Random walks in the plane

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Joint work with:

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U. of Newcastle, AU

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K.U. Leuven, BE

James Wan
U. of Newcastle, AU
Random walks in the plane

- We study random walks in the plane consisting of \( n \) steps. Each step is of length 1 and is taken in a randomly chosen direction.
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We are interested in the distance traveled in \( n \) steps. For instance, how large is this distance on average?
Long walks

- Asked by Karl Pearson in Nature in 1905

Long walks

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- For long walks, the probability density is approximately \( \frac{2x}{n} e^{-x^2/n} \)
- For instance, for \( n = 200 \):


Densities

$n = 2$

$n = 3$

$n = 4$

$n = 5$

$n = 6$

$n = 7$
Densities

\[ n = 2 \]

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Fact from probability theory: the distribution of the distance is determined by its moments.
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Represent the $k$th step by the complex number $e^{2\pi ix_k}$. The $s$th moment of the distance after $n$ steps is:

\[ W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \, d\vec{x} \]

In particular, $W_n(1)$ is the average distance after $n$ steps.
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This is hard to evaluate numerically to high precision. For instance, Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where $N$ is the number of sample points.
The \( s \)th moment of the distance after \( n \) steps:

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Moments

The $s$th moment of the distance after $n$ steps:

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$W_2(1) = \frac{4}{\pi}$
The $s$th moment of the distance after $n$ steps:

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \, dx$$

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- Sloane’s, etc.:

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W_2(2k) = \binom{2k}{k}
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\[
W_5(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2(k-j)}{k-j} \sum_{\ell=0}^{j} \binom{j}{\ell}^2 \binom{2\ell}{\ell}
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Theorem (Borwein-Nuyens-S-Wan)

\[ W_n(2k) = \sum_{a_1 + \cdots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2. \]
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- \( f_n(k) := W_n(2k) \) counts the number of abelian squares: strings \( xy \) of length \( 2k \) from an alphabet with \( n \) letters such that \( y \) is a permutation of \( x \).
Combinatorics

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- Introduced by Erdős and studied by others.
- \( f_n(k) \) satisfies recurrences and convolutions.


For integers $k \geq 0$,

\[
(k + 2)^2 W_3(2k + 4) - (10k^2 + 30k + 23)W_3(2k + 2) + 9(k + 1)^2 W_3(2k) = 0.
\]
Functional Equations for $W_n(s)$

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$$

**Theorem (Carlson)**

*If $f(z)$ is analytic for $\Re(z) \geq 0$, “nice”, and

$$
f(0) = 0, \quad f(1) = 0, \quad f(2) = 0, \quad \ldots,
$$

then $f(z) = 0$ identically.*
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\[
|f(z)| \leq A e^{\alpha |z|}, \text{ and } |f(iy)| \leq B e^{\beta |y|} \text{ for } \beta < \pi
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For integers $k \geq 0$,

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$W_n(s)$ is nice!

$$|f(z)| \leq Ae^{\alpha|z|}, \text{ and } |f(iy)| \leq Be^{\beta|y|} \text{ for } \beta < \pi$$
So we get complex functional equations like

\[(s + 4)^2 W_3(s + 4) - 2(5s^2 + 30s + 46)W_3(s + 2) + 9(s + 2)^2 W_3(s) = 0.\]
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\[(s+4)^2 W_3(s+4) - 2(5s^2 + 30s + 46)W_3(s+2) + 9(s+2)^2 W_3(s) = 0.\]

This gives analytic continuations of \(W_n(s)\) to the complex plane, with poles at certain negative integers.
$W_3(1) = 1.57459723755189 \ldots = ?$

- Easy: $W_2(2k) = \binom{2k}{k}$. In fact, $W_2(s) = \binom{s}{s/2}$.
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- Easy: $W_2(2k) = \binom{2k}{k}$. In fact, $W_2(s) = \binom{s}{s/2}$.
- Again, from combinatorics:

\[
W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = {}_3F_2 \left( \begin{array}{c} \frac{1}{2}, -k, -k \\ 1, 1 \end{array} \bigg| 4 \right) =: V_3(2k)
\]
\[ W_3(1) = 1.57459723755189 \ldots = ? \]

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- We discovered numerically that \( V_3(1) \approx 1.574597 - .126027i \).
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\[
W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \binom{3}{2, -k, -k \mid 1, 1 \mid 4} =: V_3(2k)
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**Theorem (Borwein-Nuyens-S-Wan)**

For integers \( k \) we have \( W_3(k) = \Re \binom{3/2, -k/2, -k/2 \mid 1, 1 \mid 4} \).
$W_3(1) = 1.57459723755189 \ldots = ?$

Corollary (Borwein-Nuyens-S-Wan)

$$W_3(1) = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6 \left( \frac{1}{3} \right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6 \left( \frac{2}{3} \right)$$

- Similar formulas for $W_3(3), W_3(5), \ldots$
A generating function

Recall:

\[ W_n(2k) = \sum_{a_1 + \ldots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2 \]
A generating function

- Recall:

\[ W_n(2k) = \sum_{a_1 + \cdots + a_n = k} \left( \begin{array}{c} k \\ a_1, \ldots, a_n \end{array} \right)^2 \]

- Therefore:

\[
\sum_{k=0}^{\infty} W_n(2k) \frac{(-x)^k}{(k!)^2} = \sum_{k=0}^{\infty} \sum_{a_1 + \cdots + a_n = k} \frac{(-x)^k}{(a_1!)^2 \cdots (a_n!)^2} = \left( \sum_{a=0}^{\infty} \frac{(-x)^a}{(a!)^2} \right)^n = J_0(2\sqrt{x})^n
\]
Theorem (Ramanujan’s Master Theorem)

For “nice” analytic functions $\varphi$,

$$
\int_0^{\infty} x^{\nu-1} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \varphi(k) x^k \right) \, dx = \Gamma(\nu) \varphi(-\nu).
$$
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- **Begs to be applied to**

\[
\sum_{k=0}^\infty W_n(2k) \frac{(-x)^k}{(k!)^2} = J_0(2\sqrt{x})^n
\]

by setting \( \varphi(k) = \frac{W_n(2k)}{k!} \)
Ramanujan’s Master Theorem

- We find:

\[ W_n(-s) = 2^{1-s} \frac{\Gamma(1 - s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^n(x) \, dx \]
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- A 1-dimensional representation!
  Useful for symbolical computations
  as well as for high-precision integration
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- First and more inspiredly found by David Broadhurst building on work of J.C. Kluyver


A convolution formula

Conjecture

For even $n$,

$$W_n(s) = \sum_{j=0}^{\infty} \left( \frac{s}{2} \right)^2 \binom{\frac{s}{2}}{j} W_{n-1}(s-2j).$$
Conjecture

For even \( n \),

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\]

- Inspired by the combinatorial convolution for \( f_n(k) = W_n(2k) \):

\[
f_{n+m}(k) = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right)^2 f_n(j) f_m(k - j)
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A convolution formula

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W_n(s) = \sum_{j=0}^{\infty} \left( \frac{s}{2j} \right)^2 W_{n-1}(s - 2j).
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\]

- True for even $s$
- True for $n = 2$
- Now proven up to some technical growth conditions
You will have to look at the papers to find…

- a hyper-closed form for $W_4(1)$,
- Meijer-G and hypergeometric expressions for $W_3(s)$ and $W_4(s)$,
- evaluations of derivatives including

\[ W'_3(0) = \frac{1}{\pi} \text{Cl} \left( \frac{\pi}{3} \right) , \quad W'_4(0) = \frac{7\zeta(3)}{2\pi^2} , \]

- expressions for residues at the poles of $W_n(s)$,
- …
References


Both preprints as well as this talk are/will be available from: http://arminstraub.com

THANK YOU!

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