Exploring strange functions on the computer

Continuous, nowhere differentiable functions

Weierstraß, 1872:

$$C_{a,b}(x) \ := \ \sum_{n=0}^{\infty} a^n \cos(b^n \cdot b\pi x) \quad (|a| < 1,\ b > 1)$$

is cnd if $b \in 2\mathbb{N}+1,\ ab > 1 + \frac{3}{2}\pi$.

Hardy, 1916: $C_{a,b},\ S_{a,b}$ cnd if $b \in \mathbb{R},\ b > 1,\ ab \geq 1$.

Simpler proof? (Freud, Kahane, Hata, Baouche/Dubuc,...)

Consider only $b \in \mathbb{N}$, in fact $b = 2$, and

$$S_{a,2}(x) \ = \ \sum_{n=0}^{\infty} a^n \sin(2^{n+1}\pi x) \quad (|a| < 1)$$

on $[0,1]$.

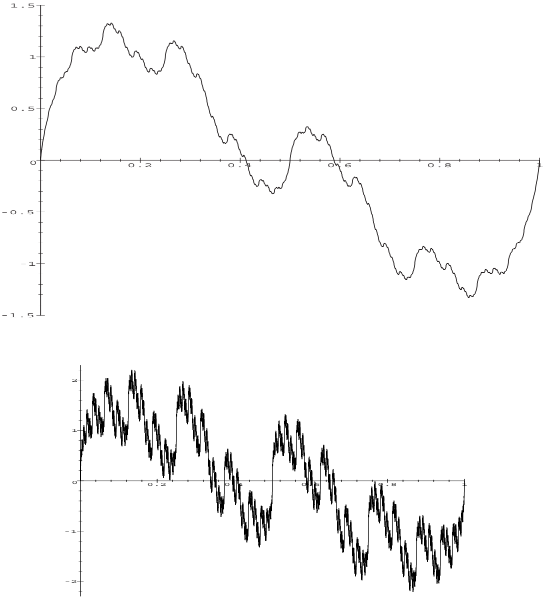


Figure 5.2. The Weierstrass functions $S_{1/2,2}$ (top) and $S_{3/4,2}$ (bottom).

Functional equations

For $S = S_{a,2}$**:**

$$\begin{aligned} S\left(\frac{x}{2}\right) &= a\,S(x) + \sin(\pi x), \\ S\left(\frac{x+1}{2}\right) &= a\,S(x) - \sin(\pi x). \end{aligned}$$

In general: System (F) consisting of

$$f\left(\frac{x}{2}\right) = a_0 f(x) + g_0(x) \qquad (\mathsf{F}_0)$$

$$f\left(\frac{x+1}{2}\right) = a_1 f(x) + g_1(x) \qquad (\mathsf{F}_1)$$

on $[0,1]$, for given $|a_0|, |a_1| < 1$, $g_0, g_1 : [0,1] \to \mathbb{R}$ and unknown $f : [0,1] \to \mathbb{R}$.

Examples: 1) $S_{a,2}$ with $a_0 = a_1 = a$ and $g_0(x) = -g_1(x) = \sin(\pi x)$.

2) $C_{a,2}$ with $a_0 = a_1 = a$ and $g_0(x) = -g_1(x) = \cos(\pi x)$.

3) $T_a(x) := \sum_{n=0}^{\infty} a^n\, d(2^n x)$, $d(x) = \mathrm{dist}(x, \mathbb{Z})$,

with $a_0 = a_1 = a$ and $g_0(x) = \dfrac{x}{2}$, $g_1(x) = \dfrac{1-x}{2}$.

Unique solutions?

$$f\left(\frac{x}{2}\right) \;=\; a_0 f(x) + g_0(x), \qquad f\left(\frac{x+1}{2}\right) \;=\; a_1 f(x) + g_1(x). \tag{F}$$

$$f \text{ solves (F)} \quad \Longrightarrow \quad f(0) = \frac{g_0(0)}{1-a_0},\; f(1) = \frac{g_1(1)}{1-a_1}$$

$$\Longrightarrow \quad f\left(\tfrac{1}{2}\right) = a_0 f(1) + g_0(1) = a_1 f(0) + g_1(0).$$

Thus: If a solution exists, then necessarily

$$a_0 \frac{g_1(1)}{1-a_1} + g_0(1) \;=\; a_1 \frac{g_0(0)}{1-a_0} + g_1(0). \tag{$*$}$$

Moreover,

$$f\left(\tfrac{1}{4}\right) = a_0 f\left(\tfrac{1}{2}\right) + g_0\left(\tfrac{1}{2}\right),\; f\left(\tfrac{3}{4}\right) = a_1 f\left(\tfrac{1}{2}\right) + g_1\left(\tfrac{1}{2}\right),$$

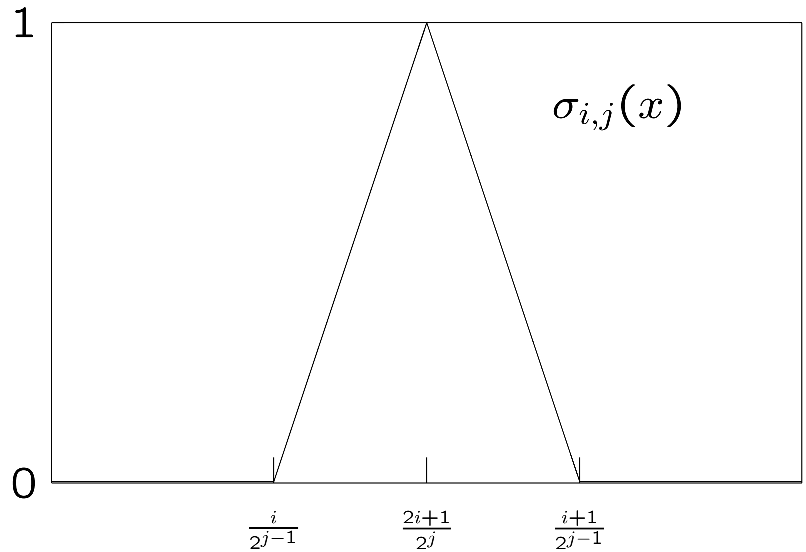
$$f\left(\tfrac{1}{8}\right) = a_0 f\left(\tfrac{1}{4}\right) + g_0\left(\tfrac{1}{4}\right),\; f\left(\tfrac{3}{8}\right) = \ldots,\; f\left(\tfrac{5}{8}\right) = \ldots,\; f\left(\tfrac{7}{8}\right) = \; a_1 f\left(\tfrac{3}{4}\right) + g_1\left(\tfrac{3}{4}\right),$$

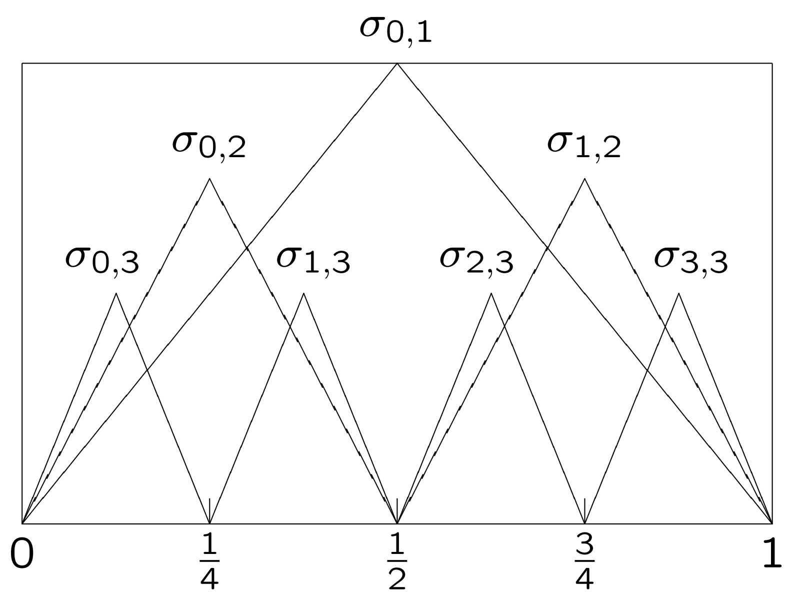
$$f\left(\tfrac{2i+1}{16}\right),$$

$$\ldots$$

$$f\left(\tfrac{2i+1}{2^n}\right).$$

Schauder basis





Schauder coefficients

Theorem (Schauder, 1930, and Faber, 1908).
Every $f \in C[0,1]$ has a unique expansion of the form

$$f(x) \;=\; \gamma_{0,0}(f)\,\sigma_{0,0}(x) + \gamma_{1,0}(f)\,\sigma_{1,0}(x) + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} \gamma_{i,n}(f)\,\sigma_{i,n}(x),$$

where the coefficients $\gamma_{i,n}(f)$ are given by

$$\gamma_{0,0}(f) \;=\; f(0), \quad \gamma_{1,0}(f) = f(1), \quad \text{and}$$

$$\gamma_{i,n}(f) \;=\; f\left(\frac{2i+1}{2^n}\right) - \frac{1}{2} f\left(\frac{i}{2^{n-1}}\right) - \frac{1}{2} f\left(\frac{i+1}{2^{n-1}}\right).$$

Theorem (Faber, 1910).
Assume that $f \in C[0,1]$ has a finite derivative at some point x_0. Then

$$\lim_{n\to\infty} 2^n \cdot \min\left\{|\gamma_{i,n}(f)| : i = 0, \ldots, 2^{n-1}-1\right\} \;=\; 0.$$

Recursion formula for solutions of (F)

Theorem.

Assume that $(*)$ holds and that g_0, g_1 are continuous.

Let f be the continuous solution of the system (F).

Then

(i) $\gamma_{0,0}(f) = f(0) = \frac{g_0(0)}{1-a_0}$ and $\gamma_{1,0}(f) = f(1) = \frac{g_1(1)}{1-a_1}$,

(ii) $\gamma_{0,1}(f) = \left(a_1 - \frac{1}{2}\right) f(0) - \frac{1}{2} f(1) + g_1(0) = \left(a_0 - \frac{1}{2}\right) f(1) - \frac{1}{2} f(0) + g_0(1)$,

(iii) $\gamma_{i,n+1}(f) = a_0 \gamma_{i,n}(f) + \gamma_{i,n}(g_0)$ for $i = 0, \ldots, 2^{n-1} - 1$,

$\gamma_{i,n+1}(f) = a_1 \gamma_{i-2^{n-1},n}(f) + \gamma_{i-2^{n-1},n}(g_1)$ for $i = 2^{n-1}, \ldots, 2^n - 1$.

Results and questions

Let $\underline{\delta}_n(f) := 2^n \cdot \min\left\{|\gamma_{i,n}(f)| : i = 0, \ldots, 2^{n-1} - 1\right\}$.

Theorem. $\underline{\delta}_n(S_{a,2}) \not\to 0$ $(n \to \infty)$ for $1 > a \geq \frac{1}{2}$.

This proves that $S_{a,2}$ is cnd for $1 > a \geq \frac{1}{2}$.

Open questions.

1) Show that, for $a = \frac{1}{2}$, $\lim\limits_{n\to\infty} \underline{\delta}_n(S_{a,2})$ exists, and find its value.

2) Show, more generally, that $\lim\limits_{n\to\infty} \underline{\delta}_n(S_{a,2})/(2|a|)^n$ exists, and determine the function $a \mapsto \lim\limits_{n\to\infty} \underline{\delta}_n(S_{a,2})/(2|a|)^n$.

A functional equation with discontinuous solution

Consider the system, for given $0 < q < 1$,

$$\begin{aligned} s\left(\frac{x}{2}\right) &= q\,s(x) - 1, \\ s\left(\frac{x+1}{2}\right) &= q\,s(x) + 1. \end{aligned}$$

This system has a unique bounded solution s_q, which is discontinuous precisely at the dyadic rationals.

Let $F_q(t) := m\{x \in [0,1] \mid s_q(x) \le t\}$, the distribution function of s_q.

It can be shown that F_q is the unique function satisfying the functional equation

$$F(t) = \frac{1}{2} F\left(\frac{t-1}{q}\right) + \frac{1}{2} F\left(\frac{t+1}{q}\right)$$

with $F_q(t) = 0$ for $t < -1/(1-q)$ and $F_q(t) = 1$ for $t > 1/(1-q)$.

Theorem (Jessen/Wintner 1935).
F_q is either absolutely continuous or singular.

Question: For which q is F_q absolutely continuous, for which q is it singular?

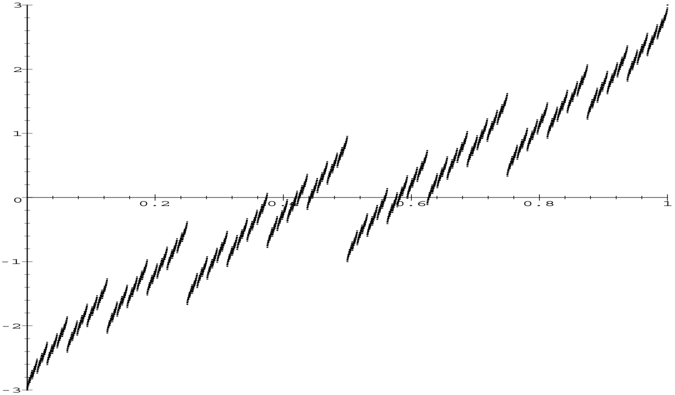


Figure 5.7. Cantor dust (the case $q = 2/3$).

Some answers

Theorem (Kershner/Wintner 1935).
For $0 < q < \frac{1}{2}$, F_q is singular (in fact, a Cantor function).

Theorem (Wintner 1935).
For $q = \frac{1}{2}$, $F_q(t) = \left\{ \begin{array}{cc} 0, & t < -2 \\ \dfrac{t+2}{4}, & -2 \le t \le 2 \\ 1, & t > 2 \end{array} \right\}$, which is absolutely continuous.
In fact, for each $q = 2^{-1/p}$, F_q is absolutely continuous.

Theorem (Erdős 1939).
If $q > \frac{1}{2}$ and $1/q$ is a Pisot number, then F_q is singular!
E.g., F_q is singular for $q = (\sqrt{5} - 1)/2 \approx 0.618033989$.
(Proof: via the Fourier-Stieltjes transform of F_q.)

Theorem (Garsia 1962).
Some explicit algebraic numbers q (besides $2^{-1/p}$) for which F_q is absolutely continuous.

Theorem (Solomyak 1995).
F_q is absolutely continuous for a.e. $q \in (\frac{1}{2}, 1)$!

Open questions and experimental approach

Open: 1) Is the set of exceptional values $q > \frac{1}{2}$ (with F_q singular) countable?
2) Is there a rational $q > \frac{1}{2}$ with F_q singular?
Is there a rational $q > \frac{1}{2}$ with F_q absolutely continuous?
3) What about $q = \frac{2}{3}$? What about other specific values?

Experimental approach: Visualize the density $f_q = F_q'$ a.e.
In fact, if F_q is absolutely continuous, then f_q is a non-trivial L_1-solution of the functional equation

$$f(t) \; = \; \frac{1}{2q}\left(f\left(\frac{t-1}{q}\right) + f\left(\frac{t+1}{q}\right)\right), \qquad (S_q)$$

on $\mathbb{R}$.
Vice versa, if a non-trivial L_1-solution f_q of (S_q) exists, then it is the density of an absolutely continuous F_q.

How to visualize f_q?

$$f(t) = \frac{1}{2q}\left(f\left(\frac{t-1}{q}\right) + f\left(\frac{t+1}{q}\right)\right) \qquad (S_q)$$

It can be shown: If a non-trivial L_1-solution f_q of (S_q) exists, then it:

- is unique up to a multiplicative constant,
- satisfies $\operatorname{supp} f_q = [-\frac{1}{1-q}, \frac{1}{1-q}]$,
- and is either positive or negative a.e. on its support.

This implies: Define an operator B_q on L_1 by

$$(B_q f)(t) = \frac{1}{2q}\left(f\left(\frac{t-1}{q}\right) + f\left(\frac{t+1}{q}\right)\right)$$

and consider the iteration $f^{(n)} := B_q f^{(n-1)}$ with some $f^{(0)} \in L_1$. Then:

If $(f^{(n)})_n$ converges in L_1, then the limit is an L_1-solution of (S_q).

If (S_q) has a non-trivial L_1-solution, then $(f^{(n)})_n$ converges *in the mean* in L_1.

A final remark about $q = 2/3$

Rescale F_q resp. f_q such that the support is $[0,1]$ instead of $[-\frac{1}{1-q}, \frac{1}{1-q}]$.

Then for $q = 2/3$, the functional equation (S_q) is equivalent to the system

$$
\begin{aligned}
f\left(\frac{x}{3}\right) &= \frac{3}{4} f\left(\frac{x}{2}\right), \\
f\left(\frac{x+1}{3}\right) &= \frac{3}{4} f\left(\frac{x}{2}\right) + \frac{3}{4} f\left(\frac{x+1}{2}\right), \\
f\left(\frac{x+2}{3}\right) &= \frac{3}{4} f\left(\frac{x+1}{2}\right)
\end{aligned}
$$

on $[0,1]$.

Does this system have a non-trivial L_1-solution?

If so, is the solution continuous?