MORPHING LORD BROWNCKER’S CONTINUED FRACTION FOR PI INTO THE PRODUCT OF WALLIS

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Introduction

Three of the oldest and most celebrated formulas for pi are:

\[ \frac{2}{\pi} = \sqrt{2 \sqrt{2 \sqrt{2 \sqrt{2 \sqrt{2 \sqrt{2 \cdots}}}}} } \]

\[ \frac{2}{\pi} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots}{2 \cdot 4 \cdot 6 \cdot 8 \cdots} \]

\[ \frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}} \]

The first is Vieta’s product of nested radicals from 1592 [6]. The second is Wallis’s product of rational numbers [7] from 1656 and the third is Lord Brouncker’s continued fraction [5,7], also from 1656. (In the remainder of the paper we will use the more convenient notation \( \frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \cdots}} \) for continued fractions.)

In a previous paper [3] the author showed that (1) and (2) are actually special cases of a more general formula.
By examining this formula for the sequence of special values \( n = 0, 1, 2, \cdots \), we observe that the product of Wallis (case \( n = 0 \)) appears to gradually morph into Vieta’s product as \( n \) approaches infinity. We illustrate this below:

\[
\begin{align*}
\text{original Wallis’s product} & : \quad \frac{2}{\pi} = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot \frac{11 \cdot 13}{12 \cdot 12} \cdots \\
\text{original Vieta’s product} & : \quad \frac{2}{\pi} \to \frac{2}{\sum_{k=1}^{\infty} \frac{2^{n+1}k - 1}{2^{n+1}k} \cdot \frac{2^{n+1}k + 1}{2^{n+1}k}.}
\end{align*}
\]

Observe that as we progress through each step of the sequence, one additional factor of Vieta’s product is added, while every other fraction, starting with the first, in the Wallis type product is removed.

We will show that Brouncker’s continued fraction (3) and the product of Wallis (2) are both special cases of the general formula

\[
\frac{4}{\pi} = W(n) \frac{1}{2n+1} \left[ (4n+1) + \frac{1^2}{2(4n+1)} + \frac{3^2}{2(4n+1)} + \frac{5^2}{2(4n+1)} + \cdots \right],
\]
in which

\[ W(n) = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n-1)(2n+1)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots 2n \cdot 2n} \]

is the partial Wallis product. Just as above, this general formula allows us to start with Lord Brouncker’s continued fraction (case \( n = 0 \)) and gradually morph it into the Wallis product as \( n \) approaches infinity.

\[
\begin{align*}
\text{\( n = 0 \):} & \quad \frac{4}{\pi} = 1 + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \cdots \quad \text{(Lord Brouncker’s continued fraction.)} \\
\text{\( n = 1 \):} & \quad \frac{4}{\pi} = \frac{1 \cdot 3}{2 \cdot 2} \times \frac{1}{3} \left[ 5 + \frac{1^2}{10} + \frac{3^2}{10} + \frac{5^2}{10} + \cdots \right] \\
\text{\( n = 2 \):} & \quad \frac{4}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} \times \frac{1}{5} \left[ 9 + \frac{1^2}{18} + \frac{3^2}{18} + \frac{5^2}{18} + \cdots \right] \\
\text{\( n = 3 \):} & \quad \frac{4}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \times \frac{1}{7} \left[ 13 + \frac{1^2}{26} + \frac{3^2}{26} + \frac{5^2}{26} + \cdots \right] \\
\text{\( \cdots \)}
\end{align*}
\]

\[
\begin{align*}
\text{\( n \to \infty \):} & \quad \frac{4}{\pi} = 2 \times \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots} \quad \text{(original product of Wallis)}
\end{align*}
\]

Observe that as we progress through each step of the sequence, one additional factor of Wallis’s product is added, while the Brouncker type continued fraction

\[
\left[ x + \frac{1^2}{2x} + \frac{3^2}{2x} + \frac{5^2}{2x} + \cdots \right]
\]

has the value of \( x \) incremented by 4.

In a recent paper [2] Lange called attention to a continued fraction for \( \pi \) resembling Brouncker’s fraction (3)
\[ \pi = 3 + \frac{1^2}{6} + \frac{3^2}{6} + \frac{5^2}{6} + \cdots. \]

We show that the general formula

\[ \pi = \frac{1}{W(n)} \times \frac{1}{(2n+1)} \left[ (4n+3) + \frac{1^2}{2(4n+3)} + \frac{3^2}{2(4n+3)} + \frac{5^2}{2(4n+3)} + \cdots \right], \]

like (5) contains Lange’s continued fraction (7) and the product of Wallis (2) as special cases. Again by examining this formula as \( n = 0,1,2,\cdots \), we can morph (7) into (2) as shown below:

\[ n = 0: \quad \pi = 3 + \frac{1^2}{6} + \frac{3^2}{6} + \frac{5^2}{6} + \cdots. \] (Lange’s continued fraction.)

\[ n = 1: \quad \pi = \frac{2 \times 2}{1 \times 3} \times \frac{3 \times 5}{2 \times 2} \left[ 7 + \frac{1^2}{14} + \frac{3^2}{14} + \frac{5^2}{14} + \cdots \right]. \]

\[ n = 2: \quad \pi = \frac{2 \times 2 \times 4 \times 4}{1 \times 3 \times 3 \times 5} \times \frac{3 \times 5}{2 \times 2} \left[ 11 + \frac{1^2}{22} + \frac{3^2}{22} + \frac{5^2}{22} + \cdots \right]. \]

\[ n = 3: \quad \pi = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6}{1 \times 3 \times 3 \times 5 \times 5 \times 7} \times \frac{3 \times 5}{2 \times 2} \left[ 15 + \frac{1^2}{30} + \frac{3^2}{30} + \frac{5^2}{30} + \cdots \right]. \]

\[ \cdots \]

\[ n \to \infty: \quad \pi = 2 \times \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6}{1 \times 3 \times 3 \times 5 \times 5 \times 7}. \] (Original product of Wallis reciprocated.)

Extensions of the formulas (6) and (8) are also given.

**Derivation of the results and more morphing**

All the results of this paper are special cases of the known formula [4, page 35]
\[
\frac{4\Gamma\left(\frac{x + y + 3}{4}\right)\Gamma\left(\frac{x - y + 3}{4}\right)}{\Gamma\left(\frac{x + y + 1}{4}\right)\Gamma\left(\frac{x - y + 1}{4}\right)} = x + \frac{1^2 - y^2}{2x} + \frac{3^2 - y^2}{2x} + \frac{5^2 - y^2}{2x} + \cdots,
\]

valid for either \(y\) an odd integer and \(x\) any complex number or \(y\) any complex number and \(\text{Re}(x) > 0\). The names of Euler, Stieltjes, and Ramanujan [1, page 140] have been associated with this result. Using the very well known formulas \(\Gamma(n + 1) = n!\),

\[
\Gamma(x) = \Gamma(x + 1) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi},
\]

we have

\[
\left(\frac{2k + 1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{2^k} \sqrt{\pi},
\]

valid for \(k = 1, 2, 3, \ldots\). With this last result and appropriate values of \(x\) and \(y\), the left hand side of (9) can be expressed in terms of rational numbers and \(\pi\). For example, if we set \(y = 0\) and \(x = 4n + 1\) in (9) we get our general formula (5) and setting \(y = 0\) and \(x = 4n + 3\) we get our general formula (8). The manipulations are simple and the reader will have no difficulty verifying our formulas.

If in (9) we set \(x = 4n + 3\) and \(y = 2(2j + 1)\) for \(j\) an integer in the range \(0 \leq j \leq n\) we get an extension of (5)

\[
\frac{4}{\pi} = \frac{(2n - 2j + 1)(2n - 2j + 3)\cdots(2n + 2j + 1)}{(2n - 2j + 2)(2n - 2j + 4)\cdots(2n + 2j + 2)} \times W(n - j) \times
\]

\[
\frac{1}{2n - 2j + 1}\left[ 4n + 3 + \frac{1^2 - 2^2 (2j + 1)^2}{2(4n + 3)} + \frac{3^2 - 2^2 (2j + 1)^2}{2(4n + 3)} + \frac{5^2 - 2^2 (2j + 1)^2}{2(4n + 3)} + \cdots \right].
\]

Let \(j = 0\) and (10) becomes

\[
\frac{4}{\pi} = \frac{(2n + 1)}{(2n + 2))} \times W(n) \times \frac{1}{2n + 1}\left[ 4n + 3 + \frac{1^2 - 2^2}{2(4n + 3)} + \frac{3^2 - 2^2}{2(4n + 3)} + \frac{5^2 - 2^2}{2(4n + 3)} + \cdots \right].
\]
We list the special cases of this formula for \( n = 0, 1, 2, \cdots \) below:

\[
\begin{align*}
n = 0: \quad & \frac{4}{\pi} = \frac{1}{2} \left[ 3 + \frac{1^2 - 2^2}{6} + \frac{3^2 - 2^2}{6} + \frac{5^2 - 2^2}{6} + \cdots \right] \\
n = 1: \quad & \frac{4}{\pi} = \frac{3}{4} \times \frac{3}{2} \times \frac{1}{3} \left[ 7 + \frac{1^2 - 2^2}{14} + \frac{3^2 - 2^2}{14} + \frac{5^2 - 2^2}{14} + \cdots \right] \\
n = 2: \quad & \frac{4}{\pi} = \frac{5}{6} \times \frac{5}{2} \times \frac{3 \cdot 5}{4} \times \frac{1}{5} \left[ 11 + \frac{1^2 - 2^2}{22} + \frac{3^2 - 2^2}{22} + \frac{5^2 - 2^2}{22} + \cdots \right] \\
n = 3: \quad & \frac{4}{\pi} = \frac{7}{8} \times \frac{7}{2} \times \frac{3 \cdot 5 \cdot 7}{4} \times \frac{1}{6} \left[ 15 + \frac{1^2 - 2^2}{30} + \frac{3^2 - 2^2}{30} + \frac{5^2 - 2^2}{30} + \cdots \right] \\
& \cdots \\
n \to \infty: \quad & \frac{4}{\pi} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdots \text{ (original Wallis product)}
\end{align*}
\]

If in (9) we set \( x = 4n + 1 \) and \( y = 2(2j + 1) \) for \( j \) an integer in the range \( 1 \leq j < n \) we get an extension of (8)

\[
\pi = \frac{(2n - 2j + 2)(2n - 2j + 4) \cdots (2n + 2j)}{(2n - 2j + 3)(n - j + 5) \cdots (2n + 2j + 1)} \times \frac{1}{W(n - j)}
\]

\[
\begin{align*}
(11) \quad & \frac{1}{2(n - j)} \left[ 4n + 1 + \frac{1^2 - 2^2 (2j + 1)^2}{2(4n + 1)} + \frac{3^2 - 2^2 (2j + 1)^2}{2(4n + 1)} + \frac{5^2 - 2^2 (2j + 1)^2}{2(4n + 1)} + \cdots \right].
\end{align*}
\]

Let \( j = 1 \) and (11) becomes

\[
\begin{align*}
\pi = \frac{(2n)(2n + 2)}{(2n + 1)(2n + 3)} \times \frac{1}{W(n - 1)} & \times \\
& \frac{1}{2(n - 1)} \left[ 4n + 1 + \frac{1^2 - 6^2}{2(4n + 1)} + \frac{3^2 - 6^2}{2(4n + 1)} + \frac{5^2 - 6^2}{2(4n + 1)} + \cdots \right].
\end{align*}
\]

Let us list this formula for various values on \( n \):
\[
\begin{align*}
n &= 2 : \pi = \frac{4 \cdot 6}{5 \cdot 7} \times \frac{2 \cdot 2}{1 \cdot 3} \times \frac{1}{2} \left[ 9 + \frac{1^2 - 6^2}{18} + \frac{3^2 - 6^2}{18} + \frac{5^2 - 6^2}{18} + \ldots \right] \\
\end{align*}
\]
\[
\begin{align*}
n &= 3 : \pi = \frac{6 \cdot 8}{7 \cdot 9} \times \frac{2 \cdot 2}{1 \cdot 3} \times \frac{4 \cdot 4}{3 \cdot 5} \times \frac{1}{4} \left[ 13 + \frac{1^2 - 6^2}{26} + \frac{3^2 - 6^2}{26} + \frac{5^2 - 6^2}{26} + \ldots \right] \\
\end{align*}
\]
\[
\begin{align*}
n &= 4 : \pi = \frac{8 \cdot 10}{9 \cdot 11} \times \frac{2 \cdot 2}{1 \cdot 3} \times \frac{4 \cdot 4}{3 \cdot 5} \times \frac{6 \cdot 6}{5 \cdot 7} \times \frac{1}{6} \left[ 17 + \frac{1^2 - 6^2}{34} + \frac{3^2 - 6^2}{34} + \frac{5^2 - 6^2}{34} + \ldots \right] \\
\end{align*}
\]
\[
\begin{align*}
n &= 5 : \pi = \frac{10 \cdot 12}{11 \cdot 13} \times \frac{2 \cdot 2}{1 \cdot 3} \times \frac{4 \cdot 4}{3 \cdot 5} \times \frac{6 \cdot 6}{5 \cdot 7} \times \frac{8 \cdot 8}{7 \cdot 9} \times \frac{1}{8} \left[ 21 + \frac{1^2 - 6^2}{42} + \frac{3^2 - 6^2}{42} + \ldots \right] \\
\end{align*}
\]

\[
\begin{align*}
n \to \infty : \pi &= 2 \times \frac{2 \cdot 2}{1 \cdot 3} \times \frac{4 \cdot 4}{3 \cdot 5} \times \frac{6 \cdot 6}{5 \cdot 7} \times \frac{8 \cdot 8}{7 \cdot 9} \ldots \quad \text{(original Wallis product reciprocated)}
\end{align*}
\]

We see that the general formulas (10) and (11) start with a generalized Brouncker type

continued fraction \[
\left[ x + \frac{1^2 - y^2}{2x} + \frac{3^2 - y^2}{2x} + \frac{5^2 - y^2}{2x} + \ldots \right]
\]

and gradually morph it into the Wallis product as \( n \) approaches infinity.

**Final remarks**

Wallis described the ingenious way in which he obtained his product (2) in [7].

He states that he showed his product to Lord Brouncker who then obtained the continued fraction (3). It appears that Brouncker never published his method of finding this continued fraction and only partially explained his reasoning to Wallis. Wallis gives some hints in [7, pages 167 - 178] as to how Brouncker proceeded but the explanation is incomplete. Stedall in [5, pages 300-310] has discussed this and made her own conjecture as to how Brouncker might have reasoned. In his discussion of this question, Wallis published a table in [7, page 172] which we reproduce here.
In the third row of this table we see the continued fractions obtained from our general formulas (5) and (8). Stedall [5] has recently called attention to these fractions that appear to have been overlooked. She also points out [5, page 307] that both Wallis and Brouncker could easily have written the value of these fractions in terms of rational numbers and pi. Thus we see that the continued fractions that we obtained from (5) and (8) are among the oldest continued fractions and their values were conjectured as early as 1656!

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References


