Three Convexity Results that Tony Did, Would or Should Like

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“Harald Bohr is reported to have remarked "Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.” (D.J.H. Garling)

The Bohrs

- One Nobel Prize
  - physics
- One Olympic Medal
  - soccer
Abstract of Convexity Talk

It is a true pleasure to honour Tony

• as a friend, a mentor, a colleague over 30 years
• I have learned a great deal of analysis, geometry, and common sense from him

In 25 years Fran seems to have changed less?
I offer three examples of convexity appearing (unexpectedly) over the years in my research.

Each example either illustrates the power of convexity, of modern symbolic computation, or of both …

I start with a few comments about computer assisted mathematics and collaboration.
D-Drive’s Nova Scotia location lends us unusual freedom when interacting globally. Many cities around the world are close enough in a chronological sense to comfortably accommodate real-time collaboration.
The reader who wants to get an introduction to this exciting approach to doing mathematics can do no better than these books.
—Notices of the AMS

I do not think that I have had the good fortune to read two more entertaining and informative mathematics texts.
—Australian Mathematical Society Gazette

This Experiments in Mathematics CD contains the full text of both Mathematics by Experiment: Plausible Reasoning in the 21st Century and Experimentation in Mathematics: Computational Paths to Discovery in electronic, searchable form. The CD includes several "smart" enhancements, such as

- Hyperlinks for all cross references
- Hyperlinks for all Internet URLs
- Hyperlinks to bibliographic references
- Enhanced search function, which assists one with a search for a particular mathematical formula or expression.

These enhancements significantly improve the usability of these files and the reader's experience with the material.
Experimental Mathematics in Action

The emerging field of experimental mathematics has expanded to encompass a wide range of studies, all unified by aggressive utilization of modern computer technology in mathematical research. This volume presents a number of case studies of experimental mathematics in action, together with some high-level perspectives.

Specific case studies include:
- analytic evaluation of integrals by means of symbolic and numeric computing techniques
- evaluation of Apery-like summations
- finding dependencies among high-dimensional vectors (with applications to factoring large integers)
- inverse scattering (reconstruction of physical objects based on electromagnetic or acoustic scattering)
- investigation of continuous but nowhere differentiable functions.

In addition to these case studies, the book includes some background on the computational techniques used in the analyses.
COXETER’S (1927) Kaleidoscope

1907-2003
The Perko Pair $10_{161}$ and $10_{162}$ are two adjacent 10-crossing knots (1900)

- first shown to be the same by Ken Perko in 1974
- and beautifully made dynamic in KnotPlot (open source)
Outline of Convexity Talk

A. Generalized Convexity of Volumes (Bohr-Mollerup).

B. Coupon Collecting and Convexity.

C. Convexity of Spectral Functions.

The talk ends when I do.

Full details are in the three reference texts.
Generalized Convexity of Volumes

A. Generalized Convexity of Gamma (Bohr-Mollerup).

Γ is usually defined for Re(x) > 0 as

\[ \Gamma(x) := \int_0^\infty e^{-t} t^{x-1} \, dt. \]  \hspace{1cm} (1)

**Theorem 1 (Bohr-Mollerup)** \( \Gamma \) is the unique function \( f : (0, \infty) \to (0, \infty) \) such that:

(a) \( f(1) = 1 \);
(b) \( f(x + 1) = xf(x) \);
(c) \( f \) is log-convex (\( x \to \log(f(x)) \) is convex).

- Application is often *automatable* in a computer algebra system, as we now illustrate:
Generalized Convexity of Volumes

A. Generalized Convexity of Gamma (Beta function).

The $\beta$–function is defined by

$$\beta(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt$$  \hspace{1cm} (1)

for $\Re(x), \Re(y) > 0$. As is often established using polar coordinates and double integrals

$$\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}.$$  \hspace{1cm} (2)

Proof Use $f := x \rightarrow \beta(x, y) \Gamma(x + y)/\Gamma(y)$. (a) and (b) are easy. For (c) we show $f$ is log-convex via Hölder’s inequality. Thus $f = \Gamma$ as required. QED
Generalized Convexity of Volumes

A. Convexity of Volumes (Blaschke-Santalo inequality).

For a convex body $C$ in $\mathbb{R}^n$ its polar is

$$C^o := \{ y \in \mathbb{R}^n : \langle y, x \rangle \leq 1 \text{ for all } x \in C\}.$$ 

Denoting $n$-dimensional Euclidean volume of $S \subseteq \mathbb{R}^n$ by $V_n(S)$, **Blaschke-Santalo** says

$$V_n(C) V_n(C^o) \leq V_n(E) V_n(E^o) = V_n^2(B_n(2)) \tag{1}$$

where maximality holds (only) for any ellipsoid $E$ and $B_n(2)$ is the Euclidean unit ball.

**Question** How to explain cases of this as convexity estimates?
Generalized Convexity of Volumes

A. Convexity of Volumes (Dirichlet Formulae).

The volume of the ball in the \( \| \cdot \|_p \)-norm, \( V_n(p) \), was first determined by Dirichlet

\[
V_n(p) = 2^n \frac{\Gamma(1 + \frac{1}{p})^n}{\Gamma(1 + \frac{n}{p})}. \tag{1}
\]

When \( p = 2 \),

\[
V_n = 2^n \frac{\Gamma\left(\frac{3}{2}\right)^n}{\Gamma\left(1 + \frac{n}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)^n}{\Gamma\left(1 + \frac{n}{2}\right)},
\]

is more concise than that usually recorded.

*Maple* code derives this formula as an iterated integral for arbitrary \( p \) and fixed \( n \).
Generalized Convexity of Volumes

A. Convexity of Volumes (Ease of Drawing Pictures).

\[ \log \Gamma(x) \]

\[ -\log V_a(1/x) \text{ for } a=4/3 \text{ and } 3 \]
A. Convexity of Volumes (‘mean’ log-convexity).

Theorem 2 [(H,A) log-concavity] The function \( V_\alpha (p) := 2^\alpha \Gamma (1 + \frac{1}{p})^{\alpha} / \Gamma (1 + \frac{\alpha}{p}) \) satisfies

\[
V_\alpha (p)^\lambda V_\alpha (q)^{1-\lambda} < V_\alpha \left( \frac{1}{\frac{\lambda}{p} + \frac{1-\lambda}{q}} \right),
\]

for all \( \alpha > 1 \), if \( p, q > 1 \), \( p \neq q \), and \( \lambda \in (0, 1) \).

\( \alpha = n, \ \frac{1}{p} + \frac{1}{q} = 1 \) with \( \lambda_1 = \lambda_2 = 1/2 \) recovers the \( p \)-norm case of Blaschke-Santalo; and the lower bound. This extends to substitution norms.

Q. How far can one take this?
"Here's where you made your mistake."
Outline of Convexity Talk

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C. Convexity of Spectral Functions.

The talk ends when I do
B. The origin of the problem.

Consider a network objective function $p_N$:

$$
p_N(q) = \sum_{\sigma \in S_N} \left( \prod_{i=1}^{N} \frac{q_{\sigma(i)}}{\sum_{j=i}^{N} q_{\sigma(j)}} \right) \left( \sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{N} q_{\sigma(j)}} \right),
$$

summed over all $N!$ permutations; so a typical term is

$$
\left( \prod_{i=1}^{N} \frac{q_i}{\sum_{j=i}^{N} q_j} \right) \left( \sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{n} q_j} \right).
$$

For example, with $N = 3$ this is

$$
q_1 q_2 q_3 \left( \frac{1}{q_1 + q_2 + q_3} \right) \left( \frac{1}{q_2 + q_3} \right) \left( \frac{1}{q_3} \right) \left( \frac{1}{q_1 + q_2 + q_3} + \frac{1}{q_2 + q_3} + \frac{1}{q_3} \right).
$$

This arose as the objective function in research into coupon collection. Ian Affleck wished to show $p_N$ was convex on the positive orthant. I hoped not!
B. Doing What is Easy.

First, we try to simplify the expression for $p_N$.

The *partial fraction decomposition* gives:

\[
\begin{align*}
p_1(x_1) &= \frac{1}{x_1}, \\
p_2(x_1, x_2) &= \frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{x_1 + x_2}, \\
p_3(x_1, x_2, x_3) &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{1}{x_1 + x_2} - \frac{1}{x_2 + x_3} - \frac{1}{x_1 + x_3} \\
&\quad + \frac{1}{x_1 + x_2 + x_3}.
\end{align*}
\]

*Partial fractions* are an arena in which computer algebra is hugely useful. Try performing the third case in (1) by hand. It is tempting to predict the “same” pattern will hold for $N = 4$. This is easy to confirm (by computer) and so we are led to:
B. A Very Convex Integrand. (Is there a direct proof?)

A year later, Omar Hijab suggested re-expressing $p_N$ as the joint expectation of Poisson distributions. This leads to:

If $x = (x_1, \cdots, x_n)$ is a point in the positive orthant $R^n_+$, then

$$p_N(x) = \left( \prod_{i=1}^{n} x_i \right) \int_{R^n_+} e^{-\langle x, y \rangle} \max(y_1, \cdots, y_n) \, dy,$$

where $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ is the Euclidean inner product.

Now $y_i \to x_i y_i$ and standard techniques show $1/p_N$ is concave, as the integrand is. [We can now ignore probability if we wish!]

Q. “inclusion-exclusion” convexity: OK for $1/g(x) > 0$, $g$ concave.
“Mathematicians are a kind of Frenchmen:
whatever you say to them they translate into their own language, and right away it is something entirely different.”

(Johann Wolfgang von Goethe)
Maximen und Reflexionen, no. 1279
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Convexity of Spectral Functions

C. Eigenvalues of symmetric matrices (Lewis and Davis).

λ(S) lists **decreasingly** the real (resp. non-negative) eigenvalues of a symmetric (resp. PSD) \( n \times n \) matrix \( S \). The **Fenchel conjugate** is the convex closed function defined by

\[
f^*(x) := \sup_y \langle y, x \rangle - f(y).
\]

**Theorem (Spectral conjugacy)** If \( f : \mathbb{R}^n \mapsto [-\infty, \infty] \) is a symmetric function, it satisfies the formula \((f \circ \lambda)^* = f^* \circ \lambda\).

**Corollary [Davis/Lewis]** Suppose \( f : \mathbb{R}^n \mapsto [-\infty, \infty] \) is symmetric. Then the “spectral function” \( f \circ \lambda \) is closed and convex (resp. differentiable) if and only if \( f \) is closed and convex (resp. differentiable).
Convexity of Spectral Functions

C. Three Amazing Examples (Lewis).

I. Log Determinant Let $\logb(x) : = - \log(x_1 x_2 \cdots x_n)$ which is clearly symmetric and convex. The corresponding spectral function is $S \mapsto - \log \det(S)$.

II. Sum of Eigenvalues Ranging over permutations, let $f_k(x) : = \max_\pi \{x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(k)}\}$. This is clearly symmetric and convex. The corresponding spectral function is $\sigma_k(S) : = \lambda_1(S) + \lambda_2(S) + \cdots + \lambda_k(S)$. In particular the largest eigenvalue, $\sigma_1$, is a continuous convex function of $S$ and is differentiable if and only if the eigenvalue is simple.
Convexity of Spectral Functions

C. Three Amazing Examples (Lewis).

III. $k$–th Largest Eigenvalue The $k$–th largest eigenvalue may be written as

$$\mu_k(S) = \sigma_k(S) - \sigma_{k-1}(S).$$

In particular, this represents $\mu_k$ as the difference of two convex continuous, hence locally Lipschitz, functions of $S$ and so we discover the very difficult result that for each $k$, $\mu_k(S)$ is a locally Lipschitz function of $S$.

- Hard analogues exist for singular values, etc.
Convexity of Barrier Functions

C. A Fourth Amazing Example (Nesterov & Nemirovskii).

IV Self-concordant Barrier Functions Let $A$ be a nonempty open convex set in $\mathbb{R}^N$. Define, for $x \in A$,

$$F(x) := \lambda_N((A - x)^o),$$

where $\lambda_N$ is $N$-dimensional Lebesque measure and $(A - x)^o$ is the polar set. Then $F$ is an essentially Fréchet smooth, log-convex, barrier function for $A$.

- Central to modern interior point methods.
- The orthant yields $lb(x) := -\sum_{k=1}^{N} \log x_k$.
- Hilbert space analog (JB-JV, CUP, 2008)?
"He was very big in Vienna."
This polished solid silicon bronze sculpture is inspired by the work of David Borwein, his sons and colleagues, on the conditional series above for salt, Madelung's constant. This series can be summed to give uncountably many constants; one is Madelung's constant for sodium chloride. (Convexity is hidden here too!)

This constant is a period of an elliptic curve, a real surface in four dimensions. There are uncountably many ways to imagine that surface in three dimensions; one has negative gaussian curvature and is the tangible form of this sculpture. (As described by the artist.)


“The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.”