COMPRESSED SENSING: A
SUBGRADIENT DESCENT METHOD
FOR MISSING DATA PROBLEMS

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Central Problem: $\ell_0$ minimization

Given a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ full-rank with $0 < m < n$, solve

$$(P_0) \quad \text{minimize} \quad \|x\|_0 \quad \text{subject to} \quad Ax = b$$

where $\|x\|_0 := \sum_j |\text{sign}(x_j)|$ with $\text{sign}(0) := 0$.

- Combinatorial optimization problem (hard to solve).
Central Problem: $\ell_0$ minimization

Solve instead

$$(\mathcal{P}_1) \quad \begin{array}{l}
\text{minimize} \\
\text{subject to}
\end{array} \quad \begin{array}{l}
\|x\|_1 \\
Ax = b
\end{array}$$

where $\|x\|_1$ is the usual $\ell_1$ norm.

Linear programming problem...easy to solve for small problems.

Try solving the problem when $x$ is a $512 \times 512$ image...
Application: Crystallography

Given

![Sampled autocorrelation transfer function with 81.576% of pixels sampled](image1.png)

(autocor. transfer function – ATF – w/ missing pixels)

Find

![True autocorrelation transfer function](image2.png)

(true ATF)
Application: Crystallography

For the **linear mapping** \( A : \mathbb{R}^n \to \mathbb{R}^m \ (m < n) \), solve

\[
\begin{align*}
\text{minimize} \quad & \| x \|_1 \\
\text{subject to} \quad & x \in C
\end{align*}
\]

where \( C := \{ x \in \mathbb{R}^n \mid Ax = b \} \).

One could apply the **Douglas-Rachford iteration**

\[
x_{n+1} = \frac{1}{2} \left( R_{f_1} R_{f_2} + I \right) (x_n)
\]

where

\[
R_{f_j} x := 2 \text{prox}_{\alpha, f_j} x - x
\]

for \( f_1(x) = \| x \|_1 \) and \( f_2(x) = \iota_C(x) \), and \( \alpha > 0 \) fixed.
Application: Crystallography

Great strategy for big problems, but it is (arbitrarily) slow to converge and accuracy is bad.
Motivation

Find a variational/geometrical *interpretation* of the Candes-Tao (2004) probabilistic criteria for when the solution to $(\mathcal{P}_1)$ is unique and exactly matches the true signal $x^*$.

- *As a by-product*, find better practical methods for solving the underlying problem.
- Try to use entropy/penalty ideas and duality and also prove some rigorous theorems.
Outline

- Dual convex regularization
- A subgradient descent algorithm with exact linesearch
- Computational results
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Fenchel duality

The **Fenchel conjugate** of $f$, denoted $f^*: X^* \to ]-\infty, +\infty]$ and defined by

$$f^*(x^*) = \sup_{x \in X} \{ \langle x^*, x \rangle - f(x) \}.$$  

For the $\ell_1$ problem, the norm is proper, convex, lsc and $b \in \text{core} (\text{Adom} f)$, so strong Fenchel duality [3] holds:

$$\inf_{x \in \mathbb{R}^n} \{ \|x\|_1 \mid Ax = b \} = \sup_{y \in \mathbb{R}^m} \{ \langle b, y \rangle - \| (A^* y) \|_1^* \}.$$  

where

$$\|x^*\|_1^* := \begin{cases} 0 & x^* \in [-1, 1] \\ +\infty & \text{else} \end{cases}$$
Elementary observations

The dual to $\mathcal{P}_1$ is

$\mathcal{D}_1$

\[
\begin{align*}
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad (A^* y)_j \in [-1, 1] \quad j = 1, 2, \ldots, n.
\end{align*}
\]

The solution includes a vertex of the polyhedron described by the constraints. The uniqueness of solutions to the primal problem depend on whether or not solutions to the dual problem reside along the edges or faces of the dual polyhedron.

If a solution $\bar{x}$ to $\mathcal{P}_1$ is unique, then

\[
m \geq \{ \text{the number of active constraints in } \mathcal{D}_1 \} = \| \bar{x} \|_0.
\]
Elementary observations

For the $\ell_0$ problem, the function is proper, lsc but not convex, so only weak Fenchel duality holds:

$$\inf_{x \in \mathbb{R}^n} \{ \|x\|_0 \mid Ax = b \} \geq \sup_{y \in \mathbb{R}^m} \{ \langle b, y \rangle - \| (A^* y) \|_0^* \}.$$ 

where

$$\|x^*\|_0^* := \begin{cases} 0 & x^* = 0 \\ +\infty & \text{else} \end{cases}$$
Elementary observations

In other words, the dual to \( (P_0) \) is

\[
\begin{align*}
(\mathcal{D}_0) \quad & \text{maximize} \quad b^T y \\
& y \in \mathbb{R}^m \\
& \text{subject to} \quad A^* y = 0.
\end{align*}
\]

The *primal problem* is a combinatorial optimization problem; the *dual problem*, however, is a linear program, which is finitely terminating.

In fact, the solution to the dual problem is trivial: \( \bar{y} = 0 \)...

which tells us nothing useful about the primal problem.
Relax/Regularize the dual and either solve this directly, or solve the corresponding regularized primal problem, or a mixture of the two.
Regularization/Relaxation: Shifted Fermi-Dirac Entropy

For $L, \epsilon > 0$, define

$$f_{\epsilon,L}^*(x) := \sum_{j=1}^{n} \left[ \epsilon \left( \frac{(L + x_j) \ln(L + x_j) + (L - x_j) \ln(L - x_j)}{2L \ln(2)} - \frac{\ln(L)}{\ln(2)} \right) \right]$$

for $x \in [-L, L]^n$

$$:= +\infty$$

for $\|x\|_\infty > L$.

Then

$$f_{\epsilon,L}^{**}(x) = \sum_{j=1}^{n} \left[ \frac{\epsilon}{\ln(2)} \ln \left( 4x_j L / \epsilon + 1 \right) - x_j L - \epsilon \right].$$

(2)

$f^*$ is proper, convex and lsc, thus $f^{***} = f^*$ and we define $f := f^{**}$. 
Regularization/Relaxation: Shifted Fermi-Dirac Entropy

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Regularization/Relaxation: Shifted Fermi-Dirac Entropy

For $L > 0$ fixed, in the limit as $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} f_{\epsilon, L}^*(y) = \begin{cases} 0 & y \in [-L, L] \\ +\infty & \text{else} \end{cases}$$

and

$$\lim_{\epsilon \to 0} f_{\epsilon, L}(x) = L|x|.$$ 

For $\epsilon > 0$ fixed we have

$$\lim_{L \to 0} f_{\epsilon, L}^*(x) = \begin{cases} 0 & y = 0 \\ +\infty & \text{else} \end{cases}$$

and

$$\lim_{L \to 0} f_{\epsilon, L}(x) := 0.$$
Regularization/Relaxation: Shifted Fermi-Dirac Entropy

- $\| \cdot \|_0$ and $f^*_{\epsilon_0}$ have the same conjugate;
- $\| \cdot \|_0^{**} \neq \| \cdot \|_0$ while $f^{***}_{\epsilon_0} = f^*_{\epsilon_0}$;
- $f_{\epsilon,L}$ and $f^*_{\epsilon,L}$ are convex and smooth on the interior of their domains for all $\epsilon, L > 0$. This is in contrast to metrics of the form $\left( \sum_j |x_j|^p \right)$ which are nonconvex for $p < 1$. 
Regularization/Relaxation: Shifted Fermi-Dirac Entropy

Solve

\[(\mathcal{D}_{L,\epsilon}) \quad \minimize_{y \in \mathbb{R}^m} f^*_{L,\epsilon}(A^* y) - \langle b, y \rangle\]

This is a convex optimization problem, so equivalently we solve

\[0 \in A \partial f^*_{L,\epsilon}(A^* y) - b\]
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\(\epsilon = 0\)

Solve

\[0 \in A \partial f^*_{L,0}(A^* y) - b\]

via subgradient descent:

*given \(y_-\), choose \(v_- \in \partial f^*_{L,0}(A^* y_-)\), \(\lambda_- > 0\) and construct \(y_+\) as*

\[y_+ = y_- + \lambda_- (b - Av_-)\]

**Issues:**

- how to choose \(v_- \in \partial f^*_{L,0}(A^* y_-)\)
- how to choose step length \(\lambda_-\).
\( \epsilon = 0: \) Choose \( v_- \in \partial f_{L,0}^*(A^*y_-) \)

Note that \( f_{L,0}^* = \nu_{[-L,L]^n} \) so

\[
\partial \nu_{[-L,L]^n}(x^*) = N_{[-L,L]}(x^*)
= \{ v \in \mathbb{R}^n \mid v_j \leq 0 \ (j \in \mathcal{J}_-), \ v_j \geq 0 \ (j \in \mathcal{J}_+), \ v_j = 0 \ (j \in \mathcal{J}_0) \}
\]

where \( \mathcal{J}_- = \{ j \in \mathbb{N} \mid x_j = -L \} \), \( \mathcal{J}_+ = \{ j \in \mathbb{N} \mid x_j = L \} \) and \( \mathcal{J}_0 = \{ j \in \mathbb{N} \mid x_j \in ]-L,L[ \} \).

Choose \( v_- \in N_{[-L,L]}(A^*y_-) \) to be the solution to

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| b - Av \|^2 \\
\text{subject to} & \quad v \in N_{[-L,L]^n}(A^*y_-)
\end{align*}
\]
\( \epsilon = 0: \) Choose \( v_- \in \partial f^*_{L,0}(A^* y_-) \)

\[
(\mathcal{P}_{v_-}) \quad \begin{array}{l}
\text{minimize} \\
\quad \underset{v \in \mathbb{R}^n}{\frac{1}{2} \| b - Av \|^2}
\end{array}
\begin{array}{l}
\text{subject to} \\
\quad v \in N_{[-L,L]^n}(A^* y_-)
\end{array}
\]

Define \( B = \{ v \in \mathbb{R}^n \mid Av = b \} \). Reformulate:

\[
(\mathcal{P}_{v_-}) \quad \begin{array}{l}
\text{minimize} \\
\quad \underset{v \in \mathbb{R}^n}{\frac{\beta}{2(1 - \beta)} \text{dist}^2(v, B) + \iota N_{[-L,L]^n}(A^* y_-)(v)}
\end{array}
\]
\[ \epsilon = 0: \text{Choose } \nu_- \in \partial f^*_{L,0}(A^*y_-) \]

**Approximate averaged alternating proximal reflections:** Choose \( \nu^{(0)} \in \mathbb{R}^n \). For \( \nu \in \mathbb{N} \) set

\[
\nu^{(\nu+1)} = \frac{1}{2} \left( R_1 \left( R_2 \nu^{(\nu)} + \epsilon_\nu \right) + \rho_\nu + \nu^{(\nu)} \right), \quad (3)
\]

where

- \( R_1 x := 2 \text{prox} \frac{\beta}{2(1-\beta)} \text{dist}(v,B)^2 x - x \)
- \( R_2 x := 2 \text{prox} \iota_{N[-L,L]^n(A^*y_-)} x - x \)
- The sequences \( \{ \epsilon_\nu \} \) and \( \{ \rho_\nu \} \) are the errors at each iteration, and assumed to be summable.
\( \epsilon = 0: \text{Choose } v_\epsilon \in \partial f_{L,0}^*(A^*y_-) \)

(L. 2008) Algorithm 3 is equivalent to: **Inexact Relaxed Averaged Alternating Reflections** (L. 2005) 
Choose \( v^{(0)} \in \mathbb{R}^n \) and \( \beta \in [1/2, 1] \). For \( \nu \in \mathbb{N} \) set 

\[

v^{(\nu+1)} = \frac{\beta}{2} \left( R_B \left( R_{N_{[-L,L]}}(A^*y_-)v^{(\nu)} + \epsilon_n \right) + \rho_n + v^{(\nu)} \right) \\
+(1 - \beta) \left( P_{N_{[-L,L]}}(A^*y_-)v^{(\nu)} + \frac{\epsilon_n}{2} \right).

\]

where \( R_B := 2P_B - I \) and likewise for \( R_{N_{[-L,L]}}(A^*y_-) \). 

The sequence \( \{v^{(\nu)}\}_{\nu=1}^{\infty} \) converges to \( \overline{v} \) where \( P_B \overline{v} \) solves \( (P_{\nu_-}) \) (L. 2008, Combettes 2004).
\[ \epsilon = 0: \textbf{Choose } \lambda \_ \]

**Exact line search**: choose the largest \( \lambda_\) that solves

\[
\min_{\lambda \in \mathbb{R}^+} f_{L,0}^*(A^*y_\_ + A^*\lambda(b - Av_\_))
\]

Note that \( f_{L,0}^*(A^*y_\_ + A^*\lambda(b - Av_\_)) = 0 \) for all \( A^*y_\_ + A^*\lambda(b - Av_\_) \in [-L, L]^n \), so really we solve

\[
\begin{align*}
\text{minimize} & \quad -\lambda \\
\text{subject to} & \quad \lambda(A^*(b - Av_\_))_j \leq L - (A^*y_\_)_j \\
& \quad \lambda(A^*(b - Av_\_))_j \geq -L - (A^*y_\_)_j \quad j = 1, \ldots, n
\end{align*}
\]

\[ (P_\lambda) \]
\[ \epsilon = 0: \text{Choose } \lambda_- \]

**Exact line search:** Define

\[ \mathcal{J}_+ = \{ j \mid (A^*(b - Av_-))_j > TOL \}, \]
\[ \mathcal{J}_- = \{ j \mid (A^*(b - Av_-))_j < -TOL \} \]

\[ \lambda_- = \min \left\{ \begin{array}{c} \min_{j \in \mathcal{J}_+} \left\{ (L - (A^*y_-)_j)/(A^*(b - Av_-))_j \right\}, \\
\min_{j \in \mathcal{J}_-} \left\{ -(L - (A^*y_-)_j)/(A^*(b - Av_-))_j \right\} \end{array} \right\} \]

Algorithm terminates when \( \mathcal{J}_+ = \mathcal{J}_- = \emptyset \).
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The solution to the dual $y$ mapped by $A^*$:

This is a **qualitative** solution to the primal problem: it tells us the location and sign of the nonzero elements of the signal.
Computational Results

The solution to the primal $\bar{x}$ determined by the solution to

$$(P_{\bar{y}}) \quad \begin{align*}
\text{minimize} & \quad \frac{1}{2} \| b - Ax \|^2 \\
\text{subject to} & \quad x \in \mathbb{R}^n \\
& \quad x \in N_{[-L,L]^n}(A^*\bar{y})
\end{align*}$$

where $\bar{y}$ is the solution to the dual problem. The $\ell_\infty$ error is $10^{-12}$.
Computational Results

Observations:

- Inner iterations can be shown to be arbitrarily slow: the solution sets to the subproblems are not metrically regular and the indicator function $\mathcal{N}_{[-L,L]^n}$ is not coercive in the sense of Lions.

- The algorithm fails when there are too few samples relative to the sparsity of the true solution.
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Work in progress:

- Characterize the recoverability of the true solution in terms of the regularity of the subproblem

\[ (P_v) \quad \begin{align*}
\min_{v \in \mathbb{R}^n} & \quad \frac{1}{2} \| b - Av \|^2 \\
\text{subject to} & \quad v \in N_{[-L, L]^n}(A^* y_-)
\end{align*} \]
Conclusion

Thanks
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References
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