Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.
In Thrall to Fibonacci

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Many years ago now, Jeff Shallit, heavily influenced by the cult of Fibonacci, noticed the continued fraction expansion

$$2^{-1} + 2^{-2} + 2^{-3} + 2^{-5} + \cdots + 2^{-F_h} + \cdots$$

$$= [ 0 , 1 , 10 , 6 , 1 , 6 , 2 , 14 , 4 , 124 , 2 , 1 , 2 , 2039 , 1 , 9 , 1 , 1 , 1 , 1 , 262111 , 2 , 8 , 1 , 1 , 1 , 3 , 1 , 536870655 , 4 , 16 , 3 , 1 , 3 , 7 , 1 , 140737488347135 , \ldots ] .$$

The increasing sequence of very large partial quotients demands explanation. That was discovered, a while later, in remarks of Michel Mendès France, with Blanchard and then with me, wherein one considers continued fractions of formal Laurent series and then specialises the variable to an appropriate integer. Indeed, Jeff and I found experimentally that

$$X^{-1} + X^{-2} + X^{-3} + X^{-5} + \cdots + X^{-F_h} + \cdots$$

$$= [ 0 , X^{-1} , X^2+2X+2 , X^3-X^2+2X-1 , -X^3+X-1 , -X , -X^4+X , -X^2 , -X^7+X^2 , -X-1 , X^2-X+1 , X^{11}-X^3 , -X^3-X , -X , X , X^{18}-X^5 , -X , X^3+1 , X , -X , -X-1 , -X+1 , -X^{29}+X^8 , X-1 , \ldots ] .$$
The continued fraction expansion \([ 0, a_1(x), a_2(x), \ldots ]\) of a formal power series \(f(x) = \sum_{h=1}^{\infty} f_h x^{-h}\), say in \(\mathbb{Q}((1/x))\), generically has all its partial quotients \(a_h(x)\) of degree one (of course I except ‘early’ partial quotients) and with coefficients increasing in complexity at frantic pace — the length of their numerators and denominators increases as \(O(h^2)\). A prime example is

\[
x^{-1} + x^{-2} + x^{-3} + x^{-5} + x^{-7} + x^{-11} + x^{-13} + x^{-17} + \cdots =
\]

\[
[ 0, x - 1, x^2 + 2x + 2, x - 1, x, x + 1, -x + 3, -\frac{1}{6}x - \frac{2}{9}, -54x - 36,
\]

\[
-\frac{1}{81}x^7 - \frac{2}{81}x, 54x + 36, -\frac{1}{30}x + \frac{14}{225}, -\frac{125}{26}x - \frac{5775}{676}, \frac{17576}{85625}x - \frac{471172}{2346125}
\]

\[
-\frac{1607095625}{21477872}x + \frac{111100749375}{109459984}, \frac{5930577406}{1541204704375}x + \frac{220062276512}{10788432930625},
\]

\[
-\frac{151038061028750}{12125065506567}x - \frac{3797528391580000}{117208966563481}, \frac{1301767671194406}{528633213600625}x + \frac{45843615713376}{10788432930625},
\]

\[
\frac{528633213600625}{8282833201083948}x - \frac{4712765099249571875}{23584677303986451601},
\]

\[
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The continued fraction expansion \([ 0 \, a_1(x) \, a_2(x) \, \ldots ]\) of a formal power series \(f(x) = \sum_{h=1}^{\infty} f_h x^{-h}\), say in \(\mathbb{Q}((1/x))\), generically has all its partial quotients \(a_h(x)\) of degree one (of course I except ‘early’ partial quotients) and with coefficients increasing in complexity at frantic pace — the length of their numerators and denominators increases as \(O(h^2)\). A prime example is

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-\frac{1}{81}x^7 - \frac{2}{81}x \; , \; 54x + 36 \; , \; -\frac{1}{30}x + \frac{14}{225} \; , \; -\frac{125}{26}x - \frac{5775}{676} \; , \; \frac{17576}{85625}x - \frac{471172}{2346125}
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Notice the early surprise reporting the number of days in the week.
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$$\sum_{h=0}^{\infty} x^{-3h} = [0, x, -x, -x^3, x, x, -x, -x^9, x, -x, -x, x^3, x, x, -x, -x^27, x, -x, -x, -x^3, x, x, -x, x^9, x, -x, \ldots]$$,
On the other hand,

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and very similarly — but here I write \( \overline{y} \) in place of \( -y \) both to save space and better to display the pattern of signs.
On the other hand,

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$$x \sum_{h=0}^{\infty} x^{-2^h} = [1, x, \overline{x}, \overline{x}, \overline{x}, x, x, \overline{x}, \overline{x}, x, \overline{x}, \overline{x}, x, x, \overline{x}, \overline{x}, \overline{x}, x, x, \overline{x}, \overline{x}, \overline{x}, \overline{x}, \ldots]$$
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Note that the sequence of signs is a paperfolding sequence.
On the other hand,

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\[ x^3 , x , x , -x , -x^{27} , x , -x , -x , -x^3 , x , x , -x , x^9 , x , -x , \ldots ] , \]

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\[ x , x , x , \overline{x} , \overline{x} , x , \overline{x} , \overline{x} , \overline{x} , x , \overline{x} , \overline{x} , \overline{x} , \ldots ] . \]

Note that the sequence of signs is a paperfolding sequence: these results are ready consequence of Michel Mendès France’s paperfolding lemma.
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Note that the sequence of signs is a paperfolding sequence: these results are ready consequence of Michel Mendès France’s paperfolding lemma; the magenta partial quotients mark the folds in the expansions.
Indeed given a sequence of positive integers \((G_h)\) satisfying \(G_{h+1}/G_h > 2\) for all \(h\), the paperfolding lemma entails that a series \(\sum_{h=0}^{\infty} x^{-G_h}\) has a folded continued fraction expansion of the genre just instanced.
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If, however, \( \frac{G_{h+1}}{G_h} < 2 \) for any, let alone for many \( h \), then it seems that there will be partial quotients with non-integer coefficients and that the non-integrality will propagate in the course of the expansion to become generic behaviour as in the prime example.
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There is, however, an astonishing class of exceptions.
Specifically, if \((G_h)\) is a linear recurrence sequence, then I can, and will present compelling evidence that the continued fraction expansion is generic unless, surprisingly, \((G_h)\) satisfies one of the recurrence relations

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G_{h+k} = G_{h+k-1} + G_{h+k-2} + \cdots + G_h,
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with appropriate initial values; to wit: Fibonacci/Lucas numbers and their higher order immediate generalisations.
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The secret behind my method is betrayed by the example

\[
g(x) = x \sum_{h=0}^{\infty} x^{-2^h} = [1 , x , \overline{x} , \overline{x} , \overline{x} , x , x , \overline{x} , \overline{x} , x , \overline{x} , \overline{x} , \overline{x} , x , x , x , x , \ldots] .
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Here, obviously, \(g(x^2) = x(g(x) - 1)\).
But (I say more about this below) multiplying a continued fraction by $x$ multiplies every second partial quotient by $x$ and *divides* each of the remaining partial quotients by $x$. 
But (I say more about this below) multiplying a continued fraction by $x$ multiplies every second partial quotient by $x$ and divides each of the remaining partial quotients by $x$. So the functional equation $g(x^2) = x(g(x) - 1)$ alleges that

$$[1, x^2, \overline{x}^2, \overline{x}^2, x^2, x^2, \overline{x}^2, x^2, \overline{x}^2, x^2, x^2, \overline{x}^2, x^2, \overline{x}^2, x^2, \overline{x}^2, \ldots] = [0, 1, \overline{x}^2, \overline{1}, \overline{x}^2, 1, x^2, \overline{1}, \overline{x}^2, 1, \overline{x}^2, \overline{1}, x^2, 1, x^2, \overline{1}, \overline{x}^2, \ldots].$$
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$$= [0, 1, \overline{x}^2, \overline{1}, \overline{x}^2, 1, x^2, \overline{1}, \overline{x}^2, 1, \overline{x}^2, \overline{1}, x^2, 1, x^2, \overline{1}, \overline{x}^2, \ldots].$$

Apparently, inserting a ripple $1\overline{1}1\overline{1}1\overline{1}1\ldots$ or, if one prefers, a fold of 1s into a continued fraction expansion, and also rippling — changing the sign of alternate — pre-existing entries, needs no more than to require one to change the zero-th entry from 1 to 0.
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\[ = [0, 1, \overline{x^2}, \overline{1}, \overline{x^2}, 1, x^2, \overline{1}, \overline{x^2}, 1, \overline{x^2}, \overline{1}, x^2, 1, x^2, \overline{1}, \overline{x^2}, \ldots]. \]

Apparently, inserting a ripple $1\overline{1}1\overline{1}1\overline{1}1\ldots$ or, if one prefers, a fold of 1s into a continued fraction expansion, and also rippling — changing the sign of alternate — pre-existing entries, needs no more than to require one to change the zero-th entry from 1 to 0. This, indeed, is my ripple lemma re-alleged below.
Implicit Folding
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Suppose we know the continued fraction expansion for $S(h)$, where

$$S(h) = x^{G_m} (x^{-G_m} + x^{-G_{m+1}} + x^{-G_{m+2}} + \cdots + x^{-G_{m+h}}).$$
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We step our knowledge up to $S(h+1)$ by the following strategy:

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We step our knowledge up to $S(h+1)$ by the following strategy:

1. Replace $m$ by $m + 1$;
2. Divide by $x^{G_{m+1} - G_m}$;
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Of course we do know that

$$S(0) = 1 = [1], \quad S(1) = [1, x^{G_{m+1} - G_m}],$$

so this process has a beginning.
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so this process has a beginning. After sufficiently many steps to have reached the \ldots we divide by $x^{G_m}$ and will then have obtained the continued fraction expansion of

$$S = x^{-G_m} + x^{-G_{m+1}} + x^{-G_{m+2}} + x^{-G_{m+3}} + x^{-G_{m+4}} + \cdots \cdots.$$
Dividing a Continued Fraction by . . .
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It’s not hard to see that

\[ M[N_a, M_b, N_c, M_d, N_e, \ldots] = N[M_a, N_b, M_c, N_d, M_e, \ldots], \]
It’s not hard to see that

\[ M[Na, Mb, Nc, Md, Ne, \ldots] = N[Ma, Nb, Mc, Nd, Me, \ldots], \]

cutely illustrating how multiplying, or dividing, divides respectively multiplies, every second partial quotient.
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\[ [1, a, b, c, d, e, \ldots] = [0, 1, \bar{a}, \bar{1}, b, 1, \bar{c}, \bar{1}, d, 1, \bar{e}, \bar{1}, \ldots]. \]
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Specifically, terminating versions of the lemma are given by

\[ [1 , a , \beta ] = [0 , 1 , \bar{a} - 1 , \bar{\beta} ] \text{ and } [1 , a , b , \gamma ] = [0 , 1 , \bar{a} , \bar{1} , b + 1 , \gamma ] . \]
It’s not hard to see that

\[ M[Na, Mb, Nc, Md, Ne, \ldots] = N[Ma, Nb, Mc, Nd, Me, \ldots], \]

cutely illustrating how multiplying, or dividing, divides respectively multiplies, every second partial quotient. Unfortunately, however, our expansions begin with the partial quotient 1, which is not divisible by much at all. If only that 1 were 0 — because zero is divisible by anything! Happily, **Ripple Lemma.** *For space-saving reasons write \(-\) for \(\ldots\). Then*

\[ [1, a, b, c, d, e, \ldots] = [0, 1, \bar{a}, \bar{1}, b, 1, \bar{c}, \bar{1}, d, 1, \bar{e}, \bar{1}, \ldots]. \]

Specifically, terminating versions of the lemma are given by

\[ [1, a, \beta] = [0, 1, \bar{a} - 1, \bar{\beta}] \quad \text{and} \quad [1, a, b, \gamma] = [0, 1, \bar{a}, \bar{1}, b + 1, \gamma]. \]

Amusingly, the ripple lemma is an immediate consequence of a simple rule for changing the sign of partial quotients in a continued fraction expansion.
A Folded Continued Fraction
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As example, set $G_n = 2^n$, noting that $G_{m+1} - G_m = G_m$. Recall we know that $S(1) = x^{G_m} (x^{-G_m} + x^{-G_{m+1}}) = [1, x^{G_m}]$. 

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We will loop: (a) Ripple the continued fraction expansion and replace $m$ by $m + 1$; (b) Divide by $x^{G_{m+1} - G_m} = x^{G_m}$, and add 1.
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We get (a) $[0, 1, x^{G_{m+1}}, 1]$; (b) $[1, x^{G_m}, x^{G_m}, x^{G_m}]$. 
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(b) $[1, x^{G_m}, \overline{x}^{G_m}, \overline{x}^{G_m}, \overline{x}^{G_m}, x^{G_m}, x^{G_m}, \overline{x}^{G_m}]$. 


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Plainly, each loop doubles the number of stable entries of the expansion: we started with, one, then had two, now have four stable entries. Yet more clearly, every entry after the zero-th will be $\pm x^{G_m}$. 

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We get (a) \([0, 1, x^{G_{m+1}}, \bar{1}]\); (b) \([1, x^{G_m}, \bar{x}^{G_m}, \bar{x}^{G_m}]\). The next loop gives (a) \([0, 1, \bar{x}^{G_{m+1}}, \bar{1}, \bar{x}^{G_{m+1}}, 1, x^{G_{m+1}}, \bar{1}]\);

(b) \([1, x^{G_m}, \bar{x}^{G_m}, \bar{x}^{G_m}, x^{G_m}, \bar{x}^{G_m}, x^{G_m}, \bar{x}^{G_m}]\).

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Plainly, each loop doubles the number of stable entries of the expansion: we started with, one, then had two, now have four stable entries. Yet more clearly, every entry after the zero-th will be \( \pm x^{G_m} \). A little less obviously, but evident if one has the correct experience, the sequence of signs \( \pm \) is a paperfolding sequence given by the pattern of creases in a piece of paper folded repeatedly in half.

We obtain the expansion of \( x^{G_m} \sum_{n \geq m} x^{-G_n} \) and, finally, divide by \( x^{G_m} \).
The General Case
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In brief, the argument just now sketched works flawlessly if the oldest exponent $G_{m+n} - 2G_{m+n-1} + G_m$ is positive for all $n$.
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In brief, the argument just now sketched works flawlessly if the oldest exponent $G_{m+n} - 2G_{m+n-1} + G_m$ is positive for all $n$, at the very least a Plan B will be required if $G_{m+k+1} - 2G_{m+k} + G_m = 0$ for some $k$. 

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The good news is that there is an obvious Plan B, but the downside is that it applies if $G_{m+k+1} - 2G_{m+k} + G_m = 0$. This relation defines the Fibonacci and Lucas numbers if $k = 2$, and their generalisations for greater $k$. 
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The good news is that there is an obvious Plan B, but the downside is that it applies if $G_{m+k+1} - 2G_{m+k} + G_m = 0$. This relation defines the Fibonacci and Lucas numbers if $k = 2$, and their generalisations for greater $k$. By the way, though more general results could be obtained from my arguments, here and from hereon I do suppose that $(G_h)$ is a linear recurrence sequence.
Inadmissible Entries
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In the special example $G_m = 2^m$ the final division leads to the expansion

$$[0, x^{G_m}, 1, x^{G_m}, 1, x^{G_m}, 1, x^{G_m}, 1, x^{G_m}, 1, \ldots]$$
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but entries in such an expansion should be polynomials of degree at least one.
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but entries in such an expansion should be polynomials of degree at least one. Analogously, the expansion $-\pi = [3, 7, 15, \overline{1}, 292, \overline{1}, \ldots]$ is filled with inadmissible negative entries.
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Analogously, the expansion $-\pi = [3, 7, 15, \bar{1}, 292, \bar{1}, \ldots]$ is filled with inadmissible negative entries. But $-\pi = [4, 1, 6, 15, 1, 292, 1, \ldots]$
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The underlying rule is that (a) \(-\beta = [0, 1, \bar{1}, 1, 0, \beta]\)
In the special example $G_m = 2^m$ the final division leads to the expansion

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but entries in such an expansion should be polynomials of degree at least one. Analogously, the expansion $-\pi = [3, 7, 15, \overline{1}, 292, \overline{1}, \ldots]$ is filled with inadmissible negative entries. But $-\pi = [4, 1, 6, 15, 1, 292, 1, \ldots]$ and one should be able to obtain that directly from the inadmissible result. Indeed, to do that, one notices above that negating a negation produces a spontaneous partial quotient $1$; hence the ripple lemma.

The underlying rule is that (a) $-\beta = [0, 1, \overline{1}, 1, 0, \beta]$ or, multiplying by $-1$, alternatively (a′) $-\beta = [0, \overline{1}, 1, \overline{1}, 0, \beta]$. We also need the fairly evident zero removal rule (b) $[\ldots, a, 0, b, \ldots] = [\ldots, a + b, \ldots]$. 
In the special example $G_m = 2^m$ the final division leads to the expansion
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The underlying rule is that (a) $-\beta = [0, 1, 1, 1, 0, \beta]$ or, multiplying by $-1$, alternatively (a') $-\beta = [0, 1, 1, 1, 0, \beta]$. We also need the fairly evident zero removal rule (b) $[\ldots, a, 0, b, \ldots] = [\ldots, a + b, \ldots]$. A check:

$-\pi = [3, 0, 1, 1, 0, 7, 15, 1, 292, \ldots] = [4, 1, 6, 15, 1, 292, \ldots]$. 
Removing Inadmissible Entries
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We saw that the final division in the example $G_h = 2^h$, $m = 0$, leads to

$$[0, x, \overline{1}, \overline{x}, \overline{1}, x, 1, \overline{x}, \overline{1}, x, \overline{1}, \overline{x}, 1, x, 1, \ldots].$$
Removing Inadmissible Entries

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Applying our removal of ±1s by negation strategy, that’s

$$[0, x, \overline{I}, 0, 1, \overline{I}, 1, 0, x, 1, \overline{x}, \overline{I}, x, 1, \overline{x}, 1, x, \overline{I}, \overline{x}, \overline{I}, \ldots].$$
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$$= [0, x - 1, x + 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, \bar{1}, x, \bar{1}, \bar{x}, 1, x, 1, \ldots].$$
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$$= [0, x - 1, x + 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, \overline{1}, x, \overline{1}, x, \overline{1}, x, \overline{1}, x, \overline{1}, \ldots]$$

$$= [0, x - 1, x + 2, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, \overline{1}, \ldots].$$
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We saw that the final division in the example $G_h = 2^h, m = 0$, leads to

$$[0, x, \overline{1}, \overline{x}, \overline{1}, x, 1, \overline{x}, \overline{1}, x, \overline{1}, \overline{x}, 1, x, 1, \ldots].$$

Applying our removal of $\pm 1$s by negation strategy, that’s

$$[0, x, \overline{1}, 0, 1, \overline{1}, 1, 0, x, 1, \overline{x}, \overline{1}, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, \overline{1}, \ldots]$$

$$= [0, x - 1, x + 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, \overline{1}, x, \overline{1}, \overline{x}, 1, x, 1, \ldots]$$

$$= [0, x - 1, x + 2, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, \overline{1}, \ldots]$$

$$= [0, x - 1, x + 2, x, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, \overline{1}, \overline{x}, 1, x, 1, \ldots]$$

$$\vdots$$

$$= [0, x + 2, x, x, x - 2, x, x + 2, x, \ldots].$$
Removing Inadmissible Entries

We saw that the final division in the example $G_h = 2^h$, $m = 0$, leads to

$$[0, x, \bar{1}, \bar{x}, \bar{1}, x, 1, \bar{x}, \bar{1}, x, 1, x, 1, \ldots].$$

Applying our removal of $\pm 1$s by negation strategy, that’s

$$[0, x, \bar{1}, 0, 1, \bar{1}, 1, 0, x, 1, \bar{x}, \bar{1}, x, 1, \bar{x}, 1, x, \bar{1}, \bar{x}, \bar{1}, \ldots]$$
$$= [0, x - 1, x + 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, \bar{1}, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$
$$= [0, x - 1, x + 2, x - 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, 1, x, \bar{1}, \bar{x}, \bar{1}, \ldots]$$
$$= [0, x - 1, x + 2, x, x - 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$
$$\vdots$$
$$= [0, x + 2, x, x, x - 2, x, x + 2, x, \ldots] =: [0, S_{x, x-2}(x + 2, x)].$$
Removing Inadmissible Entries

We saw that the final division in the example $G_h = 2^h$, $m = 0$, leads to

$$[0, x, \bar{1}, \bar{x}, \bar{1}, x, 1, \bar{x}, \bar{1}, x, \bar{1}, x, 1, x, 1, \ldots].$$

Applying our removal of $\pm 1$s by negation strategy, that’s

$$[0, x, \bar{1}, 0, 1, \bar{1}, 1, 0, x, 1, \bar{x}, \bar{1}, x, 1, \bar{x}, 1, x, \bar{1}, \bar{x}, \bar{1}, \ldots]$$

$$= [0, x - 1, x + 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, \bar{1}, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$

$$= [0, x - 1, x + 2, x - 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, 1, x, \bar{1}, \bar{x}, \bar{1}, \ldots]$$

$$= [0, x - 1, x + 2, x, x - 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$

$$= [0, x + 2, x, x - 2, x, x + 2, x, \ldots] =: [0, S_x^\infty, x - 2(x + 2, x)].$$

Here $S_p(w) = \overrightarrow{w} \overleftarrow{p} \overrightarrow{w}$ is the perturbed symmetry operator on words $w = \overrightarrow{w}$. 
Removing Inadmissible Entries

We saw that the final division in the example $G_h = 2^h$, $m = 0$, leads to

$$[0, x, \overline{1}, \overline{x}, \overline{1}, x, 1, \overline{x}, \overline{1}, x, \overline{1}, x, 1, \ldots].$$

Applying our removal of $\pm 1$s by negation strategy, that’s

$$[0, x, \overline{1}, 0, 1, \overline{1}, 1, 0, x, 1, \overline{x}, \overline{1}, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, \overline{1}, \ldots]$$

$$= [0, x - 1, x + 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, \overline{1}, x, \overline{1}, \overline{x}, 1, x, 1, \ldots]$$

$$= [0, x - 1, x + 2, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, \overline{1}, \ldots]$$

$$= [0, x - 1, x + 2, x, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, \overline{1}, \overline{x}, 1, x, 1, \ldots]$$

$$\vdots$$

$$= [0, x + 2, x, x, x - 2, x, x + 2, x, \ldots] =: [0, S^\infty_{x, x-2}(x + 2, x)].$$

Here $S_p(w) = \overrightarrow{w} \overrightarrow{p} \overleftarrow{w}$ is the perturbed symmetry operator on words $w = \overrightarrow{w}$. All this is trivial algorithmically.
Removing Inadmissible Entries

We saw that the final division in the example $G_h = 2^h$, $m = 0$, leads to

$$[0, x, \bar{1}, \bar{x}, \bar{1}, x, 1, \bar{x}, \bar{1}, x, \bar{1}, \bar{x}, 1, x, 1, \ldots].$$

Applying our removal of $\pm 1$s by negation strategy, that’s

$$[0, x, \bar{1}, 0, 1, \bar{1}, 1, 0, x, 1, \bar{x}, \bar{1}, x, 1, \bar{x}, 1, x, \bar{1}, \bar{x}, \bar{1}, \ldots]$$

$$= [0, x - 1, x + 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, \bar{1}, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$

$$= [0, x - 1, x + 2, x - 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, 1, \bar{x}, 1, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$

$$= [0, x - 1, x + 2, x, x - 1, 1, 0, \bar{1}, 1, \bar{1}, 0, x, \bar{1}, \bar{x}, 1, x, 1, \ldots]$$

$$\vdots$$

$$= [0, \overrightarrow{x+2}, x, \overrightarrow{x-2}, \overleftarrow{x+2}, x, \ldots] =: [0, S_{x,x-2}^{\infty}(x+2, x)].$$

Here $S_p(w) = \overrightarrow{w} \overrightarrow{p} \overleftarrow{w}$ is the perturbed symmetry operator on words $w = \overrightarrow{w}$.

All this is trivial algorithmically, but is painful to do by hand.
Removing Inadmissible Entries

We saw that the final division in the example $G_h = 2^h$, $m = 0$, leads to

$$\left[ 0, x, \overline{1}, \overline{x}, \overline{1}, x, 1, \overline{x}, \overline{1}, x, \overline{1}, x, 1, \ldots \right].$$

Applying our removal of $\pm 1$s by negation strategy, that’s

$$\left[ 0, x, \overline{1}, 0, 1, \overline{1}, 1, 0, x, 1, \overline{x}, \overline{1}, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, \overline{1}, \ldots \right]$$

$$= \left[ 0, x - 1, x + 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, \overline{1}, x, \overline{1}, \overline{x}, 1, x, 1, \ldots \right]$$

$$= \left[ 0, x - 1, x + 2, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, 1, \overline{x}, 1, x, \overline{1}, \overline{x}, 1, x, 1, \ldots \right]$$

$$= \left[ 0, x - 1, x + 2, x, x - 1, 1, 0, \overline{1}, 1, \overline{1}, 0, x, \overline{1}, \overline{x}, 1, x, 1, \ldots \right]$$

$$\vdots$$

$$= \left[ 0, x + 2, x, x, x - 2, x, x + 2, x, \ldots \right] =: \left[ 0, S^\infty_{x, x-2}(x + 2, x) \right].$$

Here $S_p(w) = \overrightarrow{w} \overrightarrow{p} \overleftarrow{w}$ is the perturbed symmetry operator on words $w = \overrightarrow{w}$. All this is trivial algorithmically, but is painful to do by hand, or to type.
Case $B_k$: $G_{m+k+1} - 2G_{m+k} + G_m = 0$

For temporary convenience I write $G^{(n)}$ in place of $G_{m+n} - 2G_{m+n-1} + G_m$ to emphasise that a partial quotient $x^{G^{(n)}}$ is $n$ loops old.
Case $B_k$: \[ G_{m+k+1} - 2G_{m+k} + G_m = 0 \]

For temporary convenience I write $G_n$ in place of $G_{m+n} - 2G_{m+n-1} + G_m$ to emphasise that a partial quotient $x^{G_n}$ is $n$ loops old. For example, after three loops we have

\[ S(3) = x^G (x^{-G} + x^{-G+1} + x^{-G+2} + x^{-G+3}) = [1, x^{G(1)}, \bar{x}^{G(2)}, \bar{x}^{G(1)}, \bar{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)}]; \]
Case B<sub>k</sub>: \( G_{m+k+1} - 2G_{m+k} + G_m = 0 \)

For temporary convenience I write \( G_{(n)} \) in place of \( G_{m+n} - 2G_{m+n-1} + G_m \) to emphasise that a partial quotient \( x^{G_{(n)}} \) is \( n \) loops old. For example, after three loops we have

\[
S(3) = xG_m(x^{-G_m} + x^{-G_{m+1}} + x^{-G_{m+2}} + x^{-G_{m+3}})
\]

\[
= [1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}];
\]

and if I begin the next loop with a partial ripple, also

\[
S(3) = [0, 1, \overline{x}^{G(1)}, \overline{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{1}, \overline{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}].
\]
Case $B_k$: $G_{m+k+1} - 2G_{m+k} + G_m = 0$

For temporary convenience I write $G(n)$ in place of $G_{m+n} - 2G_{m+n-1} + G_m$ to emphasise that a partial quotient $x^{G(n)}$ is $n$ loops old. For example, after three loops we have

$$S(3) = x^{G_m}(x^{-G_m} + x^{-G_{m+1}} + x^{-G_{m+2}} + x^{-G_{m+3}})$$

$$= [1, x^{G(1)}, \bar{x}^{G(2)}, \bar{x}^{G(1)}, \bar{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)}];$$

and if I begin the next loop with a partial ripple, also

$$S(3) = [0, 1, \bar{x}^{G(1)}, \bar{1}, \bar{x}^{G(2)}, 1, x^{G(1)}, \bar{1}, \bar{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)}].$$

Indeed, in the special case $G_h = F_h$, the Fibonacci numbers:

$F_{h+2} = F_{h+1} + F_h$, thus Case $B_2$, we have $G(3) = 0$ and we may forthwith complete the loop to obtain

$$S(4) = [1, x^{G(1)}, \bar{x}^{G(2)}, \bar{x}^{G(1)}, \bar{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)}, 0, x^{F_{m+1}}, x^{G(3)}, \bar{x}^{F_{m+1}}].$$
Finally, swallowing the zero yields

\[ S(4) = [1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, \]
\[ x^{G(1)}, x^{G(2)}, x^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} ] . \]
Finally, swallowing the zero yields

\[ S(4) = [ 1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} ]. \]

The next loop requires that we first insert two partial ripples, giving

\[ S(4) = [ 0, 1, \overline{x}^{G(1)}, \overline{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{1}, \overline{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}}, \overline{1} ]. \]
Finally, swallowing the zero yields

\[ S(4) = [ 1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, \]
\[ x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} ]. \]

The next loop requires that we first insert two partial ripples, giving

\[ S(4) = [ 0, 1, \overline{x}^{G(1)}, \overline{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{1}, \overline{x}^{G(3)} + 1, \]
\[ x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}}, \overline{1} ], \]

and then complete the loop to obtain

\[ S(5) = [ 1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}, 0, \]
\[ x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} + x^{F_{m+2} + F_{m-1}}, 0, x^{G(1)}, x^{2F_{m}}, \overline{x}^{G(1)} ]. \]
Finally, swallowing the zero yields

\[ S(4) = [1, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}}]. \]

The next loop requires that we first insert two partial ripples, giving

\[ S(4) = [0, 1, \overline{x}^{G(1)}, \overline{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{1}, \overline{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}}, \overline{1}]. \]

and then complete the loop to obtain

\[ S(5) = [1, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}, 0, x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} + x^{F_{m+2} + F_{m-1}}, 0, x^{G(1)}, x^{2F_{m}}, \overline{x}^{G(1)}]. \]

It’s not yet totally obvious that the expansion has stabilised and is growing steadily in length.
Finally, swallowing the zero yields

\[ S(4) = [1, x^{G(1)}, \overline{x}^{G(2)}, x^{G(1)}, \overline{x}^{G(3)},
\]

\[ x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} ] . \]

The next loop requires that we first insert two partial ripples, giving

\[ S(4) = [0, 1, \overline{x}^{G(1)}, \overline{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{1}, \overline{x}^{G(3)} + 1,
\]

\[ x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}}, \overline{1} ] , \]

and then complete the loop to obtain

\[ S(5) = [1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}, 0,
\]

\[ x^{F_{m+1}}, x^{G(3)}, \overline{x}^{F_{m+1}} + x^{F_{m+2}+F_{m-1}}, 0, x^{G(1)}, x^{2F_{m}}, \overline{x}^{G(1)} ] . \]

It’s not yet totally obvious that the expansion has stabilised and is growing steadily in length. However, the parity of the occurrences of \( \pm x^{G(3)} = \pm 1 \) remains friendly to our cause.
Finally, swallowing the zero yields

\[ S(4) = [1, \ x^G(1), \ \overline{x}^G(2), \ \overline{x}^G(1), \ \overline{x}^G(3), \]

\[ x^G(1), \ x^G(2), \ \overline{x}^G(1) + x^{F_{m+1}}, \ x^G(3), \ \overline{x}^{F_{m+1}} ]. \]

The next loop requires that we first insert two partial ripples, giving

\[ S(4) = [0, 1, \ \overline{x}^G(1), \ \overline{1}, \ \overline{x}^G(2), \ 1, \ x^G(1), \ \overline{1}, \ \overline{x}^G(3) + 1, \]

\[ x^G(1), \ x^G(2), \ \overline{x}^G(1) + x^{F_{m+1}}, \ 0, \ x^G(3), \ x^{F_{m+1}}, \ \overline{1} ], \]

and then complete the loop to obtain

\[ S(5) = [1, \ x^G(1), \ \overline{x}^G(2), \ \overline{x}^G(1), \ \overline{x}^G(3), \ x^G(1), \ x^G(2), \ \overline{x}^G(1), \ 0, \]

\[ x^{F_{m+1}}, \ x^G(3), \ \overline{x}^{F_{m+1}} + x^{F_{m+2}+F_{m-1}}, \ 0, \ x^G(1), \ x^{2F_{m}}, \ \overline{x}^G(1) ]. \]

It’s not yet totally obvious that the expansion has stabilised and is growing steadily in length. However, the parity of the occurrences of \( \pm x^G(3) = \pm 1 \) remains friendly to our cause; the same will hold for \( x^{G(k+1)} \) in Case B\( k \).
Back to work, we first have

\[ S(5) = [0, 1, \overline{x}^{G(1)}, \overline{\mathbf{1}}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{\mathbf{1}}, \overline{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}} + \overline{x}^{F_{m+2} + F_{m-1}} + \overline{x}^{G(1)}, \overline{\mathbf{1}}, x^{2F_{m}}, 1, x^{G(1)}, \overline{\mathbf{1}}]. \]
Back to work, we first have

\[ S(5) = [0, 1, \overline{x}^{G(1)}, \overline{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \overline{1}, \overline{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \]
\[ \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}} + \overline{x}^{F_{m+2} + F_{m-1}} + \overline{x}^{G(1)}, \overline{1}, x^{2F_{m}}, 1, x^{G(1)}, \overline{1}]. \]

and, completing the loop,

\[ S(6) = [1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \]
\[ \overline{x}^{F_{m+1}} + x^{F_{m+2} + F_{m-1}} + x^{G(1)}, x^{2F_{m}} + \overline{x}^{F_{m+2} + 2F_{m}} + \overline{x}^{G(2)}, \]
\[ \overline{x}^{G(1)}, x^{F_{m+2}}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}]. \]
Back to work, we first have

\[ S(5) = [0, 1, x_{G(1)}, 1, x_{G(2)}, 1, x_{G(1)}, 1, x_{G(3)} + 1, x_{G(1)}, x_{G(2)}, x_{G(1)} + x_{Fm+1}, 0, x_{G(3)}, x_{Fm+1} + x_{Fm+2 + Fm-1} + x_{G(1)}, 1, x_{2Fm}, 1, x_{G(1)}, 1]. \]

and, completing the loop,

\[ S(6) = [1, x_{G(1)}, x_{G(2)}, x_{G(1)}, x_{G(3)}, x_{G(1)}, x_{G(2)}, x_{G(1)} + x_{Fm+1}, x_{G(3)}, x_{Fm+1} + x_{Fm+2 + Fm-1} + x_{G(1)}, x_{2Fm} + x_{Fm+2 + 2Fm} + x_{G(2)}, x_{G(1)}, x_{Fm+2}, x_{G(1)}, x_{G(2)}, x_{G(1)}]. \]

I remark that in Case B, the exponents do not age smoothly as they do in Case A. First, their expected behaviour is that they decrease (rather than non-strictly increase).
Back to work, we first have

\[ S(5) = [0, 1, \bar{x}^{G(1)}, \bar{1}, \bar{x}^{G(2)}, 1, x^{G(1)}, \bar{1}, \bar{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}} + \bar{x}^{F_{m+2} + F_{m-1}} + \bar{x}^{G(1)}, \bar{1}, x^{2F_{m}}, 1, x^{G(1)}, \bar{1}]\]

and, completing the loop,

\[ S(6) = [1, x^{G(1)}, \bar{x}^{G(2)}, \bar{x}^{G(1)}, \bar{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \bar{x}^{F_{m+1}} + x^{F_{m+2} + F_{m-1}} + x^{G(1)}, x^{2F_{m}} + \bar{x}^{F_{m+2} + 2F_{m}} + \bar{x}^{G(2)}, \bar{x}^{G(1)}, x^{F_{m+2}}, x^{G(1)}, x^{G(2)}, \bar{x}^{G(1)}] \]

I remark that in Case B, the exponents do not age smoothly as they do in Case A. First, their expected behaviour is that they decrease (rather than non-strictly increase). Second, the vanishing of \( G_{(k+1)} \) may change the positional parity of exponents, leading them to grow exponentially, rather than to politely shrink away.
Back to work, we first have

\[ S(5) = [0, 1, \overline{x}^{G(1)}, \bar{1}, \overline{x}^{G(2)}, 1, x^{G(1)}, \bar{1}, \overline{x}^{G(3)} + 1, x^{G(1)}, x^{G(2)}, \]

\[ \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}} + \overline{x}^{F_{m+2}+F_{m-1}} + \overline{x}^{G(1)}, \bar{1}, x^{2F_m}, 1, x^{G(1)}, \bar{1}]. \]

and, completing the loop,

\[ S(6) = [1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)} + x^{F_{m+1}}, x^{G(3)}, \]

\[ \overline{x}^{F_{m+1}} + x^{F_{m+2}+F_{m-1}} + x^{G(1)}, x^{2F_m} + \overline{x}^{F_{m+2}+2F_m} + \overline{x}^{G(2)}, \]

\[ \overline{x}^{G(1)}, x^{F_{m+2}}, x^{G(1)}, x^{G(2)}, \overline{x}^{G(1)}]. \]

I remark that in Case $B_k$ the exponents do not age smoothly as they do in Case A. First, their expected behaviour is that they decrease (rather than non-strictly increase). Second, the vanishing of $G_{(k+1)}$ may change the positional parity of exponents, leading them to grow exponentially, rather than to politely shrink away. Notice that the expansion is struggling to display some folded symmetry, and surely will succeed in doing it after the next loop. Several of the initial partial quotients are now fixed.
Indeed, it now is safe to use ...s and to remark that

\[
S(6) = \left[ \ldots, \bar{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, x^{F_{m+1}} + \bar{x}^{F_{m+2}} + F_{m-1} + \bar{x}^{G(1)} + \bar{1}, x^2F_m + \bar{x}^{F_{m+2}+2F_m} + \bar{x}^{G(2)} + 1, \bar{x}^{G(1)} + \bar{1}, x^{F_{m+2}} + 1, \bar{x}^{G(1)} + \bar{1}, x^{G(2)} + 1, x^{G(1)} + \bar{1} \right]
\]
Indeed, it now is safe to use . . .s and to remark that

\[ S(6) = \ldots, \overline{x}^{G(1)} + x^{F_{m+1}}, 0, x^{G(3)}, \]

\[ x^{F_{m+1}} + \overline{x}^{F_{m+2} + F_{m-1}} + \overline{x}^{G(1)}, 1, x^{2F_{m}} + \overline{x}^{F_{m+2} + 2F_{m}} + \overline{x}^{G(2)}, 1, \]

\[ x^{G(1)}, 1, x^{F_{m+2}}, 1, \overline{x}^{G(1)}, 1, x^{G(2)}, 1, x^{G(1)}, 1 \]

and therefore, filling in the . . .s,

\[ S(7) = [1, x^{G(1)}, \overline{x}^{G(2)}, \overline{x}^{G(1)}, \overline{x}^{G(3)}, x^{G(1)}, x^{G(2)}, \]

\[ \overline{x}^{F_{m+1}} + x^{F_{m+2} + F_{m-1}} + x^{G(1)}, x^{2F_{m}} + \overline{x}^{F_{m+2} + 2F_{m}} + \overline{x}^{G(2)}, \]

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Indeed, it now is safe to use \ldots s and to remark that

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nicely displaying a (highly perturbed) folding.
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\right.
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nicely displaying a (highly perturbed) folding.

I can now announce what is in effect the main lemma of my talk.
Indeed, it now is safe to use ...s and to remark that

\[ S(6) = \left[ \ldots, x_{G(1)}^{F_{m+1}}, 0, x_{G(3)} \right], \]
\[ x_{F_{m+1}} + x_{F_{m+2} + F_{m-1}} + x_{G(1)}^{\overline{m} \overline{0}}, x_{2F_{m}} + x_{F_{m+2} + 2F_{m}} + x_{G(2)}^{\overline{1}}, 1, \]
\[ x_{G(1)}^{\overline{1}}, x_{F_{m+2}}, 1, x_{G(1)}^{\overline{0}}, x_{G(2)}, 1, x_{G(1)}, \overline{1} \]

and therefore, filling in the ...s,

\[ S(7) = \left[ 1, x_{G(1)}, x_{G(2)}, x_{G(1)}^{\overline{0}}, x_{G(3)}, x_{G(1)}, x_{G(2)}, \right], \]
\[ x_{F_{m+1}}^{\overline{0}}, x_{F_{m+2} + F_{m-1}}^{\overline{0}}, x_{G(1)}^{\overline{1}}, x_{2F_{m}}^{\overline{0}}, x_{F_{m+2} + 2F_{m}}^{\overline{0}}, x_{G(2)}^{\overline{1}}, \]
\[ x_{G(1)}^{\overline{1}}, x_{F_{m+2}}^{\overline{1}}, x_{F_{m+4}}^{\overline{0}}, x_{G(3)}^{\overline{0}}, x_{G(1)}^{\overline{1}}, x_{G(2)}^{\overline{1}}, x_{G(1)}^{\overline{0}}, x_{F_{m+2} + F_{m}}^{\overline{0}}, \]
\[ x_{G(1)}^{\overline{1}}, x_{G(2)}^{\overline{1}}, x_{G(1)}^{\overline{0}}, x_{G(1)}^{\overline{0}}, x_{G(1)}^{\overline{0}}, x_{G(2)}^{\overline{1}}, x_{G(1)}^{\overline{0}} \]

nicely displaying a (highly perturbed) folding.

I can now announce what is in effect the main lemma of my talk: surprisingly, the twenty-two partial quotients comprising \( S(7) \) are precisely the partial quotients commencing the expansion of \( S(\infty) \).
It is now time to recall that \( G(1) = F_{m-1} \), \( G(2) = F_{m-2} \), \( G(3) = 0 \), thus obtaining, when \( k = 2 \)

\[
S(\infty) = [1, x^{F_{m-1}}, \overline{x}^{F_{m-2}}, \overline{x}^{F_{m-1}}, \overline{x}^{0}, x^{F_{m-1}}, x^{F_{m-2}}, \\
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\]
\[
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\]
\[
\bar{x}^{F_{m-1}}, x^{F_{m+2}} + \bar{x}^{F_{m+4}} + x^0, x^{F_{m-1}}, x^{F_{m-2}}, \bar{x}^{F_{m-1}}, x^{F_{m+2}+F_m},
\]
\[
x^{F_{m-1}}, \bar{x}^{F_{m-2}}, \bar{x}^{F_{m-1}}, x^0, x^{F_{m-1}}, x^{F_{m-2}}, \bar{x}^{F_{m-1}}, \ldots \ldots \]}

I had retained the general notation in an effort to emphasise that my remarks apply not just to the Fibonacci/Lucas numbers but generally to their higher order generalisations.
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\]

I had retained the general notation in an effort to emphasise that my remarks apply not just to the Fibonacci/Lucas numbers but generally to their higher order generalisations. The point is that particular case \( k = 2 \) amply illustrates the complexity of the more general one and that for each \( k \) there is a minimal \( n = n(k) \) so that the continued fraction expansion of \( S(n) \) is a prefix of that of \( S(\infty) \). We have here noticed \( n(2) = 7 \).
Case C: A bad, bad, thing
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Experimentally, one sees that the almost any mistype or omission in giving data to PARI-GP seems invariably to yield a generic continued fraction expansion: the logarithmic height of the coefficients of the partial quotients grows at quadratic pace.
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Recall that rippling is made possible by the identities

\[
\begin{align*}
[A+1, B, \gamma] &= [A + 1, 0, \bar{\gamma}, 1, \bar{\gamma}, 0, \bar{B}, \bar{\gamma}] = [A, 1, \bar{B} + 1, \bar{\gamma}].
\end{align*}
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\[
[A + c, B, \gamma] = c[A/c + 1, cB, \gamma/c]
= c[A/c, 1, cB + 1, \overline{\gamma}/c] = [A, 1/c, c^2B + c, \overline{\gamma}/c^2].
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If \( c \neq \pm 1 \) this makes a mess which propagates through the continued fraction expansion.
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Therefore, for an arbitrary nonzero constant \(c\), we have

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\]

\[
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\]

If \(c \neq \pm 1\) this makes a mess which propagates through the continued fraction expansion precisely so as to produce the generic quadratic growth in the size of the numerical coefficients of the partial quotients.
If, ever, $G_{m+h+1} - 2G_{m+h} + G_m$ is negative whereas $G_{m+h} - 2G_{m+h-1} + G_m$ is positive then the relevant loop will first require division by some smaller power of $x$ than the $(G_{m+1} - G_m)$th, followed by another ripple and division by the remaining power of $x$. That ‘further ripple’ will surely involve moving some constant $c$ different from $\pm 1$, very likely provoking a propagating ‘mess’ as suggested above.
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I was interested to discover that the $S(h)$ actually are fairly tolerant of error. I experimented on $S(h) = x^5(x^{-5} + x^{-8} + x^{-13} + x^{-21} + \cdots + x^{-F_{h+5}})$ and thought it might be lucky to replace the exponent $F_7 = 13$ by $14$. That made a smaller mess than I had expected (its propagation was rather muted), but dividing that by $x^5$ provided 47 pages of chaos, the first of which is my ‘A Small Mistake’ picture.
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However, while $S(h)$ always does, $x^{-3} + x^{-5} + x^{-8} + x^{-13} + x^{-21} + \cdots$, for example, does not have a specialisable continued fraction expansion. If $G_m \neq 1$, one cannot be sure of being able to safely divide the expansion for $S(h)$ by $X^{G_m}$, further restricting the class of specialisable examples.
Notes and References


We discuss in detail the continued fraction expansion of binary decimals

\[ s_a = 2 \sum_{h=0}^{\infty} (-1)^{a_h} 2^{-2^h}, \quad a = 0.a_1 a_2 a_3 \ldots \]

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Computations by Jeff Shallit provoke the conjecture discussed today; we give a proof of the cases B_2 and B_3 by a rather more intricate method than that suggested here.
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Computations by Jeff Shallit provoke the conjecture discussed today; we give a proof of the cases B_2 and B_3 by a rather more intricate method than that suggested here.


I discuss the subject generally, explain the ripple lemma, and propose the argument detailed today for the cases B_k.