Effective Computation of Bessel Functions, Part II

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For any complex pair \((p, q)\) and real numbers \(\alpha, \beta \in (-\pi, \pi)\), let

\[
I(p, q, \alpha, \beta) := \int_{\alpha}^{\beta} e^{-iq\omega} e^{p\cos \omega} d\omega.
\]

Then we have the absolutely convergent representation

\[
I(p, q, \alpha, \beta) = \frac{ie^p}{q} \sum_{k=0}^{\infty} \frac{r_{k+1}(-2i)}{k!} \int_{\sin \frac{\alpha}{2}}^{\sin \frac{\beta}{2}} x^k e^{-2px^2} dx,
\]

where

\[
\begin{align*}
r_{2m+1}(\nu) &:= \nu \prod_{j=1}^{m} \left( \nu^2 + (2j - 1)^2 \right), \\
r_{2m}(\nu) &:= \prod_{j=1}^{m} \left( \nu^2 + (2j - 2)^2 \right).
\end{align*}
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These are, you may recall, the coefficients in the series expansion of \(\exp(\arcsin x)\).
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In particular, for the case where \((\alpha, \beta) = (-\pi/2, \pi/2)\), we have

\[
I(p, q) := I(p, q, -\pi/2, \pi/2) = \frac{2ie^p}{q} \sum_{k=0}^{\infty} \frac{r_{2k+1}(-2iq)}{(2k)!} B_k(p),
\]

with

\[
B_k(p) := \int_0^{1/\sqrt{2}} x^{2k} e^{-2px^2} \, dx = \frac{1}{2^{k+1}\sqrt{2}} \int_0^1 e^{-pu} u^{k-\frac{1}{2}} \, du
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\[
= -\frac{e^{-p}}{p2^{k+1}\sqrt{2}} + \left(k - \frac{1}{2}\right) \frac{B_{k-1}(p)}{2}.
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\mathcal{I}(p, q) := \mathcal{I}(p, q, -\pi/2, \pi/2) = \frac{2i e^p}{q} \sum_{k=0}^{\infty} \frac{r_{2k+1}(-2iq)}{(2k)!} B_k(p),
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For integral order, we have from the Laguerre paper

\[ J_n(z) = \frac{1}{2\pi} \left( e^{-in\pi/2} \mathcal{I}(iz, n) + e^{in\pi/2} \mathcal{I}(-iz, n) \right), \]

and

\[ I_n(z) = \frac{1}{2\pi} \left( \mathcal{I}(z, n) + \cos(\pi n) \mathcal{I}(-z, n) \right). \]

As Jon mentioned, we want to use the integral representations to get expressions for general \( \nu \).
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As Jon mentioned, we want to use the integral representations to get expressions for general \( \nu \).
The integral representations are:

\[ J_\nu(z) = \frac{1}{\pi} \int_{0}^{\pi} \cos(\nu t - z \sin t) \, dt - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} e^{-\nu t - z \sinh t} \, dt, \]

\[ Y_\nu(z) = \frac{1}{\pi} \int_{0}^{\pi} \sin(z \sin t - \nu t) \, dt \]
\[ - \frac{1}{\pi} \int_{0}^{\infty} \left( e^{\nu t} + e^{-\nu t} \cos \nu \pi \right) e^{-z \sinh t} \, dt, \]

\[ I_\nu(z) = \frac{1}{\pi} \int_{0}^{\pi} e^{z \cos t} \cos \nu t \, dt - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} e^{-z \cosh t - \nu t} \, dt, \]

and

\[ K_\nu(z) = \int_{0}^{\infty} e^{-z \cosh t} \cosh \nu t \, dt = \frac{1}{2} \int_{-\infty}^{\infty} e^{-z \cosh t - \nu t} \, dt. \]
The integrals on $[0, \pi]$ can be expressed in terms of the $I$ function. Specifically,

$$J_{\nu}(z) = \frac{1}{2\pi} \left( e^{-i\nu\pi/2}I(iz, \nu) + e^{i\nu\pi/2}I(-iz, \nu) \right) - \frac{\sin \nu \pi}{\pi} \int_0^\infty \ldots$$

$$Y_{\nu}(z) = \frac{1}{2\pi i} \left( e^{-i\nu\pi/2}I(iz, \nu) - e^{i\nu\pi/2}I(-iz, \nu) \right) - \frac{1}{\pi} \int_0^\infty \ldots$$
\[\begin{align*}
l_\nu(z) &= \frac{1}{2} \left( \mathcal{I}(z, \nu) + e^{i\nu\pi} \mathcal{I}(-z, \nu, 0, \pi/2) + e^{-i\nu\pi} \mathcal{I}(-z, -\nu, 0, \pi/2) \right) \\
&\quad - \frac{\sin \nu\pi}{\pi} \int_0^\infty \ldots \\
&\quad = \frac{1}{2\pi} \left( \mathcal{I}(z, \nu) + \cos \nu\pi \mathcal{I}(-z, \nu) - \sin \nu\pi \mathcal{I}^*(z, \nu) \right) \\
&\quad - \frac{\sin \nu\pi}{\pi} \int_0^\infty \ldots ,
\end{align*}\]

where

\[\mathcal{I}^*(z, \nu) = \frac{2e^z}{\nu} \sum_{n=0}^{\infty} \frac{r_{2n+2}(2i\nu)}{(2n+1)!} B_{n+\frac{1}{2}}(z).\]
To get the generalizations we want, we basically just need to evaluate the infinite integrals.

Let us look at the integrals in the $J$ and $Y$ cases. A change of variables plus integration by parts gives us

$$
\int_{0}^{\infty} e^{-\nu t - z \sinh t} \, dt = \frac{1}{\nu} - \frac{Z}{\nu} \int_{0}^{\infty} e^{-zs} e^{-\nu \text{arcsinh } s} \, ds.
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The expansion of $e^{-\nu \arcsinh s}$ about $s = 0$, used in the finite case to obtain the series, is only valid on $[0, 1)$. 
For large $s$, it makes sense to expand about infinity!

The series, valid on $(1, \infty)$, is

\[ s^\nu e^{-\nu \operatorname{arcsinh} s} = \sum_{n=0}^{\infty} \frac{A_n(\nu)}{s^{2n}}, \]

where $A_0(\nu) = 2^{-\nu}$ and for $n \geq 1$,

\[ A_n = \frac{(\nu + 2n - 2)(\nu + 2n - 1)}{4n(n + \nu)} A_{n-1}, \]

from which we easily obtain

\[ A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu} (\nu + n + 1)_{n-1}}{2^{2n} n!}. \]
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Note that when $\nu$ is a negative integer, we have problems with the recurrence.

When $n = \lfloor (1 - \nu)/2 \rfloor$, the numerator is 0. When $n = -\nu$, the denominator is zero.

In this case, $A_n(\nu) = (-1)^{\nu+1} A_{n+\nu}(-\nu)$ for $n \geq -\nu$
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Localize!

For fixed $k$, $f_k(s) := e^{-\nu \arcsinh(k+s)}$ satisfies the second order differential equation

$$f''_k(s) = \frac{1}{k^2 + 1 + 2ks + s^2} \left( \nu^2 f_k(s) - (k + s)f'_k(s) \right).$$

So if we set

$$e^{-\nu \arcsinh(k+s)} = \sum_{n=0}^{\infty} \frac{a_n(k, \nu)}{n!} s^n,$$

then we have the recurrence relation

$$a_{n+2} = \frac{1}{k^2 + 1} \left( (\nu^2 - n^2) a_n - k(2n + 1) a_{n+1} \right),$$

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$$a_0 = (k + \sqrt{k^2 + 1})^{-\nu}, \quad a_1 = -\frac{\nu a_0}{\sqrt{k^2 + 1}}.$$
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We can subdivide $[0, \infty)$ into the intervals $[0, 1/2], [1/2, 3/2], \ldots, [N - 1/2, N + 1/2], [N + 1/2, \infty)$ and on each interval expand $e^{-\nu \arcsinh s}$ at $k$, the centre of the interval.

Each of these series has radius of convergence $\sqrt{k^2 + 1}$ and so we may interchange summation and integration.

For the infinite interval at the end, we use the expansion about infinity.
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For the infinite interval at the end, we use the expansion about infinity.
Thus for any positive integer $N$, we have

$$\int_0^\infty e^{-zs} e^{-\nu \arcsinh s} \, ds =$$

$$\sum_{n=0}^\infty \left( \frac{a_n(0, \nu)}{n!} \alpha_n(z) + \beta_n(z) \sum_{k=1}^N e^{-kz} \frac{a_n(k, \nu)}{n!} \right) + A_n(\nu) G_n(N + \frac{1}{2}, z, \nu),$$

where

$$\alpha_n(z) := \int_0^{1/2} e^{-zs} s^n \, ds = -\frac{e^{-z/2}}{2^n z} + \frac{n}{z} \alpha_{n-1}(z),$$

$$\beta_n(z) := \int_{-1/2}^{1/2} e^{-zs} s^n \, ds = \frac{e^{z/2}}{(-2)^n z} - \frac{e^{-z/2}}{2^n z} + \frac{n}{z} \beta_{n-1}(z),$$
and

\[ G_n(\theta, z, \nu) := \frac{e^{-\theta z}}{\theta^{2n+\nu-1}} \int_0^\infty e^{-\theta z s} (1 + s)^{-2n-\nu} ds \]

\[ = \frac{1}{(\nu + 2n - 1)(\nu + 2n - 2)} \times \left( \frac{e^{-\theta z (\nu + 2n - 2 - \theta z)}}{\theta^{2n+\nu-1}} + z^2 G_{n-1}(\theta, z, \nu) \right). \]
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\[ \left( \frac{e^{-\theta z}(\nu + 2n - 2 - \theta z)}{\theta^{2n+\nu-1}} + z^2 G_{n-1}(\theta, z, \nu) \right). \]
So we have found a representation for the Bessel functions in terms of several sums:

Sums involving $I$ from the integral on $[0, \pi]$, where each summand looks like

$$\frac{r_{n+1}(2\nu)}{n!} B_{n(+1/2)}(z),$$

sums from the subdivisions of the real line on the infinite integral, where a typical summand is

$$\frac{a_n(k, \nu)}{n!} \beta_n(z) e^{-kz},$$

and the sum from the tail, where each summand is

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$$A_n(\nu) G_n(N + \frac{1}{2}, z, \nu).$$
Let us first look at
\[ \frac{r_{n+1}(2\nu)}{n!} \].

For simplicity we consider the case \( n \) even, \( n = 2m \). Then this is
\[
\prod_{j=1}^{m} \left( 1 - \frac{1}{2j} - \frac{4\nu^2}{(2j - 1)(2j)} \right),
\]
which is bounded and decreasing for \( m > 2|\nu|^2 \). Similarly for odd \( n \).

Also, (for arbitrary \( n \))
\[
B_n(z) = \frac{1}{2^{n+3/2}} \int_0^1 e^{-zu} u^{n-1/2} du
\]
so it is bounded by
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|B_n(z)| \leq \frac{\max(1, e^{-\text{Re}(z)})}{2^{n+3/2}}.
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\]
Thus the terms of type

\[ \frac{r_{n+1}(2\nu)}{n!} B_n(z) = O_{\nu,z}(2^{-n}), \]

where the big-\(O\) constant can be explicitly computed.
For terms of the type

\[ \frac{a_n(k, \nu)}{n!} \beta_n(z) e^{-kz}, \]

note that \( a_n(k, \nu)/n! \) are the Taylor coefficients, and so they are \( O\left(\frac{1}{(k^2 + 1)^{n/2}}\right) \) from the radius of convergence. We can fairly easily get a weaker but explicit geometric bound using the recurrence relation for \( a_n(k, \nu) \).

\( \beta_n(z) \) is the \( n \)-th moment of the exponential, and can be explicitly computed. A simple estimate yields

\[ |\beta_n(z) e^{-kz}| \leq \frac{e^{-(k-1/2) \Re(z)}}{2^n}. \]
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For terms of the type

\[ A_n(\nu) G_n(N + \frac{1}{2}, z, \nu), \]

we can get a bound

\[ |A_n(\nu)| \leq \frac{\nu 2^{\lceil |\nu| \rceil - \nu - 1}}{n} \]

from the explicit formula,

and use bounds for the incomplete gamma function to get explicit big-O constants for the bound

\[ G_n(N + \frac{1}{2}, z, \nu) = O_{\nu,z}((N + 1/2)^{-\text{Re}(\nu)-2n}). \]
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$$G_n(N + \frac{1}{2}, z, \nu) = O_{\nu, z}((N + 1/2)^{-\text{Re}(\nu)-2n}).$$
Putting it all together, we see that the (slowest) sums converge like $2^{-n}$, and with explicit big-$O$ constants we may determine how many terms are needed for a specific accuracy.
Other features to note:

- For each type of sum, the summands are all computable via recursion.
- The most difficult computation involved are the computation of $B_0$ and $G_0$, each of which involves an incomplete gamma evaluation. It should be noted that this can be done via continued fractions, so this scheme can be thought of as a continued fraction evaluation scheme for Bessel functions.
- The sum involving $A_n G_n$ is bounded like $O_\nu (e^{-z(N+1/2)})$ by estimating the integral of the tail. So one can avoid the computation of $G_0$ altogether by choosing a large enough $N$. 
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- For each type of sum, the summands are all computable via recursion.
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Our computation scheme has some advantages over the traditional ascending-asymptotic switching scheme:

- Our series are all uniformly geometrically convergent, whereas some asymptotic formulas are divergent series, and some are only algebraically convergent (i.e., like $n^{-\alpha}$ rather than $2^{-n}$).

- Each summand in our series is a product of functions that depend only on $\nu$ or only on $z$, and thus these values can be stored and recycled for one-$\nu$-many-$z$ or one-$z$-many-$\nu$ computations. Note also that each of these functions is eventually decreasing.

The following table compares the performance between the ascending series, the standard divergent asymptotic series, and our series for $J_{\nu}$ with the choice $N = 1$.  

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Effective Computation of Bessel Functions, Part II
Table: Comparison between various series for $J_\nu(z)$.

<table>
<thead>
<tr>
<th>$(\nu, z)$</th>
<th>$M$</th>
<th>Ascending Series</th>
<th>Asymptotic Series</th>
<th>Exp-arc Series</th>
</tr>
</thead>
</table>
| $\nu = 6.2$  
$z = 100$ | 10  | $10^{22}$ | $10^{-32}$ | $10^{-5}$ |
| 50         |     | $10^{41}$ | $10^{-76}$ | $10^{-18}$ |
| 100        |     | $10^{22}$ | $10^{-89}$ | $10^{-33}$ |
| 150        |     | $10^{-19}$ | $10^{-79}$ | $10^{-49}$ |
| 200        |     | $10^{-75}$ | $10^{-55}$ | $10^{-64}$ |
| $\nu = 12.3$  
$z = 50$ | 10  | $10^{18}$ | $10^{-23}$ | $10^2$ |
| 30         |     | $10^{17}$ | $10^{-41}$ | $10^{-10}$ |
| 50         |     | $10^6$ | $10^{-45}$ | $10^{-17}$ |
| 70         |     | $10^{-11}$ | $10^{-42}$ | $10^{-23}$ |
| 100        |     | $10^{-45}$ | $10^{-28}$ | $10^{-33}$ |
| $\nu = 12.3$  
$z = 75 + 57i$ | 10  | $10^{27}$ | $10^{-4}$ | $10^{13}$ |
| 50         |     | $10^{38}$ | $10^{-48}$ | $10^{-17}$ |
| 100        |     | $10^{14}$ | $10^{-59}$ | $10^{-33}$ |
| 120        |     | $10^{-2}$ | $10^{-56}$ | $10^{-39}$ |
| 150        |     | $10^{-31}$ | $10^{-47}$ | $10^{-48}$ |
| 200        |     | $10^{-89}$ | $10^{-20}$ | $10^{-64}$ |
Thank you for your attention!

A preprint is available at the AARMS docserver

http://locutus.cs.dal.ca:8088/archive/00000371/