrac{1}{(c)_t}$ and $g(t)=a^t$, we get:

$$a^{1/2-c/2}e^a\cdot J_{c-1}(2\sqrt{a})\cdot\Gamma(c)=\sum_{n=0}^{\infty}\frac{{}_1F_1(-n,c,1)}{n!}a^n\ .$$

2) If $f(t)=x^t$, $g(t)=(1/2)_{-t}$, then

$$\cosh(2\sqrt{x})=\sqrt{\pi}\sum_{n=0}^{\infty}I_{-1/2+n}(2)\frac{(x-1)^n}{n!}\ .$$

Let $f(t)=J_0(t/4)$ then $\int_a^b f(t)dt$ for *a*=-*e*, *b*=*π*

$$\sum_{n=0}^{M}(-1)^n\frac{a_n(f)}{n!}\sum_{m=0}^{n}{}_1S_n^{(m)}\frac{b^{m+1}-a^{m+1}}{m+1}$$ για *M = 100*:

$$\text{Timing}\left[N\left[\sum_{n=0}^{100}\frac{\text{af}[n]}{n!}\ (-1)\text{^}n\sum_{m=0}^{n}\text{StirlingS1}[n,m]\ \frac{b\text{^}(m+1)-a\text{^}(m+1)}{m+1}-\int_a^b f[t]\,dt,\ 100\right]\right]$$

{10.235 Second, 1.09686995670647943464181471697200709711040915746375249175664 4 × 10^-60}

Let f be analytic in $\mathbb{R}$, then if

$$a_n(f)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k)\ ,$$

and

$$0 < \left| \sum_{n=0}^{\infty} a_n(f) \right| < \infty ,$$

then

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \cdot \frac{(-x)_n}{n!} ,$$

Where $(x)_n = \dfrac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)(x+2)\cdot \ldots \cdot (x+n-1)$

A kind of Proof. (for the complete Proposition see [M,B])

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n .$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\sum_{m=0}^{n} {}_2S_n^m x(x-1)\ldots(x-m+1) \right).$$

But

$$ {}_2S_n^m = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n \qquad (1)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\sum_{m=0}^{n} \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n x(x-1)\ldots(x-m+1) \right)$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\sum_{m=0}^{n} \frac{1}{m!} \sum_{k=0}^{m} (-1)^m \binom{m}{k} k^n (-x)_m \right)$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} (-1)^m \binom{m}{k} k^n (-x)_m \right)$$

$$= \sum_{m=0}^{\infty} (-x)_m \sum_{k=0}^{m} \frac{(-1)^m}{(m-k)!k!} f(k) .$$

Corollary

f, *g* analytic:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f(k) g(k) = \sum_{k=0}^{\infty} \frac{a_k(f)}{k!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} g(k+l) ,$$

Proof.

$$f(x) = \sum_{m=0}^{\infty} \frac{a_m(f)}{m!} \cdot (-x)_m$$

$$f(x)=\sum_{m=0}^{\infty}\frac{a_m(f)}{m!}\cdot\frac{\Gamma(-x+m)}{\Gamma(-x)}$$

Hence

$$f(x)\Gamma(-x)=\sum_{m=0}^{\infty}\frac{a_m(f)}{m!}\cdot\Gamma(-x+m)$$

Hence

$$f(-ix)\Gamma(ix)=\sum_{m=0}^{\infty}\frac{a_m(f)}{m!}\cdot\Gamma(ix+m).$$

$$g(-xi)f(-xi)\Gamma(xi)=\sum_{s=0}^{\infty}\frac{a_s(f)}{s!}\cdot\Gamma(xi+s)g(-xi)\,. \tag{α}$$

Integrating we get the result

(remember that)

$$\int_{-\infty}^{\infty} f(t)(M\Psi)(x+it)dt=2\pi\lim_{r\to 1^-}\sum_{m=0}^{\infty}\frac{\Psi^{(m)}(0)}{m!}f(i\cdot(x+m))r^m$$

Let f analytic Taylor in $\mathbb{R}$, then $a,b\in\mathbb{R},\ a<b$:

$$\int_a^b f(t)dt=\sum_{n=0}^{\infty}(-1)^n\frac{a_n(f)}{n!}\sum_{m=0}^{n}{}_1S_n^{(m)}\frac{b^{m+1}-a^{m+1}}{m+1}\ .$$

Proof.

Let $a,b\in\mathbb{R}$, $a<b$ then:

$$\int_a^b f(t)dt=\sum_{n=0}^{\infty}\frac{a_n(f)}{n!}\int_a^b(-x)_n dx\,. \tag{β}$$

${}_1S_n^{(m)}$ Stirling first kind then:

$$(-x)_n=(-1)^n x(x-1)...(x-n+1)=(-1)^n\sum_{m=0}^{n}{}_1S_n^{(m)}x^n\,. \tag{γ}$$

From (β),(γ) we get the proof.

Some New Results on Prime Sums

Nikos Bagis
Department of Informatics
Aristotle University of Thessaloniki
54124 Thessaloniki, Greece

Abstract

In this work we cosider sums of primes. We set as a base a generalization of Euler's prime number theorem and we use the values of Riemann Zeta function for the approximation. We also give the truncation error of these approximations

keywords Number Theory; Prime Sums; Elliptic Theta Functions; Euler-Totient Constant

1 Finite Sums and Products

Theorem 1
Let f be analytic in $A \supseteq (-1,1]$ with $f(0)=0$. Let also a_n is an arbitrary sequence such that $|a_n|<1$, for $n=1,2,3,....$ Then we have

$$\sum_{k\leq x} f(a_k) = -\sum_{n=1}^{\infty} \log\left(\prod_{k\leq x}(1-a_k^n)\right)\frac{1}{n}\sum_{d|n}\frac{f^{(d)}(0)}{\Gamma(d)}\mu(n/d) \tag{1}$$

Where x is a positive integer and μ is the Moebius Mu function defined by

$$\mu(n) = \left\{ \begin{array}{c} 1,\ if\ n=1 \\ (-1)^r,\ if\ k = the\ product\ of\ r\ distinct\ primes \\ 0,\ othewize \end{array} \right\} \tag{2}$$

Proof.

$$-\sum_{n=1}^{\infty} c_n \log\left(\prod_{k\leq x}(1-a_k^n)\right) = -\sum_{n=1}^{\infty}\frac{c_n}{n}\sum_{k=1}^{x}\log(1-a_k^n) = \sum_{n=1}^{\infty}\frac{c_n}{n}\sum_{k=1}^{x}\sum_{m=1}^{\infty}\frac{1}{m}a_k^{mn} =$$

$$= \sum_{k=1}^{x}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{c_n}{nm}a_k^{mn} = \sum_{k=1}^{x}\sum_{r=1}^{\infty}\frac{1}{r}a_k^r\sum_{d|r}c_d \tag{3}$$

Let now

$$\frac{1}{r}\sum_{d|r}c_d = \frac{f^{(r)}(0)}{r!}$$

From the Moebius Inversion Theorem (see Theorem (ii)) of this article, or for more details one can see [Ap] chapter 2. we get that

$$c_r = \sum_{d|r}\frac{f^{(d)}(0)}{\Gamma(d)}\mu(r/d)$$

Hence equation (3) becomes

$$-\sum_{n=1}^{\infty}\log\left(\prod_{k\leq x}(1-a_k^n)\right)\frac{1}{n}\sum_{d|n}\frac{f^{(d)}(0)}{\Gamma(d)}\mu(n/d) = \sum_{k=1}^{x}\sum_{r=1}^{\infty}\frac{f^{(r)}(0)}{r!}(a_k)^r = \sum_{k=1}^{x}f(a_k)$$

2 The extension of Euler's Theorem

Theorem 2.
If f is analytic in $A \supseteq (-1,1]$ and $f(0)=0$, then we have

$$A(s) := \sum_{p-prime} f\left(\frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\log(\zeta(sn))}{n} \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu(n/d) \tag{4}$$

Where the first sum is over all prime, and ζ is the Riemann's Zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $s > 1$.

Proof.
Set $a_n = 1/p_n^s$, where p_n is the n-th prime, $s > 1$ and $x = \infty$. Then using Euler's Theorem

$$1/\zeta(s) = \prod_{p-prime} \left(1 - \frac{1}{p^s}\right)$$

, $s > 1$ the result follows easily from Theorem 1.

Proposition 1.
The Truncation Error of (3) is

$$E(f,M,s) := \left| \sum_{n=M+1}^{\infty} \frac{\log(\zeta(ns))}{n} \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu(n/d) \right| \tag{5}$$

then

$$E(f,M,s) \leq \frac{2^{s+1}(2^s+1)(s+1)M^2}{\pi(2^s-1)^3(sM+s-1)2^{sM}} \int_0^{2\pi} \left|f(e^{it})\right| dt \tag{6}$$

Proof.
Let

$$E = \left| \sum_{n=M+1}^{\infty} \frac{\log(\zeta(ns))}{n} \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu(n/d) \right|$$

First from the Poisson formula we have

$$\frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-itk} dt$$

thus

$$\left| \frac{f^k(0)}{\Gamma(k)} \right| \leq \frac{k}{2\pi} \int_0^{2\pi} \left|f(e^{it})\right| dt$$

$$\left|\sum_{d|n}\frac{f^{(d)}(0)}{\Gamma(d)}\mu(n/d)\right| \le \sum_{k=1}^{n}\left|\frac{f^{(k)}(0)}{\Gamma(k)}\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{k=1}^{n}k$$

and the error becomes

$$E \le \frac{1}{4\pi}\int_0^{2\pi}\left|f(e^{it})\right|dt\left|\sum_{n=M+1}^{\infty}(n+1)\log(\zeta(sn))\right| \le$$

from the inequality $\log(x) \le x-1$ we arive at

$$E \le \frac{1}{4\pi}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=M+1}^{\infty}(n+1)\left(\frac{1}{2^{sn}}+\frac{1}{3^{sn}}+\frac{1}{4^{sn}}+...\right) \tag{7}$$

We will try to estimate the series

$$A(x) := \frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+...,$$

$x = sn$

$$A(x) = \frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+... \le \frac{1}{2^x}+\int_2^{\infty}\frac{1}{t^x}dt =$$

$$= \frac{1}{2^{sn}}+\frac{2}{(sn-1)2^{sn}} = \frac{(sn+1)}{2^{sn}(sn-1)}$$

Thus we can write

$$E \le \frac{1}{4\pi}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=M+1}^{\infty}(n+1)\frac{sn+1}{2^{sn}(sn-1)}$$

observe that $n+1 \le 2n$, $1/(sn-1) \le 1/(s(M+1)-1)$, thus

$$E \le \frac{2}{4\pi(sM+s-1)}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=M+1}^{\infty}\frac{n(sn+1)}{2^{sn}} \le$$

$$E \le \frac{s+1}{2\pi(sM+s-1)}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=M+1}^{\infty}\frac{n^2}{2^{sn}} =$$

$$= \frac{s+1}{2\pi(sM+s-1)2^{sM}}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=1}^{\infty}\frac{(n+M)^2}{2^{sn}}$$

$$= \frac{s+1}{2\pi(sM+s-1)2^{sM}}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=1}^{\infty}\frac{n^2+2nM+M^2}{2^{sn}} \le$$

$$\le \frac{s+1}{2\pi(sM+s-1)2^{sM}}\int_0^{2\pi}\left|f(e^{it})\right|dt\sum_{n=1}^{\infty}\frac{4n^2M^2}{2^{sn}} \le$$

$$\leq \frac{2^{s+1}(2^s+1)(s+1)M^2}{\pi(2^s-1)^3(sM+s-1)2^{sM}} \int_0^{2\pi} \left| f(e^{it}) \right| dt$$

and we get the estimate

General observations and motivations

i) Let $\phi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}$, $|q| < 1$, then

$$\sum_{p-prime} \log\left(\phi\left(\frac{1}{p^s}\right)\right) = \sum_{n=1}^{\infty} \frac{\log(\zeta(sn))}{n} \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu(n/d) \tag{8}$$

where $f(x) = \log(\phi(x))$. We try to calculate

$$X(n) := \frac{1}{n} \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu(n/d)$$

From ([Be3], pg. 36) we have the following representation of ϕ

$$\phi(q) = \frac{(-q,-q)_\infty}{(q,-q)_\infty}$$

where $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.
Hence

$$\log(\phi(q)) = \sum_{n=1}^{\infty} \log(1-(-q)^n) - \sum_{n=1}^{\infty} log(1+(-q)^n) =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{m}(-q)^{nm} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m}(-q)^{nm} =$$

$$= \sum_{n=1}^{\infty} (-1)^n \left(\sum_{d|n} \frac{(-1)^d - 1}{d} \right) q^n$$

Hence

$$X(n) = \frac{1}{n} \sum_{d|n} (-1)^d \left(\sum_{\delta|d} ((-1)^{d/\delta} - 1)\delta \right) \mu(n/d)$$

which one can see numerically that

$$X_2(n) = \left\{ \begin{array}{c} 2,\ n \equiv 1 mod4 \\ -3,\ n \equiv 2 mod4 \\ 2,\ n \equiv 3 mod4 \\ -1,\ n \equiv 0 mod4 \end{array} \right\} \tag{9}$$

We prove that X is mod4 periodic:

It is known ([Be3], pg.36) that holds

$$\sum_{k=-\infty}^{\infty} q^{k^2} = \prod_{n=0}^{\infty} \frac{(1+q^{2n+1})(1-q^{2n+2})}{(1-q^{2n+1})(1+q^{2n+2})} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{2n})}{(1-q^{2n-1})(1+q^{2n})}$$

Hence from the relations

$$\prod_{p-prime}\left(1-\frac{1}{p^s}\right) = \frac{1}{\zeta(s)}, s>1$$

$$\prod_{p-prime}\left(1+\frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)}, s>1$$

$$\prod_{p-prime}\phi\left(\frac{1}{p^s}\right) =$$

$$= \prod_{n=1}^{\infty}\prod_{p-prime}\left(1+\frac{1}{p^{(2n-1)s}}\right)\left(1-\frac{1}{p^{2ns}}\right)\left(1-\frac{1}{p^{(2n-1)s}}\right)^{-1}\left(1+\frac{1}{p^{2ns}}\right)^{-1} =$$

$$= \prod_{n=1}^{\infty} \frac{\zeta^2((2n-1)s)\zeta^2(4ns)}{\zeta(2(2n-1)s)\zeta^2(2ns)} = \ldots = \prod_{n=1}^{\infty} \frac{\zeta^2(ns)\zeta^2(4ns)}{\zeta^5(2ns)}$$

Having in mind the above we can write

$$\prod_{p-prime}\phi\left(\frac{1}{p^s}\right) = \prod_{n=1}^{\infty} \frac{\zeta^2(ns)\zeta^2(4ns)}{\zeta^5(2ns)} = \prod_{n=1}^{\infty} \zeta(sn)^{X_2(n)} \tag{10}$$

where $s>1$

For to prove the periodicity of X_2 we go backwards. The only thing that left is the uniqueness of the expansion but this follows from the continuity of $\zeta(s)$, when $s>1$ (see [Kor] pg.15).

Note that we work with $\sum_{k=-\infty}^{\infty} q^{k^2}$ and not with $\sum_{k=-\infty}^{\infty} q^{k^2}$.

ii) Also one can see

Set $\psi(q) = \sum_{k=-\infty}^{\infty} q^{k(k+1)/2}$, $|q|<1$ then

$$\frac{1}{n}\sum_{d|n} \frac{\mu(n/d)}{\Gamma(d)} \frac{\partial^d}{\partial x^d} \left(\log(\psi(x))\right)_{x=0} = -(-1)^n \tag{11}$$

We will show that if $\theta(q)$ is a theta function, obeying some certain conditions then $X(n)$ is a periodic sequence having values only 0 and 1.

From [An] pg. 178 we have

Theorem(i).(The Jacobi triple product identity)

There holds the following factorization formula

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+hn} = \prod_{n=0}^{\infty} (1-q^{2kn+2k})(1-q^{2kn+k-h})(1-q^{2kn+k+h}) \tag{12}$$

From [Ap] we have

Theorem(ii).(The Moebius inversion formula)
If f, g are arithmetic functions then we have

$$\sum_{d|n} f(d)\mu(n/d) = g(n) \Leftrightarrow f(n) = \sum_{d|n} g(d) \tag{13}$$

Theorem 3. If we set $\theta(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{kn^2+hn}$ with $k, h \in \mathbf{N}$, $k > h > 0$ then holds the following relation

$$\frac{1}{n}\sum_{d|n}\frac{\mu(n/d)}{\Gamma(d)}\left(\frac{\partial^d}{\partial x^d}\log(\psi(x))\right)_{x=0} = X_{k,h}(n) \tag{14}$$

where

$$X_{k,h}(n) = \left\{\begin{array}{c} 1,\ n \equiv 0 mod p \\ 1,\ n \equiv k+h mod p \\ 1,\ n \equiv k-h mod p \\ 0,\ else \end{array}\right\} \tag{15}$$

Proof.
It is

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{kn^2+hn} = \prod_{n=0}^{\infty}(1-q^{2kn+2k})(1-q^{2kn+k-h})(1-q^{2kn+k+h}) \tag{16}$$

Observe that

$$\theta(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{kn^2+hn} = \prod_{n=1}^{\infty}(1-q^n)^{X_{k,h}(n)} \tag{17}$$

we can write

$$\log(\theta(q)) = \sum_{n=1}^{\infty} X_{k,h}(n)\log(1-q^n) = -\sum_{n=1}^{\infty} X_{k,h}(n)\sum_{m=1}^{\infty}\frac{1}{m}q^{mn} =$$

$$= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}X_{k,h}(n)q^{mn}$$

or

$$\log(\theta(q)) = -\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{X_{k,h}(d)}{n/d}\right)q^n = -\sum_{n=1}^{\infty}\left(\sum_{d|n}X_{k,h}(d)d\right)\frac{q^n}{n}$$

Thus

$$\frac{1}{n!}\left(\frac{\partial^n\log(\theta(q))}{\partial q^n}\right)_{q=0} = -\frac{1}{n}\sum_{d|n}X_{k,h}(d)d$$

An equivalent form using Theorem(ii) is

$$-\frac{1}{n}\sum_{d|n}\frac{\mu(n/d)}{\Gamma(d)}\left(\frac{\partial^d \log(\theta(q))}{\partial q^d}\right)_{q=0} = X_{k,h}(n) \tag{18}$$

3 The Euler Totient constant

When n is a positive integer, Euler's Totient function $\phi(n)$, is defined to be the number of positive integers not greater than n and relatively prime to n. Interesting constants emerge if we consider the sum of reciplocals of $\phi(n)$. Landau proved that

$$\sum_{n=1}^{N}\frac{1}{\phi(n)} = a\log(N) + b + O\left(\frac{\log(N)}{N}\right)$$

where

$$a = \sum_{k=1}^{\infty}\frac{\mu(k)^2}{k\phi(k)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{315\zeta(3)}{2\pi^4} = 1.9435964368\ldots$$

and

$$b = \frac{315\zeta(3)}{2\pi^4} - \sum_{k=1}^{\infty}\frac{\mu(k)^2\log(k)}{k\phi(k)} = -0.06057\ldots$$

In the above formulas, $\zeta(x)$ is the Riemannn's zeta function and $\zeta(3)$ is Apery's constant.
An alternative expression for the constant b is

$$b = \frac{315\zeta(3)}{2\pi^4}\left(\gamma - \sum_{p-prime}\frac{\log(p)}{p^2-p+1}\right) = -0.06057\ldots$$

We will show how to estimate the value of prime series, such as

$$\sum_{p-prime}\frac{\log(p)}{p^2-p+1} \tag{19}$$

For this we set

$$f_1(x) = \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2}\log(1-x+x^2) \tag{20}$$

then

$$b = \frac{315}{2\pi^4}\zeta(3)\left(\gamma + \sum_{n=2}^{\infty}\frac{\zeta'(n)}{\zeta(n)}\sum_{d|n}\frac{f_1^{(d)}(0)}{\Gamma(d)}\mu(n/d)\right)$$

We can calculate (19), setting f_1 in (4) and differentiating with respect to s. Then

$$\lim_{s\to 1^+}\sum_p {f'}_1\left(\frac{1}{p^s}\right)\frac{\log(p)}{p^s} = \sum_p\frac{\log(p)}{p^2-p+1}$$

Proposition 3.
If we set

$$E_1 = E(f_1, M) := \left| \sum_{n=M+1}^{\infty} \frac{\zeta^{'}(n)}{\zeta(n)} \sum_{d|n} \frac{f_1^{(d)}(0)}{\Gamma(d)} \mu(n/d) \right|$$

when $M >> 1$ then

$$E(f_1, M) \leq (2M+4)\zeta^{'}(M+1)$$

where $\zeta^{'}$ is the derivative of the Riemann's Zeta function.

Proof.
If $a(k) = f_1^{'}(0)k/k!$, then $f(0) = 0$ and

$$a(n) = \left\{ \begin{array}{c} 0,\ n \equiv 1 mod 6 \\ 1,\ n \equiv 2 mod 6 \\ 1,\ n \equiv 3 mod 6 \\ 0,\ n \equiv 4 mod 6 \\ -1,\ n \equiv 5 mod 6 \\ -1,\ n \equiv 0 mod 6 \end{array} \right\} \tag{21}$$

Note. The values of $a(k)$ follow from
$h(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{x}\right) + 1/2\log(1-x+x^2)\right) = x/(x^2-x+1)$,
$h(1/x) = h(x)$ and
$h(x) = x/(x-p_1)(x-p_2) = \left(\sum_{k=0}^{\infty}(xp_1)^k - \sum_{k=0}^{\infty}(xp_2)^k\right)/(p_1-p_2) = \ldots$
p_1, p_2 are the roots of $x^2 - x + 1 = 0$.
Also one can see that $a(k) = \frac{2}{\sqrt{3}}\sin((k-1)\pi/3)$, for $k = 1, 2, 3, \ldots$
Hence

$$\left| \sum_{d|n} \frac{f_1^{(d)}(0)\mu(n/d)}{\Gamma(d)} \right| < \sum_{d|n} |\mu(d)| \leq d(n) \leq n$$

where $d(n) = \sum_{d|n} 1$. Thus

$$E(f_1, M) \leq \sum_{n=M+1}^{\infty} \left| \frac{\zeta^{'}(n)}{\zeta(n)} \right| d(n) \leq \sum_{n=M+1}^{\infty} |\zeta'(n)|\, n = \sum_{n=M+1}^{\infty} \sum_{k=2}^{\infty} \frac{\log(k)}{k^n} n =$$

$$\sum_{k=2}^{\infty} \frac{\log(k)}{k} \sum_{n=M}^{\infty} \frac{n+1}{k^n} = \sum_{k=2}^{\infty} \frac{\log(k)}{k^M} \left(\frac{k - M + kM}{(k-1)^2} \right) \leq$$

$$\sum_{k=2}^{\infty} \frac{\log(k)}{k^M} \left(\frac{4}{k} + \frac{2M}{k} \right) = (2M+4)\zeta^{'}(M+1)$$

4 Finite Sums

Theorem 4.
Let x be an integer greater than 1, s is real, $s > 1$ and g analytic function in $A \supseteq (-1, 1]$, such that $g^{(a)}(0)/a!$ are bounded above. Then

$$\sum_{n \leq x} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{l} g\left(\frac{1}{p_k^{snl}}\right) = \sum_{k=1}^{\infty} g\left(\frac{1}{p_k^s}\right) + O\left(xg\left(\frac{1}{2^{s(x+1)}}\right)\right) \tag{22}$$

when $x >> 1$

Proof.

$$\sum_{k=1}^{\infty} \frac{1}{p_k^s} = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^{sn}}\right)\right) =$$

$$= -\sum_{n=1}^{M} \frac{\mu(n)}{n} \log\left(\prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^{sn}}\right)\right) - \sum_{n=M+1}^{\infty} \frac{\mu(n)}{n} \log\left(\prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^{sn}}\right)\right) =$$

$$= -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{n=1}^{\infty} \log\left(1 - \frac{1}{p_k^{sn}}\right) + \sum_{n=M+1}^{\infty} \frac{\mu(n)}{n} \log(\zeta(sn)) =$$

$$= \sum_{n=1}^{M} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{p_k^{snl}} + \sum_{n=M+1}^{\infty} \frac{\mu(n)}{n} \log(\zeta(sn))$$

Thus we get

$$\sum_{k=1}^{\infty} \frac{1}{p_k^s} - \sum_{n=1}^{M} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{lp_k^{snl}} = \sum_{n=M+1}^{\infty} \frac{\mu(n)}{n} \log(\zeta(sn))$$

Hence

$$\left|\sum_{k=1}^{\infty} \frac{1}{p_k^s} - \sum_{n=1}^{M} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{lp_k^{snl}}\right| =$$

$$\left|\sum_{n=M+1}^{\infty} \frac{\mu(n)}{n} \log(\zeta(sn))\right| \leq$$

$$\frac{C2^s(2^s+1)(s+1)M^2}{(2^s-1)^3(sM+s-1)2^{sM}} \leq$$

$$C\frac{M}{2^{s(M+1)}}$$

Let now $g^{(a)}(0)/a!$, $a = 1, 2, 3, \ldots$ are bounded above. We have

$$\frac{g^{(a)}(0)}{a!}\left(\sum_{k=1}^{\infty} \frac{1}{p_k^{sa}} - \sum_{n=1}^{M} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{lp_k^{snla}}\right) = O\left(\frac{M}{2^{sa(M+1)}} \frac{g^{(a)}(0)}{a!}\right) \tag{23}$$

We sum the above estimation with respect to a. Since $g^{(a)}(0)/a!$ are bounded above we get the result.

5 Asymptotic Sums of Primes

In what follows in this section we need to prove and use the following

Lemma.

$$\log\left(\prod_{p\leq x}\left(1-\frac{1}{p^s}\right)^{-1}\right)=\log(\zeta(s))+O\left(\frac{x}{x^s\log(x)}\right) \tag{24}$$

and

$$\sum_{p\leq x}\frac{\log(p)}{p^s-1}=-\frac{\zeta^{'}(s)}{\zeta(s)}+O\left(\frac{1}{x^{s-1}}\right) \tag{25}$$

when $x>>1$, $s>1$

Proof.
a) First we begin with

$$-\sum_{p-prime}\log\left(1-\frac{1}{p^s}\right)=\log(\zeta(s))$$

or

$$-\sum_{p\leq x}\log\left(1-\frac{1}{p^s}\right)-\log(\zeta(s))=\sum_{p>x}\log\left(1-\frac{1}{p^s}\right)$$

But

$$\sum_{p>x}\log\left(1-\frac{1}{p^s}\right)=O\left(\sum_{p\geq x}\frac{1}{p^s}\right)=O\left(\sum_{n=\pi(x)}^{\infty}\frac{1}{p_n^s}\right)=$$

$$=O\left(\sum_{n=\pi(x)}^{\infty}\frac{1}{(n\log(n))^s}\right)=O\left(\frac{x}{x^s\log(x)}\right)$$

Where we have used
$C_1n\log(n)<p_n<C_2n\log(n)$, $A_1n/\log(n)<\pi(x)<A_2n/\log(n)$,
when $n>>1$. For some costants A_1,A_2,C_1,C_2.
b) From [Be1] pg. 129 we have

$$\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{\log(k)}{k^s}=\sum_{p-prime}\frac{\log(p)}{p^s-1}=\sum_{p\leq x}\frac{\log(p)}{p^s-1}+\sum_{p>x}\frac{\log(p)}{p^s-1} \tag{26}$$

We only have to estimate $\sum_{p>x}\frac{\log(p)}{p^s-1}$, for large x.

$$\sum_{p>x}\frac{\log(p)}{p^s-1}=O\left(\sum_{p>x}\frac{p}{p^s}\right)=O\left(\sum_{n=\pi(x)}\frac{p_n}{p_n^s}\right)=O\left(\sum_{n=\pi(x)}\frac{1}{(n\log(n))^{s-1}}\right)=$$

$$= O\left(\frac{1}{x^{s-1}}\right)$$

We begin with

Theorem 5.
Let f be analytic in $(-1,1]$, $f(0)=0$ and $c_n(f) := \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)}\mu(n/d)$. If $c_n(f)/n$ is bounded above then

$$\sum_{p\le x} f\left(\frac{1}{p^s}\right) = A(s) + O\left(\frac{1}{x^{s-1}\log(x)} f^{'}\left(\frac{1}{x^s}\right)\right) \quad (27)$$

Where $s>1$ and $x >> 1$.

Proof. From

$$\log\left(\prod_{p\le x}\left(1-\frac{1}{p^s}\right)^{-1}\right) = \log(\zeta(s)) + O\left(\frac{1}{x^{s-1}\log(x)}\right)$$

and from Theorem 1 we have

$$\sum_{p\le x} f\left(\frac{1}{p^s}\right) = -\sum_{n=1}^{\infty}\log\left(\prod_{p\le x}\left(1-\frac{1}{p^{sn}}\right)\right)\frac{c_n(f)}{n} =$$

$$= \sum_{n=1}^{\infty}\frac{c_n(f)}{n}\log\left(\zeta(sn) + O\left(\frac{x}{x^{sn}\log(x)}\right)\right) =$$

$$= \sum_{p} f\left(\frac{1}{p^s}\right) + \sum_{n=1}^{\infty} O\left(\frac{x}{x^{sn}\log(x)}\right)\frac{c_n(f)}{n}$$

Hence

$$\left|\sum_{p\ge x} f\left(\frac{1}{p^s}\right)\right| = \sum_{n=1}^{\infty} O\left(\frac{x}{x^{sn}\log(x)}\right)\frac{c_n(f)}{n} =$$

$$= O\left(\frac{x}{\log(x)}\sum_{n=1}^{\infty}\frac{c_n(f)}{e^{s\log(x)n}-1}\right) = O\left(\frac{x}{\log(x)}\sum_{n=1}^{\infty}\left(\sum_{d|n} c_d(f)\right)e^{-s\log(x)n}\right) =$$

$$= O\left(\frac{x}{\log(x)}\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{\Gamma(n)}e^{-s\log(x)n}\right) = O\left(\frac{x}{x^s\log(x)} f^{'}\left(\frac{1}{x^s}\right)\right)$$

Where we have used i)

$$\sum_{n=1}^{\infty}\frac{A_n}{e^{ny}-1} = \sum_{n=1}^{\infty}\left(\sum_{d|n} A_d\right)e^{-ny}$$

and ii)

$$A_n = \sum_{d|n} c_d(f) \Leftrightarrow c_n(f) = \sum_{d|n} A_d \mu(n/d)$$

$$\Leftrightarrow \sum_{d|n} \frac{f(d)(0)}{\Gamma(d)} \mu(n/d) = \sum_{d|n} A_d \mu(n/d)$$

or

$$A_n = \frac{f^{(n)}(0)}{\Gamma(n)}$$

Corollary 1.
Let f be analytic in $A \supseteq (-1,1]$, $f(0) = 0$ and $c_n(f)/n$ is bounded above then

$$\left| \sum_p f\left(\frac{1}{p^s}\right) \right| < +\infty$$

when $s > 1$.

Corollary 2.
Let f be analytic in $A \supseteq (-1,1]$, $f(0) = 0$ and $c_n(f)/n$ is bounded above then

$$\sum_{p \leq x} f\left(\frac{1}{p^s}\right) = f(0)\pi(x) + A(s) + O\left(\frac{x}{x^s \log(x)} f'\left(\frac{1}{x^s}\right)\right)$$

where $s > 1$ and $\pi(x) = \sum_{p \leq x} 1$.
In the same way as in Theorem 5 using the asymptotic formula (25) we get

Theorem 6.
Let f be analytic in $A \supseteq (-1,1]$, $f(0) = 0$ and $c_n(f)/n$ is bounded above then

$$\sum_{p \leq x} f^{'}\left(\frac{1}{p^s}\right) \frac{\log(p)}{p^s} = -A^{'}(s) + O\left(\frac{1}{x^{s-1}} f^{'}\left(\frac{1}{x^s}\right)\right)$$

when $s > 1$ and $x >> 1$.

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A Pochhammer type sampling reconstruction of analytic functions

A. Melas
University of Athens Department of Mathematics Panepistimiopolis Athens 15784 Greece.
amelas@math.uoa.gr
N. Bagis
Aristotle University of Thessaloniki Department of Informatics Thessaloniki Greece
bagkis@hotmail.com

Abstract

We give a characterization of the functions f analytic on the half plane $Re(z) > \sigma$ that can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \beta_n \frac{(-z)_n}{n!}$ for some $\beta_n \in \mathbf{C}$. It will follow that the β_n depend on the values of f on the integers so this gives a sampling reconstruction of these functions. Several examples of this representation are given.

1 Introduction and The Main Theorem

Every polynomial can be written as a product of linear factors. Entire functions behave in much the same way: if $f(z)$ is entire then f can reconstructed by its zeros (Under some certain conditions).
Also a polynomial can always be recovered by the values of the polynomial in the set of integers. Can we recover entire functions under some conditions from their values on the set of integers? One known example of this idea, is the Shannon's Sampling Theorem for band limited functions (see [2]). Many results have been published and come to light in this area.
Here we give a new not band limited type reconstruction.
We will study here the class U of functions f that are analytic in some half-plane of the form
$\{z \in \mathbf{C} : \mathrm{Re(z)} > \sigma\}$, $\sigma > 0$ and such that there exist $\beta_n \in \mathbf{C}$, $n = 1, 2, \ldots$ with

$$f(z) = \sum_{n=0}^{\infty} \beta_n \frac{(-z)_n}{n!} \tag{1.1}$$

uniformly on compact subsets of a possibly smaller half plane $Re(z) > \sigma' \geq \sigma$. Here $(w)_n = w(w+1)\ldots(w+n-1)$ is the Pochhammer symbol. Our main result is a characterization of such functions. To describe it we define the following set:

$$A = \left\{ \gamma \in \mathbf{C} : |\mathrm{Im}(\gamma)| < \frac{\pi}{2}, |1 - \mathrm{e}^{\gamma}| \leq 1 \right\} =$$
$$= \left\{ \alpha + i\beta \in \mathbf{C} : |\beta| < \frac{\pi}{2}, \alpha \leq \log(2\cos(\beta)) \right\}$$

Then we will show the following:
Theorem 1. (a) The following are equivalent
(i) f belongs to U.
(ii)There exist $\sigma_1 > 0$, a polynomial $p(z)$ and a finite complex Borel measure μ supported on A such that:

$$f(z) = p(z) \int_A e^{(z-\sigma)\gamma} d\mu(\gamma) \tag{1.2}$$

(iii) There exist $\sigma_2 > 0$ and constants $N, C > 0$ such that:

$$|f(z + \sigma_2)| \leq C(2x)^x \left|z^{-z}\right| (1 + |z|^N) \tag{1.3}$$

whenever $x = Re(z) > 0$.
(b) If f belongs to U and (1.1) holds on $Re(z) \geq \sigma'$ then we can define (or redefine) the values the values $f(k)$ for k integer with $0 \leq k \leq \sigma'$ in such a way that:

$$\beta_n = \sum_{k=0}^{n} (-1)^k \tbinom{n}{k} f(k) \tag{1.4}$$

for all $n \geq 0$.
The other values of $f(k)$, $k > \sigma'$ are those of f. Actually the above theorem

provides a sampling reconstruction of f by it values on the integers. Note that the values $f(k)$ for $0 \le k \le \sigma'$ may be different from the values of the function f, or its analytic continuation has at those points as we will show by examples. Also the equivalence (ii) and (iii) is more or less well known using analytic functions on unbounded supports but we will give here a proof here for completeness. The main fact in this paper is of course the equivalence of (i) with (ii) or (iii).

2 The series representation

First we will show the following:
Proposition 1.
Suppose z_0 is not a nonnegative integer and that the series $\sum_{n=0}^{\infty} \beta_n \frac{(-z)_n}{n!}$ has bounded partial sums. Then the series

$$f(z) = \sum_{n=0}^{\infty} \beta_n \frac{(-z)_n}{n!} \tag{2.1}$$

converges uniformly on compact subsets of $\{z \in \mathbf{C} : \mathrm{Re(z)} > \mathrm{Re(z_0)}\}$ and defines an analytic function there. Moreover one can define the values $f(k)$ for integers k with $0 \le k \le Re(z_0)$ (if any) such that

$$\beta_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) \tag{2.2}$$

for all $n \ge 0$.
Proof.
Let $K \subseteq \{z \in \mathbf{C} : \mathrm{Re(z)} > \mathrm{Re(z_0)}\}$ be compact and let M_k, ϵ_k be such that $\epsilon_k + Re(z_0) \le Re(z) \le M_k$ whenever $z \in K$. Consider now a $z \in K$. Then for $n > M_k + 1$ we have:

$$\beta_n \frac{(-z)_n}{n!} = \beta_n \frac{(-z_0)_n}{n!} \frac{\Gamma(-z_0)}{\Gamma(-z)} \frac{\Gamma(n-z)}{\Gamma(n-z_0)}$$

so since $\frac{1}{\Gamma(-z)}$ is bounded on K it suffices by Abel's test to prove that $\sum_{n > M_k+1} |b_n(z) - b_{n+1}(z)|$ converges uniformly on K where $b_n = \frac{\Gamma(n-z)}{\Gamma(n-z_0)}$. But since $Re(n-z), Re(n-z_0) > 1$ Stirling's formula implies

$$|b_n(z)| \le c_1 \left| \frac{e^{-n+z}(n-z)^{n-z-1/2}}{e^{-n+z_0}(n-z_0)^{n-z_0-1/2}} \right| \le C_k n^{-(Re(z)-Re(z_0))} \le C_k n^{-\epsilon_k}$$

where $C_k > 0$ is a constant that depends only on K.
Hence for any $n > M_k + 1$:

$$|b_n(z) - b_{n+1}(z)| = |b_n(z)| \left| 1 - \frac{n-z}{n-z_0} \right| \le C_k \frac{|z-z_0|}{|n-z_0|} n^{-\epsilon_k} \le \frac{C'_k}{n^{1+\epsilon_k}}$$

and this completes the proof of the uniform convergence.
Next define $f(k)$ for $0 \le k \le Re(z_0)$ so that (2.2) holds that is $f(0) = \beta_0, f(1) = f(0) - \beta_0, \dots, (-1)^k f(k) = \beta_k - \sum_{m=0}^{k-1} (-1)^m \binom{n}{k} f(m)$, for $k \le Re(z_0)$. Then we will show that (2.2) holds for all n.
Indeed supposing it holds for $0, 1, 2, \dots, n-1$ where $n > Re(z_0)$, (2.1) gives

$$f(n) = \sum_{k=0}^{n} \beta_k \frac{(-n)_k}{k!} = \sum_{k=0}^{n-1} \beta_k (-1)^k \binom{n}{k} + (-1)^n \beta_n =$$

$$\sum_{k=0}^{n-1} \sum_{m=0}^{k} (-1)^m \binom{k}{m} f(m) (-1)^k \binom{n}{k} + (-1)^n \beta_n$$

hence

$$\beta_n = (-1)^n f(n) - \sum_{m=0}^{n-1} \left[\sum_{k=m}^{n-1} (-1)^{m+k+n} \binom{n}{k} \binom{k}{m} \right] f(m) =$$

$$(-1)^n f(n) - \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} f(m) \sum_{k=m}^{n-1} (-1)^{n+k} \binom{n-m}{k-m}$$

and this gives (2.2) since

$$\sum_{k=m}^{n-1} (-1)^{n+k} \binom{n-m}{k-m} = -1$$

To proceed further we need the following:
Lemma 1.
Let $z \in C$ be such that $Re(z) > 0$ and for $a \in \mathbf{C}$ with $|1-a| \le 1$ define:

$$S_N(a) = \sum_{n=0}^{N} (1-a)^n \frac{(-z)_n}{n!}$$

Then $S_N \to a^z$ as $N \to \infty$ and there exists a constant $C_z > 0$ independent of a such that:

$$|a^z - S_N(a)| \le C_z$$

for all such a.
Proof.
Considering the function $F(w) = w^z = \exp(z \log(w))$ analytic on $\mathbf{C} - (-\infty, 0]$ (where log is real and positive on $(1, +\infty)$) we have

$$F(w) - \sum_{n=0}^{N} (w-1)^n \frac{F^{(n)}(1)}{n!} = \frac{(w-1)^{N+1}}{2\pi i} \int_{\gamma_R} \frac{F(\zeta) d\zeta}{(\zeta - 1)^{N+1} (\zeta - w)}$$

where γ_R is the line segment from 0 to $-R$ followed by the circle centered at 0 and of radius R followed by the line segment from $-R$ to 0 on the next sheet of the Riemann surface of $\log(w)$ and then taking $R \to \infty$ we get if $N > Re(z)$ that

$$|a^z - S_N(a)| = \left| \frac{(1-a)^{N+1}}{2\pi i} \left(e^{i\pi z} - e^{-i\pi z}\right) \int_0^\infty \frac{z^z}{(t+1)^{N+1}(t+a)} \right| \leq$$

$$\frac{|\sin(\pi z)|}{\pi} \int_0^\infty \frac{t^{Re(z)-1}}{(t+1)^{N+1}} dt$$

since $Re(z) > 0$ and $|t+a| > t$ whenever $t > 0$ and $|1-a| \leq 1$.
This proves the Lemma.
The following will be needed here and in the examples.

Proposition 2.
Let $A_1, A_2 \subseteq A$ be such that $A_1 + A_2 \subseteq A$. If $s_1, s_2 \in \mathbf{R}$ and

$$f_1(z) = \int_A e^{(z-s_1)\gamma} d\mu_1(\gamma), f_2(z) = \int_A e^{(z-s_2)\gamma} d\mu_2(\gamma) \tag{2.3}$$

on $Re(z) > s_1$ and $Re(z) > s_2$ respectively where μ_1, μ_2 are finite complex Borel measures on A_1, A_2 respectively then there exists a Borel measure μ on A such that:

$$f_1(z) f_2(z) = \int_A e^{(z-s)\gamma} d\mu(\gamma) \tag{2.4}$$

on $x > s = \max(s_1, s_2)$.

Proof.
Suppose that $s_1 > s_2$. Then by Fubini‘s theorem we can take $\mu = \mu_1 * (e^{(s_1-s_2)}\mu_2)$ which is supported on $A_1 + A_2 \subseteq A$ and note that whenever $Re(z) > s$:

$$\int_{A_1} \int_{A_2} \left| e^{(z-s)(\gamma_1+\gamma_2)} \right| \left| e^{(s_1-s_2)\gamma} \right| d\left|\mu_1\right|(\gamma) d\left|\mu_2\right|(\gamma) < +\infty$$

Now we can prove the implication $(i) \Rightarrow (ii)$ in Theorem 1.

Proposition 3.
Let the function f be given by (2.1) where the series converges uniformly on compact subsets of $Re(z) > \sigma$. Choose an integer $s \geq 0$ such that $s > 3/2 + \sigma$. Then
(i) There exist a polynomial $p_1(z)$ of degree $\leq s-1$ and a function $h \in L^2(0, 2\pi)$ such that:

$$f(z) = p_1(z) + z(z-1)\dots(z-s+1) \int_0^{2\pi} (1-e^{i\theta})^{z-s} h(\theta) d\theta$$

for every z with $Re(z) > s$.
(ii) There exists a polynomial $p(z)$ of degree s and a finite Borel measure μ on A such that:

$$f(z) = p(z) \int_\gamma e^{(z-s)\gamma} d\mu(\gamma)$$

whenever $Re(z) > s$.
Proof.
(i) By (2.1) we can write:

$$f(z) = p_1(z) + \sum_{n=s}^{\infty} \beta_n \frac{(-z)\dots(-z+s-1)(-z+s)\dots(-z+n-1)}{n(n-1)\dots(n-s+1)(n-s)!} =$$

$$p_1(z) + (-1)^s z(z-1)\dots(z-s+1) \sum_{m=0}^{\infty} \frac{\beta_{m+s}}{(m+1)\dots(m+s)} \frac{(-z+s)_m}{m!}$$

where $\deg p_1(z) \leq s-1$.
Choose $\overline{\sigma} > \sigma$ with $s > 3/2 + \overline{\sigma}$ that is not a nonnegative integer. Then Stiring's formula gives for all large n:

$$\left| \frac{(-\overline{\sigma})_n}{n!} \right| \geq cn^{-1-\overline{\sigma}}$$

Since the series defining $f(\overline{\sigma})$ converges $\left(\beta_n \frac{(-\overline{\sigma})_n}{n!}\right)_{n\geq 0}$ must be bounded hence $|\beta_n| \leq Cn^{1+\overline{\sigma}}$ for all n and some constant $C > 0$. This implies that:

$$\sum_{m=0}^{\infty} \left| \frac{\beta_{m+s}}{(m+1)\dots(m+s)} \right|^2 \leq C_s \sum_{m=0}^{\infty} m^{2(1+\overline{\sigma}-S)} < +\infty$$

therefore there exists $h \in \mathrm{L}^2(0, 2\pi)$ such that:

$$\frac{\beta_{m+s}}{(m+1)\dots(m+s)} = \int_0^{2\pi} e^{im\theta} h(\theta) d\theta$$

for all $m \geq 0$.
But for $Re(z) > s$, Lemma 1 implies that $\sum_{m=0}^{N} e^{im\theta} \frac{(-z+s)_m}{m!}$ converges to $(1-e^{i\theta})^{z-s}$ boudedly on $(0, 2\pi)$ and hence by the dominated convergence theorem we have:

$$\sum_{m=0}^{\infty} \frac{\beta_{m+s}}{(m+1)\dots(m+s)} \frac{(-z+s)_m}{m!} = \lim_{N\to\infty} \int_0^{2\pi} \sum_{m=0}^{N} e^{im\theta} \frac{(-z+s)_m}{m!} h(\theta) d\theta$$

$$= \int_0^{2\pi} (1-e^{i\theta})^{z-s} h(\theta) d\theta$$

This proves (i).
(ii) Since the mapping $(0, 2\pi) \ni \theta \to \log(1-e^{i\theta}) \in \partial A$ is continuous one to one and onto $(-1)^s h(\theta) d\theta$ induces a finite complex Borel measure μ_0 on $\partial A \subseteq A$. Hence (i) gives:

$$f(z) = p_1(z) + z(z-1)\dots(z-s+1) \int_{\partial A} e^{(z-s)\gamma} d\mu_0(\gamma)$$

and since

$$\frac{1}{z^{k+1}} = \frac{1}{k!}\int_{-\infty}^{\infty} e^{-tz}t^k dt$$

$(Re(z) > 0)$ for any integer $k \geq 0$, (ii) follows from Proposition 2, since $(-\infty, 0) + \partial A \subseteq A$.

3 The equivalence $(ii) \Leftrightarrow (iii)$

The implication $(ii) \Leftrightarrow (iii)$ in Theorem 1 is easy. Clearly from (1.2):

$$|f(z+\sigma_1)| = C(1+|z|^N)\int_A e^{Re(z\gamma)} d\,|\mu|\,(\gamma)$$

where $Re(z) > 0$, $N = \deg p$.
Thus it suffices to prove that $e^{Re(z\gamma)} \leq (2x)^x\,|z^{-z}|$ whenever $x = Re(z) > 0$ and $\gamma \in A$. But setting $z = x+iy, \gamma = \alpha + i\beta$ where $|\beta| < \pi/2$, $e^\alpha \leq 2\cos(\beta)$, we get: $e^{Re(z\gamma)} \leq (2\cos(\beta))^x e^{-\beta y}$ which is maximized on $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ when $\tan(\beta) = -y/x$ giving:

$$e^{Re(z\gamma)} \leq \left(\frac{2x}{\sqrt{x^2+y^2}}\right)^x e^{y\arctan\left(\frac{y}{x}\right)} = (2x)^x\left|z^{-z}\right|$$

To give a self contained proof of the implication $(iii) \Rightarrow (ii)$ we first prove the following:
Lemma 2.
Suppose f is analytic on $Re(z) > 0$ and satisfies

$$|f(z)\Gamma(z)| \leq C\left(\frac{2x}{e}\right)^x \frac{1}{|z|^2} \tag{3.1}$$

for some constant C, whenever $x = Re(z) \geq 1/3$. Then given $\sigma > 1/2$ there exists a finite complex measure μ on A such that:

$$f(z) = \int_A e^{(z-\sigma)\gamma} d\mu(\gamma)$$

on $Re(z) > \sigma$.
Proof.
For $x, t > 0$ let

$$\widetilde{H}(t,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+it)\Gamma(x+it)y^{-x-it}dt \tag{3.2}$$

By Cauchy's Theorem and (3.1) we easily get that $\widetilde{H}$ is independent of x as long as $x \geq 1/3$ so we may write $\widetilde{H}(t,x) = H(t)$ for $x \geq 1/3$ and $t > 0$.
If $t \geq 1$ we have:

$$2\pi\,|H(t)| = 2\pi\left|\widetilde{H}(t,t/2)\right| \leq C\int_{-\infty}^{\infty} t^{-1/3}\frac{dy}{|1/3+iy|^2} \leq C_2 t^{-1/3}$$

Hence $e^{t/2}H(t)X_{(0,+\infty)}(t)$ is in $L^2(\mathbf{R})$ and therefore there exists $g \in L^2(\mathbf{R})$ such that:

$$H(t) = \int_{-\infty}^{\infty} e^{-it\xi - t/2} g(\xi) d\xi \tag{3.3}$$

for $t > 0$ which actually means:

$$H(t) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} e^{-t(1/2+i\xi)} e^{-\epsilon|\xi|^2} g(\xi) d\xi \tag{3.4}$$

If $Re(z) > 1/2$ the Mellin transform inversion formula together with the estimate $H(t) \le C \min\left(\frac{e^{-t/2}}{z}, t^{-1/3}\right)$ implies that:

$$f(z)\Gamma(z) = \int_0^{\infty} t^{z-1} H(t) dt = \int_0^{\infty} t^{z-1} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} e^{-t(1/2+i\xi)} e^{-\epsilon|\xi|^2} g(\xi) d\xi dt \tag{3.5}$$

if we could take the limit outside the integral then we would get:

$$f(z)\Gamma(z) = \lim_{\epsilon \to 0^+} \int_0^{\infty} t^{z-1} \int_{-\infty}^{\infty} e^{-t(1/2+i\xi)} e^{-\epsilon|\xi|^2} g(\xi) d\xi dt =$$

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \left(\int_0^{\infty} t^{z-1} e^{-t(1/2+i\xi)} dt \right) e^{\epsilon|\xi|^2} g(\xi) d\xi =$$

$$= \lim_{\epsilon \to 0^+} \Gamma(z) \int_{-\infty}^{\infty} \frac{1}{(1/2+i\xi)^z} e^{-\epsilon|\xi|^2} g(\xi)\ d\xi = \Gamma(z) \int_{-\infty}^{\infty} \frac{g(\xi)}{(1/2+i\xi)^z} d\xi$$

because $Re(z) > 1/2$ and $g \in L^2(\mathbf{R})$ imply that $\int_{-\infty}^{\infty} \left| \frac{g(\xi)}{(1/2+i\xi)^z} \right| d\xi < +\infty$ so we can apply the dominated convergence in the last step.
Let $\sigma > 1/2$ be fixed.
Since the mapping $\mathbf{R} \ni \xi \to \gamma = \log\left(\frac{1}{1/2+i\xi}\right) \in \partial A$ is one-to-one onto and continuous $(1/2+i\xi)^{-\sigma} g(\xi) d\xi$ induces a finite measure μ on ∂A such that:

$$f(z) = \int_{\partial A} e^{(z-\sigma)\gamma} d\mu(\gamma)$$

on $Re(z) > \sigma$
Hence to complete the proof of the Lemma it suffices to prove that the limit in (3.5) can be taken outside the integral. This can be done as follows:
We have

$$|\widehat{g}(\xi)| = 2\pi \left| e^{t/2} H(t) \right| \le C \min\left(\frac{1}{t}, \frac{1}{t^{1/3}}\right) \le C(1 + |t|^{-1/3})$$

, for $t > 0$ and obviously (being 0) for $t < 0$. Hence:

$$\left| \int_{-\infty}^{\infty} e^{-it\xi} e^{-\delta|\xi|^2} g(\xi) d\xi \right| = \left| \widehat{e^{-\delta|.|^2}} * \widehat{g}(t) \right| \le \frac{C}{\epsilon} \int_{-\infty}^{\infty} e^{-\frac{-(t-u)^2}{4\delta}} (1 + |u|^{-1/3}) du \le$$

$$C + 2C\int_0^\infty e^{-s^2}\frac{1}{|1-\lambda s|^{1/3}}ds$$

where $\lambda = \frac{\sqrt{\epsilon}}{|t|} > 0$.
Since $\int_0^\infty \left|t^{z-1}e^{-t/2}t^{-1/3}\right| dt < +\infty$ when $Re(z) > 1/2$ our claim will follow once we proved that the function $r(\lambda) = \int_0^\infty e^{-s^2}\frac{ds}{|1-\lambda s|^{1/3}}$ is (uniformly) bounded on $\lambda \in (0, +\infty)$. But

$$\left[\int_{2/\lambda}^\infty + \int_0^{1/(2\lambda)}\right] e^{-s^2}\frac{ds}{|1-\lambda s|^{1/3}} \leq 2\int_0^\infty e^{-s^2}ds < +\infty$$

$$\int_{1/\lambda}^{2/\lambda} e^{-s^2}\frac{ds}{|1-\lambda s|^{1/3}} = \frac{3}{2\lambda}\int_{1/\lambda}^{2/\lambda} e^{-s^2}d(\lambda s - 1)^{2/3} =$$

$$= \frac{3}{2/\lambda}\left(e^{-4/\lambda^2} + \int_{1/\lambda}^{2/\lambda} 2se^{-s^2}(\lambda s-1)^{2/3}ds\right) \leq \frac{C}{\lambda^3}e^{-1/\lambda^2}$$

and a similar inequality holds for $\int_{1/(2\lambda)}^{1/\lambda} e^{-s^2}\frac{ds}{|1-\lambda s|^{1/3}}$. This completes the proof since $1/\lambda^3 e^{-1/\lambda^2}$ is bounded on $(0, +\infty)$.
Now the implication $(iii) \Rightarrow (ii)$ can be proved as follows:
Supposing f satisfies (1.3), in view of Stirling's, formula, the function:

$$\widetilde{f}(z) = \frac{1}{(z+1)^{N+3}}f(z+\sigma_2)$$

satisfies (3.1) on $x = Re(z) \geq 1/3$. Hence by Lemma 2 there exists μ_0 such that:

$$f(z+\sigma_2) = (z+1)^{n+3}\int_A e^{(z-\sigma)\gamma}d\mu_0(\gamma)$$

where $r > 2/3 > 1/3$ when $Re(z) > \sigma$.
Hence:

$$f(z) = (z-\sigma_2+1)^{N+3}\int_A e^{(z-\sigma-\sigma_2)\gamma}d\mu(\gamma)$$

on $Re(z) > \sigma + \sigma_2 = \sigma_1$ hence f satisfies (1.2).
This completes the proof of the equivalence $(ii) \Leftrightarrow (iii)$.

4 The implication $(ii) \Rightarrow (i)$

Here we complete the proof of Theorem 1 by showing that if f satisfies (1.2) then it is in **U**.
Clearly we may assume $p(z) = z^m$ for some integer $m \geq 0$. First we have the following:

Lemma 3.
If $\gamma \in \mathbf{C}$ and $1 \leq s \leq n$ are integers then:

$$(1-e^{\gamma})^n - \sum_{k=0}^{s-1} \binom{n}{k} (-1)^k e^{k\gamma} = (-1)^s s \binom{n}{s} e^{\gamma s} \int_0^1 (1-te^{\gamma})^{n-s}(1-t)^{s-1} dt \quad (4.1)$$

Proof.
For $\gamma \in \mathbf{R}$ this follows from Taylor's theorem hence by analytic continuation it holds for all $\gamma \in \mathbf{C}$.
Lemma 4.
For each m, s integers there exist constants $c_{m,r}(s)$ for $r = 0, 1, \ldots, m$ such that:

$$\left(\frac{\partial}{\partial\gamma}\right)^m e^{\gamma s}(1-te^{\gamma})^b = \sum_{r=0}^{m} c_{m,r}(s)b(b-1)\ldots(b-r+1)(1-te^{\gamma})^{b-r}t^r e^{(s+r)\gamma} \quad (4.2)$$

whenever $\gamma, b \in \mathbf{C}$, $t \in \mathbf{R}$ are such that $1 - te^{\gamma} \notin (-\infty, 0]$.
Moreover we have:

$$\sum_{r=0}^{m}(-1)^r c_{m,r}(s)w(w-1)\ldots(w-r+1) = (w+s)^m \quad (4.3)$$

whenever $w \in \mathbf{C}$.
Proof.
The equation (4.2) can be proved by an easy induction on m which actually implies the recursion formula:

$$c_{m+1,r}(s) = -c_{m,r+1}(s) + (s+r)c_{m,r}(s) \quad (4.4)$$

where $c_{m,-1}(s) = c_{m,m+1}(s) = 0$.
To prove (4.3) we take $t = -1$ in (4.2) and $\mathbf{R} \ni \gamma \to +\infty$ to get for any $n > 0$ integer:

$$\sum_{r=0}^{m}(-1)^r c_{m,r}(s)n(n-1)\ldots(n-r+1) = \lim_{\gamma\to\infty}\left(e^{-(n+s)\gamma}\left(\frac{\partial}{\partial\gamma}\right)^m e^{\gamma s}(1+e^{\gamma})^n\right) =$$

$$= \lim_{\gamma\to\infty}\left(e^{-(n+s)\gamma}\sum_{k=0}^{\infty}\binom{n}{k}(k+s)^m e^{(k+s)\gamma}\right) = (n+s)^m$$

Hence as a polynomial equation (4.3) has infinitely many roots, thus it is an identity.
Now we have to prove the following:
Proposition 4.
Let μ be a finite Borel measure on A, $\sigma \geq 0$ and $p(z)$ a polynomial of degree $m \geq 0$. Then for the function:

$$f(z) = p(z)\int_A e^{(z-\sigma)\gamma} d\mu(\gamma) \quad (4.5)$$

we have:
(i) f is defined and analytic on the half-plane $Re(z) > \sigma$.
(ii) If we give arbitrary values $f(k)$ for k integers for $0 \le k \le \sigma$ and let

$$a_n(f) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) \tag{4.5a}$$

for $n = 0, 1, 2, \ldots$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(f) \frac{(-z)_n}{n!} \tag{4.6}$$

whenever $Re(z) > [\sigma] + 1 + m$.
Proof.
(i) Given $z_0 = x_0 + iy_0 \in \mathbf{C}$ with $x_0 > \sigma$ let $\epsilon = 1/2(x_0 - \sigma) > 0$ and let $M > 0$ be such that $\epsilon M > \frac{\pi}{2}(|y_0| + \epsilon)$ and write:

$$A_1 = A \cap \{Im(\gamma) \ge -M\}, A_2 = A \cap \{Im(\gamma) < -M\}$$

Then A_1 is compact hence $f_1(z) = p(z) \int_A e^{(z-\sigma)\gamma} d\mu(\gamma)$ is entire.
Now let

$$g(z) = \int_{A_2} e^{(z-\sigma)\gamma} d\mu(\gamma)$$

for z inside the square centered at z_0 with side 2ϵ.
If $\gamma = \alpha + i\beta \in A_2$, $z = x + iy$ is in this square then $Re(z-\sigma)\gamma = (x-\sigma)\alpha - y\beta \le -\epsilon M + (|y_0| + \epsilon)\pi/2$ thus
$\int_{A_2} \left|e^{(z-\sigma)}\right| d\,|\mu|\,(\gamma) \le |\mu|\,(A_2) < +\infty$. This in view of Morera's theorem implies that g and thus f is analytic.
(ii) It is sufficient to assume that $p(z) = z^m$. Let $s = [\sigma] + 1 \ge 1$ and define $f(k) = \lambda_k$ arbitrary for $0 \le k \le s$ (if any). Then $a_n(f) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \lambda_k$ if $n < s$ and let for $n \ge s$:

$$a_n(f) = \int_A \sum_{k=s}^{n} \binom{n}{k} (-1)^k e^{(k-\sigma)\gamma} k^m d\mu(\gamma) + \sum_{k=0}^{s-1} \binom{n}{k} (-1)^k \lambda_k =$$

$$\int_A \left\{ \left(\frac{\partial}{\partial\gamma}\right)^m \left[(1 - e^\gamma)^m - \sum_{k=0}^{s-1} \binom{n}{k} (-1)^k e^{k\gamma} \right] \right\} e^{-\sigma\gamma} d\mu(\gamma) + \sum_{k=0}^{s-1} \binom{n}{k} (-1)^k \lambda_k$$

This holds for all n since $\binom{n}{k} = 0$ for $k > n$. Hence by Lemma 3 we have:

$$a_n(f) =$$

$$\int_A \int_0^1 \left(\frac{\partial}{\partial\gamma}\right)^m (-1)^s s \binom{n}{s} e^{\gamma s} (1 - te^\gamma)^{n-s} (1-t)^{s-1} dt e^{\sigma\gamma} d\mu(\gamma) + \sum_{k=0}^{s-1} \binom{n}{k} (-1)^k \lambda_k$$

Now using Lemma 4 with $b = n - s$ and summing for $n = 0, 1, \ldots, N$ we get if $z \in \mathbf{C}$ is such that $Re(z) > s + m$

$$\sum_{n=0}^{N} a_n(f) \frac{(-z)_n}{n!}$$

$$= (-1)^s s \int_A e^{(s-\sigma)\gamma} \sum_{n=0}^{N} \sum_{r=0}^{m} c_{n,r}(s)(n-s)\dots(n-s-r+1) \binom{n}{s} \frac{(-z)_n}{n!} \times$$

$$\times \int_0^1 (1-te^\gamma)^{n-s-r} t^r e^{t\gamma} (1-t)^{s-1} dt d\mu(\gamma) + \sum_{r=0}^{s-1} (-1)^k \lambda_k \sum_{n=0}^{N} \binom{n}{k} \frac{(-z)_n}{n!}$$

By Lemma 1

$$\lim_{N\to\infty} \left(\sum_{n=0}^{N} \frac{(-z)_n}{n!} \right) = \lim_{N\to\infty} \left(\sum_{j=0}^{N-k} \frac{(-z+k)_j}{j!} \right) \frac{(-z)_k}{k!} = 0$$

as long as $Re(z) > k$. Hence the last sum tends to 0 as $N \to \infty$.
Fixing r with $0 \le r \le m$ we have:

$$= \sum_{n=0}^{N} s(n-s\dots(n-s-r+1) \binom{n}{s} \frac{(-z)_n}{n!} (1-te^\gamma e^\gamma)^{n-sr} =$$

$$= \frac{(-z)_{s+r}}{(s-1)!} \sum_{n=s+r}^{N} (1-te^\gamma)^{n-s-r} \frac{(-z+s+r)_{n-s-r}}{(n-s+r)!} =$$

$$= \frac{(-z)_{s+r}}{(s-1)!} \sum_{n=0}^{N-s-r} (1-te^\gamma)^n \frac{(-z+s+r)_n}{n!}$$

Using Lemma 1 and since $Re(z-s-r) > 0$ for $0 \le r \le m$ the series $\sum_{n=0}^{N-s-r} (1-te^\gamma)^n \frac{(-z+s+r)_n}{n!}$ converges boundedly to $t^{z-s-r} e^{(z-s-r)\gamma}$ on $t \in [0,1]$, $\gamma \in A$ (noting that $|1-te^\gamma| \le 1$ for all such t, γ). Hence using that $s - \sigma > 0$ the dominated convergence implies that:

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(f) \frac{(-z)_n}{n!} =$$

$$= \sum_{r=0}^{m} c_{m,r}(s) \frac{(-1)^s}{(s-1)!(-z)_{s+r}} \int_A e^{(s-\sigma)\gamma} \int_0^1 t^{(z-s-r)\gamma} t^r e^{r\gamma} (1-t)^{s-1} dt d\mu(\gamma) =$$

$$= \sum_{r=0}^{m} c_{m,r}(s) \frac{(-1)^s}{(s-1)!} (-z)_{s+r} \int_A e^{(z-\sigma)\gamma} \int_0^1 t^{(z-s+1)-1} (1-t)^{s-1} dt d\mu(\gamma) =$$

$$= \sum_{r=0}^{m} c_{m,r}(s) \frac{(-1)^s}{(s-1)!} (-z)_{s+r} \frac{\Gamma(s)\Gamma(z-s+1)}{\Gamma(z+1)} \int_A e^{(z-\sigma)\gamma} d\mu(\gamma) =$$

$$= \int_A e^{(z-\sigma)\gamma} d\mu(\gamma) \sum_{r=0}^{m} (-1)^r c_{m,r}(s)(z-s)\dots(z-s-r+1) =$$

$$= z^m \int_A e^{(z-\sigma)\gamma} d\mu(\gamma)$$

by Lemma 4. This completes the proof.

5 Examples and Applications

Here we start with the following:
Proposition 5.
Suppose f is as in Proposition 4 and for some $\sigma' \in \mathbf{R}$ we have:
(i) f can be analytically continued on $Re(z) > \sigma'$ (if $\sigma < \sigma'$).
(ii) For any z_0 with $Re(z_0) > \sigma'$ the series $\sum_{n=0}^{\infty} a_n(f)\frac{(-z_0)_n}{n!}$ has bounded partial sums (for some choice of $f(k)$'s for $0 \leq k \leq \sigma$). Then (4.6) holds for any $z \in \mathbf{C}$ with $Re(z) > \sigma'$.
Proof.
Follows from Proposition 1 and the principle of analytic continuation.
Example 1.
Let $f(z) = 1/z$ and $Re(z) > 0$. Defining $f(0)$ to be 0 and since $f(z) = \int_0^\infty e^{-tz}dt$ Proposition 4 can be applied to give

$$1/z = \sum_{n=0}^{\infty} a_n(f)\frac{(-z)_n}{n!}$$

on $Re(z) > 1$. But this can be improved since for $n > 0$:

$$a_n(f) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1}{k} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \int_0^1 (1-x)^{k-1}dx = \int_0^1 \frac{x^n - 1}{1-x}dx =$$

$$-\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = -\log(n) + \gamma + O\left(\frac{1}{n}\right)$$

Since for $Re(z) > 0$ the series $\sum_{n=1}^{\infty} \log(n)\frac{(-z)_n}{n!}$ converges absolutely (by Stirling formula $\left|\frac{(-z)_n}{n!}\right| \leq Cn^{-1-Re(z)}$) we get:

$$\frac{1}{z} = -\sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\frac{(-z)_n}{n!}$$

on $Re(z) > 0$.
Example 2.
$f(z) = 2^z$ is entire and $a_n(f) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^k = (-1)^n$ for all n.
However:

$$2^z = \sum_{k=0}^{\infty} \frac{(-z)_n}{n!}(-1)^n$$

for $Re(z) \geq -1$. But the series doesn't converge at any $z \in \mathbf{R}$ with $z < -1$. Hence (ii) in Proposition 5 is necessary.
For this we have the following:
Proposition 6. If μ is supported on a compact subset K of the interior A^0 of A then

$$f(z) = p(z)\int_K e^{z\gamma}d\mu(\gamma)$$

is entire and

$$f(z)=\sum_{n=0}^{\infty}a_n(f)\frac{(-z)_n}{n!}$$

for every $z\in\mathbf{C}$ where $a_n(f)$ are given by (4.5a).

Proof.

Follows from Proposition 5 since for $p(z)=z^m$ we have:

$$|a_n(f)|=\left|\int_K\left(\frac{\partial}{\partial\gamma}\right)^m\sum_{k=0}^{n}\tbinom{n}{k}(-1)^k e^{k\gamma}d\mu(\gamma)\right|=\left|\int_K\left(\frac{\partial}{\partial\gamma}\right)(1-e^{\gamma})d\mu(\gamma)\right|\leq$$

$$\sum_{r=0}^{m}\left|c_{m,r}(0)n^r\int_K|1-e^{\gamma}|^{n-\gamma}\,d\,|\mu|\,(\gamma)\right|\leq Cn^m\lambda^{n-m}\,|\mu|\,(K)$$

where $\lambda=\max_{\gamma\in K}|1-e^{\gamma}|<1$. Hence for every $l>0$ the series $\sum_{n=0}^{\infty}|a_n(f)|\,n^l$ converges implying by Stirling‘s formula that $\sum_{n=0}^{\infty}a_n(f)\frac{(-z)_n}{n!}$ converges absolutely for every $z\in\mathbf{C}$.

For analytic functions involving Laplace transforms, of the form

$$f(z)=p(z)\int_0^{\infty}e^{-(z-\sigma)t}d\mu(t)$$

where $Re(z)>\sigma\geq 0$ and μ is a Borel complex measure we have the following improvement of Proposition 4, concerning the range where (4.6) holds.

Proposition 7.

Suppose μ is supported on $(-\infty,0]$. Then (4.6) holds whenever $Re(z)>[\sigma]+1$.

Proof.

We will use Proposition 5. Suppose $p(z)=z^m$. Since $\left|\sum_{n=0}^{N}\frac{(-z)_n}{n!}\right|$ is bounded in N for any z with $Re(z)>0$ (Lemma 1) in order to verify the property (ii) in Proposition 5, in view of Abel‘s test and since (with $s=[\sigma]+1$):

$$\sum_{n=0}^{N}a_n(f)\frac{(-z)_n}{n!}=q(z)+r(z)\sum_{n=s}^{N}\frac{a_n(f)}{n(n-1)\dots(n-s+1)}\frac{(-z+s)_n}{(n-s)!}$$

(p,q polynomials) it suffices to show that

$$\sum_{n=s}^{\infty}\left|\frac{a_n(f)}{n(n-1)\dots(n-s+1)}-\frac{a_{n+1}(f)}{(n+1)n\dots(n-s+2)}\right|<+\infty$$

But by the proof of Proposition 4 (ii) we have (supposing the choice $\lambda_n=0$):

$$\frac{a_n(f)}{n(n-1)\dots(n-s+1)}=$$

$$=\int_0^{\infty}\int_0^1\left(\frac{\partial}{\partial\gamma}\right)^m\frac{(-1)^s}{(s-1)!}e^{-ts}(1-ue^{-t})^{n-s}(1-u)^{s-1}due^{\sigma t}d\mu(t)$$

Hence

$$e_n = (s-1)! \left| \frac{a_n(f)}{n(n-1)\dots(n-s+1)} - \frac{a_{n+1}(f)}{(n+1)n\dots(n-s+2)} \right| \leq$$

$$\sum_{r=0}^{m} |c_{m,r}(s)| \int_0^\infty \int_0^1 u^r e^{-rt}(1-u)^{s-1}(n-s)\dots(n-s-r-1)\times$$

$$\times \left[(1-ue^{-t})^{n-s-r} - (1-ue^{-t})^{n+1-s-r}\right] due^{-(s-\sigma)t} d\,|\mu|\,(t) \leq$$

$$\leq \sum_{r=0}^{m} |c_{m,r}(s)| \int_0^\infty \int_0^1 u^{r+1}(1-u)^{s-1}\times$$

$$\times e^{-(\gamma+1)t}(n-s)\dots(n-s-r+1)(1-ue^{-t})^{n-s-r} due^{-(s-\sigma)t} dt$$

But since $|1-ue^{-t}| < 1$ on $0 < u < 1, t > 0$ we have

$$\sum_{n=s+r}^{\infty} \int_0^\infty \int_0^1 u^{r+1}(1-u)^{s-1} e^{-(\gamma+1)t} \left(\frac{1}{1-(1-ue^{-t})}\right)^{r+1} e^{-(s-\sigma)t} dud\,|\mu|\,(t) \leq$$

$$\leq \int_0^\infty e^{-(s-\sigma)t} d\,|\mu|\,(t) < +\infty$$

since $s > \sigma$, for any $r = 0, 1, 2, \dots, m$
hence $\sum_n e_n < +\infty$ and this completes the proof.
Examples
1) Since $\sqrt{z} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-tz} \frac{dt}{\sqrt{t}}$ on $Re(z) > 0$ (branch positive on $(0, +\infty)$) we get from the previous proposition that:

$$\sqrt{z} = \sum_{n=0}^{\infty} a_n \frac{(-z)_n}{n!}$$

for $Re(z) > 1$ where

$$a_n = \sum_{k=0}^{\infty} \binom{n}{k} (-1)^k \sqrt{k}$$

2) More generally since

$$z^{-a-1} = \frac{1}{\Gamma(a+1)} \int_0^\infty e^{-tz} t^a dt$$

whenever $a > -1/2$ these functions are in U.
3) The formula

$$J_a(z) = \frac{(z/2)^a}{\Gamma(a+1)\Gamma(1/2)} \int_{-1}^1 e^{izs}(1-s^2)^a \frac{ds}{\sqrt{1-s^2}}$$

$a > 1/2$, $z \in \mathbf{C}$ for the Bessel functions combined with the above example and the Proposition 2 imply that J_a are in U.
4) Consider the function

$$f(z) = \frac{1}{\Gamma(z+1)}$$

By Stirling's formula

$$f(z) = \frac{e^{z+1}}{\sqrt{2\pi}(z+1)^{z+1/2}} e^{J(z)}$$

where $|J(z)| \to 0$ as $z \to \infty$ with $Re(z) > 0$ hence the estimate (1.3) can be directly verified. Hence with

$$\beta_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k!}$$

we have the representation

$$\frac{1}{\Gamma(z+1)} = \sum_{n=0}^{\infty} \beta_n \frac{(-z)_n}{n!}$$

with $Re(z) > 1$.
5) Consider the measure

$$\mu = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1+\sigma}} \delta(-\log(k))$$

where δ_a denotes the Dirac mass at a and $\sigma > 0$: Clearly μ is supported on A and is finite hence for $Re(z) > \sigma$ and

$$f(z) = \int_A e^{(z-\sigma)\gamma} d\mu(\gamma) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1+z}} = (2^{-z} - 1)\zeta(1+z)$$

is in U and has an analytic continuation to an entire function.
6) More generally if $F : \overline{D(0,r)} \to \mathbf{C}$ is analytic then for any polynomial $p(z)$ the function $f(z) = p(z)F(e^{-z})$ is in U. Indeed writing $F(w) = \sum_{k=0}^{\infty} c_k w^k$ where $\sum_{k=0}^{\infty} |c_k| r^k < +\infty$ we can write

$$F(e^{-z}) = \sum_{k=0}^{\infty} c_k e^{-kz} = \int_A e^{(z-\sigma)\gamma} d\mu(\gamma)$$

for $Re(z) > \sigma$ where $\mu = \sum_{k=0}^{\infty} c_k e^{-k\sigma} \delta_{-k}$ is a finite measure if $\sigma > \log\left(\frac{1}{r}\right)$.
For example $\frac{z^m e^z}{e^z+1}$, $z^m\sqrt{1+e^{-z}}$, $e^{e^{-z}}$, etc define functions in U.
7) The function $f(z) = e^{-\lambda z^2}$, $\lambda > 0$ is not in U. This can be verified by using Theorem 1 and resulting (1.3). Indeed estimate (1.3) cannot hold for $z = a + it$ as $t \to +\infty$ no matter what $a > 0$ is.

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