Ramanujan’s Arithmetic-Geometric Mean

Continued Fractions and Dynamics

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“I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate. ⋯

The spoken word and the written word are quite different arts. ⋯

I feel that to collect an audience and then read one’s material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car.”*

(Sir Lawrence Bragg)

- What would he say about reading overheads?

- *These overheads and a companion set by Russell Luke are on my web page.*

*From page 76 of *Science*, July 5, 1996.*
• G. N. Watson (1886–1965), on reading Ramanujan’s work, describes:

>a thrill which is indistinguishable from the thrill I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing ‘Day,’ ‘Night,’ ‘Evening,’ and ‘Dawn’ which Michelangelo has set over the tomb of Guiliano de’Medici and Lorenzo de’Medici.
1. Abstract

The Ramanujan AGM continued fraction

\[ R_\eta(a, b) = \frac{a}{\eta + \frac{b^2}{\eta + \frac{4a^2}{\eta + \frac{9b^2}{\eta + \ldots}}}} \]

enjoys attractive algebraic properties such as a striking arithmetic-geometric mean relation & elegant links with elliptic-function theory.

- The fraction presents a computational challenge, which we could not resist.

Published in Experimental Mathematics [Co-Lab Preprints #27, #29] and in The Ramanujan Journal [#253, #261].
In Part I: we show how to rapidly evaluate $\mathcal{R}$ for any positive reals $a, b, \eta$. The problematic case being $a \approx b$—then subtle transformations allow rapid evaluation.

- On route we find, e.g., that for rational $a = b$, $\mathcal{R}_\eta$ is an L-series with a 'closed-form.'

- We ultimately exhibit an algorithm yielding $D$ digits of $\mathcal{R}$ in $O(D)$ iterations.*

In Part II of this talk, we address the harder theoretical and computational dilemmas arising when (i) parameters are allowed to be complex, or (ii) more general fractions are used.

*The big-$O$ constant is independent of the positive-real triple $a, b, \eta$. 
PART I. Entry 12 of Chapter 18 of Ramanujan’s Second Notebook [BeIII] gives the beautiful:

\[ R_\eta(a, b) = \frac{a}{\eta + \frac{b^2}{\eta + \frac{4a^2}{\eta + \frac{9b^2}{\eta + \ldots}}}} \quad (1.1) \]

which we interpret—in most of the present treatment—for real, positive \( a, b, \eta > 0 \).

Remarkably, for such parameters, \( R \) satisfies an AGM relation

\[ R_\eta \left( \frac{a + b}{2}, \sqrt{ab} \right) = \frac{R_\eta(a, b) + R_\eta(b, a)}{2} \quad (1.2) \]
1. (1.2) is one of many relations we develop for computation of $R_{\eta}$.

2. “The hard cases occur when $b$ is near to $a$,” including the case $a = b$.

3. We eventually exhibit an algorithm uniformly of geometric/linear convergence across the positive quadrant $a, b > 0$.

4. Along the way, we find attractive identities, such as that for $R_{\eta}(r, r)$, with $r$ rational.

5. Finally, we consider complex $a, b$—obtaining theorems and conjectures on the domain of validity for the AGM relation (1.2).
Research started in earnest when we noted $\mathcal{R}_1(1,1)$ ‘seemed close to’ $\log 2$.

Such is the value of experiment: one can be led into deep waters.

Discussed in Ch. 1 of *Experimentation in Mathematics*.

- A useful simplification is

  $$\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta),$$

  as can be seen by ‘cancellation’ of the $\eta$ elements down the fraction.

  Such manipulations are valid because the continued converges.
To prove convergence we put $a/R_1$ in RCF (reduced continued fraction) form:

$$R_1(a, b) = \frac{a}{[A_0; A_1, A_2, A_3, \ldots]}$$  \hspace{1cm} (2.1)

$$:= \frac{a}{A_0 + \frac{1}{A_1 + \frac{1}{A_2 + \frac{1}{A_3 + \ldots}}}}$$

where the $A_i$ are all positive real.

*It is here [Ramanujan’s work on elliptic and modular functions] that both the profundity and limitations of Ramanujan’s knowledge stand out most sharply.*

(G.H. Hardy)
Inspection of $R$ yields the RCF elements explicitly and gives the asymptotics of $A_n$: For even $n$

$$A_n = \frac{n!^2}{(n/2)!^4} 4^{-n} \frac{b^n}{a^n} \sim \frac{2}{\pi n} \frac{b^n}{a^n}.$$  

For odd $n$

$$A_n = \frac{((n-1)/2!)^4}{n!^2} 4^{n-1} \frac{a^{n-1}}{b^{n+1}} \sim \frac{\pi}{2abn} \frac{a^n}{b^n}.$$  

• This representation leads immediately to:

**Theorem 2.1:** For any positive real pair $a, b$ the fraction $R_1(a, b)$ converges.

**Proof:** An RCF converges iff $\sum A_i$ diverges. (This is the Seidel–Stern theorem [Kh,LW].)

In our case, such divergence is evident for every choice of real $a, b > 0$.

We later show a different fraction for $R(a)$, and other computationally efficient constructs.
• Note for $a = b$, divergence of $\sum A_i$ is only \textit{logarithmic} — a true indication of slow convergence (we wax more quantitatively later).

• \textbf{Our interest started} with asking how, for $a > 0$, to (rapidly) evaluate

$$R(a) := R_1(a, a)$$

and thence to prove suspected identities.

“\textit{But this is the simplified version for the general public.}”
3. Hyperbolic-elliptic Forms

Links between standard Jacobi theta functions

\[ \theta_2(q) := \sum q^{(n+1/2)^2}, \quad \theta_3(q) := \sum q^{n^2} \]

and elliptic integrals let us establish various results. We start with:

**Theorem 3.1:** For real \( y, \eta > 0 \) and \( q := e^{-\pi y} \)

\[ \eta \sum_{k \in D} \frac{\text{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_2^2(q), \theta_3^2(q)), \]

\[ \eta \sum_{k \in E} \frac{\text{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_3^2(q), \theta_2^2(q)), \]

where \( D, E \) denote respectively the odd, even integers.

Consequently, the Ramanujan AGM identity (1.2) holds for positive triples \( \eta, a, b \).
**Proof:** The sech relations are proved—in equivalent form—in Berndt’s treatment (Vol II, Ch. 18) of *Ramanujan’s Notebooks* [BeIII].

For the AGM, assume $0 < b < a$. The assignments

$$\theta_2^2(q)/\theta_3^2(q) := b/a$$

$$\eta := \theta_2^2(q)/b$$

are possible (since $b/a \in [0, 1)$, see [BB]) and implicitly define $q, \eta$, and together with

$$\theta_2^2(q) + \theta_3^2(q) = \theta_3^2(\sqrt{q}),$$

$$2\theta_2(q)\theta_3(q) = \theta_2^2(\sqrt{q})$$

and repeated use of the sech sums above yield

$$R_1 \left( \theta_3^2(q)/\eta, \theta_2^2(q)/\eta \right) + R_1 \left( \theta_2^2(q)/\eta, \theta_3^2(q)/\eta \right)$$

$$= 2R_1 \left( \theta_3^2(\sqrt{q})/(2\eta), \theta_2^2(\sqrt{q})/(2\eta) \right).$$
Since
\[ \theta_2^2(q) = \eta b, \quad \theta_3^2(q) = \eta a \]
the AGM identity (1.2) holds for all pairs with \( a > b > 0 \).

The case \( 0 < a < b \) is handled by symmetry, or on starting by setting
\[ \theta_2^2(q)/\theta_3^2(q) := a/b. \]

- The wonderful sech identities above stem from classical work of Rogers, Stieltjes, Preece, and of course Ramanujan [BeIII] in which one finds the earlier work detailed.

- In the literature, the proof given for the AGM identity has been claimed for various complex \( a, b \) sometimes over ambitiously.*

*Mea culpa indirectly.
• These prior sech series can be used in turn to establish two numerical series involving the complete elliptic integral

\[ K(k) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\theta)}} \, d\theta. \]

Below we write \( K := K(k) \), \( K' := K(k') \) with \( k' := \sqrt{1 - k^2} \).

**Theorem 3.2:** For \( 0 < b < a \) and \( k := b/a \) we have

\[ R_1(a, b) = \frac{\pi a K}{2} \sum_{n \in \mathbb{Z}} \frac{\text{sech} \left( n\pi \frac{K'}{K} \right)}{K^2 + \pi^2 a^2 n^2}. \] (3.1)

Correspondingly, for \( 0 < a < b \) and \( k := a/b \) we have

\[ R_1(a, b) = 2\pi b K \sum_{n \in D} \frac{\text{sech} \left( n\pi \frac{K'}{2K} \right)}{4K^2 + \pi^2 b^2 n^2}. \] (3.2)

*\( K(k) \) is fast computable via the AGM iteration.
**Proof:** The series follow from the assignments \( \theta_2^2(q)/\eta := \max(a,b) \), \( \theta_3^2(q)/\eta := \min(a,b) \) and Jacobi’s *nome* relations

\[
e^{-\pi K'/K} = q, \quad K(k) = \frac{\pi}{2} \theta_3^2(q)
\]

inserted appropriately into Theorem 3.1. 

- The sech-elliptic series (3.1-2) allow fast computation of \( \mathcal{R}_1 \) for \( b \) not too near \( a \).

- To get \( D \) digits for \( \mathcal{R}_1(a,b) \) one requires \( O(D K/K') \) summands.

- So, another motive for the following analysis was slow convergence of the sech-elliptic forms for \( b \approx a \).
4. Relations for $\mathcal{R}(a)$

- Recalling that $\mathcal{R}(a) := \mathcal{R}_1(a, a)$, we next derive relations for the hard case $b = a$.

Interpreting (3.1) as a Riemann-integral in the limit as $b \to a^-$ (for $a > 0$), gives a slew of relations involving the digamma function [St,AS]

$$\psi := \frac{\Gamma'}{\Gamma}$$

and the Gaussian hypergeometric function

$$F = \, _2F_1(a, b; c; \cdot).$$

- The following identities are presented in an order that can be serially derived:
For all $a > 0$:

\[
\mathcal{R}(a) = \int_0^\infty \frac{\text{sech} \left( \frac{\pi x}{2a} \right)}{1 + x^2} \, dx
\]

\[
= 2a \sum_{k=1}^\infty \frac{(-1)^{k+1}}{1 + (2k - 1) a}
\]

\[
= \frac{1}{2} \left( \psi \left( \frac{3}{4} + \frac{1}{4a} \right) - \psi \left( \frac{1}{4} + \frac{1}{4a} \right) \right)
\]

\[
= \frac{2a}{1 + a} F \left( \frac{1}{2a} + \frac{1}{2}, 1; \frac{1}{2a} + \frac{3}{2}; -1 \right)
\]

\[
= 2 \int_0^1 \frac{t^{1/a}}{1 + t^2} \, dt
\]

and

\[
\mathcal{R}(a) = \int_0^\infty e^{-x/a} \text{sech}(x) \, dx.
\]
The big step. We noted that the sech-elliptic series (3.1) (also (3.2)) will converge slowly when $b \approx a$, yet in Sections 4, 5 we successfully addressed the case $b = a$.

We now establish a series representation when $b < a$ but $b$ is very near to $a$.

We employ the wonderful fact that sech is its own Fourier transform, in that

$$\int_{-\infty}^{\infty} e^{i \gamma x} \text{sech}(\lambda x) \, dx = \frac{\pi}{\lambda} \text{sech} \left( \frac{\pi \gamma}{2\lambda} \right).$$

Using this relation, one can perform a Poisson transform of the sech-elliptic series (3.1).

- The success of the transform depends on

$$I(\lambda, \gamma) := \int_{-\infty}^{\infty} \frac{\text{sech} \lambda x}{1 + x^2} e^{i \gamma x} \, dx.$$
One may obtain the differential equation:

$$-\frac{\partial^2 I}{\partial \gamma^2} + I = \frac{\pi}{\lambda} \text{sech} \left( \frac{\pi \gamma}{2\lambda} \right)$$

and solve it—after some *machinations*—to yield

$$I(\lambda, \gamma) = \frac{\pi}{\cos \lambda} e^{-\gamma}$$

$$+ \frac{2\pi}{\lambda} \sum_{d \in D^+} \frac{(-1)^{d-1}/2 e^{-\pi d \gamma/(2\lambda)}}{1 - \pi^2 d^2/(4\lambda^2)}.$$

where $D^+$ denotes the positive odd integers.

- When $\lambda = \pi D/2$ for some odd $D$, the $1/\cos \lambda$ pole conveniently cancels a corresponding pole in the summation, and the result can be inferred either by avoiding $d = D$ in the sum and inserting a precise residual term

$$\Delta I = \pi (-1)^{(D-1)/2} e^{-\gamma(\gamma + 1/2)/\lambda},$$

or more simply by taking a numerical limit as $\lambda \to \pi D/2$. 

37
When $\gamma \to 0$ we can recover from the sum, via analytic relations for $\psi(z)$, the $\psi$-function form of the integral of $(\text{sech}\lambda x)/(1 + x^2)$.

Via **Poisson transformation** of (3.1) we thus obtain, for $0 < b < a$,

$$
\mathcal{R}_1(a, b) = \mathcal{R}\left(\frac{\pi a}{2K'}\right) + \frac{\pi}{\cos\frac{K'}{a} e^{2K/a} - 1} + 8\pi a K' \sum_{d \in D^+} \frac{(-1)^{(d-1)/2}}{4K'^2 - \pi^2 d^2 a^2} \frac{1}{e^{\pi d K/K'} - 1}
$$

(6.1)

where $k := b/a$, $K := K(k)$, $K' := K(k')$, and $D^+$ again denotes the positive odd integers.

- A similar Poisson transform obtains from (3.2) in the case $b > a > 0$.

- Such transforms appear recondite, but we have what we desired: *convergence is rapid for $b \approx a$: because $K/K' \sim \infty$.*
8. A Uniformly Convergent Algorithm

• We may now give a complete algorithm to evaluate the original Ramanujan fraction $R_\eta(a, b)$ for positive real parameters.

• Convergence is uniform—for any positive real triple $\eta, a, b$ we obtain $D$ good digits in no more than $cD$ computational iterations, where $c$ is independent of the size of $\eta, a, b$.*

0. Observe that $R_\eta(a, b) = R_1(a/\eta, b/\eta)$ so that with impunity we may assume $\eta = 1$ and subsequently evaluate only $R_1$.

*By iterations here we mean either continued-fraction recurrence steps, or series-summand additions.
Algorithm for $\mathcal{R}_\eta(a, b)$ with real $\eta, a, b > 0$:

1. If $(a/b > 2$ or $b/a > 2)$ return the original fraction (1.1), or equivalently (2.1);

2. If $(a = b)$ {
   if $(a = p/q$ rational) return finite form (5.1);
   else return the Gauss RCF (4.2) or rational-zeta form (4.1) or (4.3) or some other scheme such as rapid $\psi$ computations; }

3. If $(b < a)$ {
   if $(b$ is not too close to $a)^*$, return sech-elliptic result (3.1); else return Poisson-transform result (6.1); }

4. (We have $b > a$) Return, as in (1.2),
   \[ 2\mathcal{R}_1 \left( \frac{(a + b)/2}{\sqrt{ab}} \right) - \mathcal{R}_1(b, a). \]

*Say, $|1 - b/a| > \varepsilon > 0$ for any fixed $\varepsilon > 0$. 

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• It is an implicit tribute to Ramanujan's ingenuity that the final step (4) of the algorithm allows entire procedure to go through for all positive real parameters.

• One may avoid step (4) by invoking a Poisson transformation of (3.2), but Ramanujan's AGM identity is finer!
9. About Complex Parameters

PART II. Complex parameters $a$, $b$, $\eta$ are complex, as we found via extensive experimentation.

- We attack this stultifying scenario by assuming $\eta = 1$ and defining
  \[ \mathcal{D} = \{(a, b) \in \mathbb{C}^2 : R_1(a, b) \text{ converges}\}, \]
  meaning the convergents of the original fraction (1.1) have a well-defined limit.

- There are literature claims [BeIII] that
  \[ \{(a, b) \in \mathbb{C}^2 : \text{Re}(a), \text{Re}(b) > 0\} \subseteq \mathcal{D}, \]
  i.e., that convergence occurs whenever both parameters have positive real part.

- **This is false**—as we shall show.
• The exact identification of $D$ is very delicate.

• $R_1(a, b)$ typically diverges for $|a| = |b|$: we observed numerically* that

$$R_1(1/2 + \sqrt{-3}/2, 1/2 - \sqrt{-3}/2)$$

and $R_1(1, i)$ have ‘period two’—as is generic—while $R_1(t \imath, t \imath)$ is ‘chaotic’ for $t > 0$.

*After a caution on checking only even terms!
• An observation that led to the results below is that we have implicitly used, for positive reals $a \neq b$ and perforce for the Jacobian parameter

$$q := \frac{\min(a, b)}{\max(a, b)} \in [0, 1),$$

the fact that

$$0 \leq \frac{\theta_2(q)}{\theta_3(q)} < 1.$$

• If, however, one plots complex $q$ with this ratio of absolute value less than one, a complicated fractal structure emerges, as shown in the Figures below—this leads to the theory of modular forms [BB].

• Thence the sech relations of Theorem 2.1 are suspect for complex $q$. 
• Numerically, the identities appear to fail when $|\theta_2(q)/\theta_3(q)|$ exceeds unity as graphed in white for $|q| < 1$:
• Such *fractal behaviour* is ubiquitous.

• Where $|\theta_4(q)/\theta_3(q)| > 1$ in first quadrant.*

*Colours show gradations between zero and one.*
Conjecture 9.0 (Analytic continuation): Consider complex pairs \((a, b)\). Then

i. If \(|a| > |b|\) the original fraction \(R_1(a, b)\) exists and agrees with the sech series (3.1).

ii. If \(|a| < |b|\) the original fraction \(R_1(a, b)\) exists and agrees with the sech series (3.2).

**Theorem 9.1:** \(R(a) \coloneqq R_1(a, a)\) converges iff \(a \notin I\). That is, the fraction diverges if and only if \(a\) is pure imaginary. Moreover, for \(a \in \mathbb{C}\setminus I\) the fraction converges to a holomorphic function of \(a\) in the appropriate open half-plane.

**Theorem 9.2:** \(R_1(a, b)\) converges for all real pairs; that is whenever \(\text{Im}(a) = \text{Im}(b) = 0\).
Theorem 9.3: (i) The even/odd parts of $R_1(1, i)$ (e.g.) converge to distinct limits. (ii) There are $\text{Re}(a), \text{Re}(b) > 0$ such that $R_1(a, b)$ diverges.

★ Define

- $\mathcal{H} := \{z \in \mathbb{C} : \left| \frac{2\sqrt{z}}{1+z} \right| < 1\}$,

- $\mathcal{K} := \{z \in \mathbb{C} : \left| \frac{2z}{1+z^2} \right| < 1\}$.

Theorem 9.4: If $a/b \in \mathcal{K}$ then both $R_1(a, b)$ and $R_1(b, a)$ converge.

Theorem 9.5: $\mathcal{H} \subset \mathcal{K}$ (properly).

The results so far combine to give a region of validity for the AGM relation, in the form of:

Theorem 9.6: If $a/b \in \mathcal{H}$ then $R_1(a, b) \& R_1(b, a)$ converge, and the arithmetic mean $(a + b)/2$ dominates the geometric mean $\sqrt{ab}$ in modulus.
• Now to the problematic remainders of the AGM relation (1.2) ⋯.

• We performed “scatter diagram” analysis to find computationally where the AGM relation holds in the parameter space.

★ The results were quite spectacular! And lead to the Theorems above.
Our resulting **tough conjecture** with
\[ C' := \{ z \in C : |z| = 1, z^2 \neq 1 \}, \]
led to:

***Theorem 9.11***: The precise domain of convergence for \( R_1(a, b) \) is
\[ D_0 = \{ (a, b) \in C \times C : (a/b \notin C') \text{ or } (a^2 = b^2, b \notin I) \}. \]
In particular, for \( a/b \in C' \) we have divergence.

Moreover, **provably**, \( R_1(a, b) \) converges to an analytic function of both \( a \) or \( b \) on the domain
\[ D_2 := \{ (a, b) \in C \times C : |a/b| \neq 1 \} \subset D_0. \]

- Note, we are not harming Theorems 9.4–9.6 because neither \( \mathcal{H} \) nor \( \mathcal{K} \) intersects \( C' \).

- The **“bifurcation”** of Theorem 9.11 is very subtle.
**Theorem 9.12:** The restriction $a/b \in \mathcal{H}$ implies the truth of the AGM relation (1.2) with all three fractions converging.

**Proof:** For $a/b \in \mathcal{H}$, the ratio $(a + b)/(2\sqrt{ab}) \notin \mathcal{C}'$ and via 9.11 we have sufficient analyticity to apply Berndt's technique.

- A picturesque take on Theorems 9.4–9.6 and 9.12 follows:

  Equivalently, $a/b$ belongs to the closed exterior of $\partial \mathcal{H}$, which in polar-coordinates is given by the **cardioid-knot**

  $$r^2 + (2\cos \phi - 4)r + 1 = 0$$

  drawn in the complex plane ($r = |a/b|$).  

63
• $a/b \in \mathcal{H}$: the arithmetic mean dominating the geometric mean in modulus.

A cardioid-knot, on the (yellow) exterior of which we can prove the truth of the Ramanujan AGM relation (1.2), (9.1).


A key component of our proofs, actually valid in any $B^*$ algebra, is:

**Theorem 9.13.** Let $(a_n)$, $(b_n)$ be sequences of $k \times k$ complex matrices.

Suppose that $\prod_{j=1}^{n} a_j$ converges as $n \to \infty$ to an invertible limit while $\sum_{j=1}^{\infty} \|b_j\| < \infty$. Then

$$\prod_{j=1}^{n} (a_j + b_j)$$

also converges to a finite complex matrix.

- Theorem 9.13 appears new even in $C^1$!

- It allows one to linearize nonlinear recursions—ignoring $O\left(\frac{1}{n^2}\right)$ terms for convergence purposes.

- This is how the issue of the dynamics of $(t_n)$ arose.
10. Visual Dynamics from a ‘Black Box’

• Six months later we had a beautiful proof using genuinely new dynamical results. Starting from the dynamical system $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left( 1 - \frac{1}{n} \right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for $n$ even, odd respectively—or is much more general.*

• Numerically all one learns is that is tending to zero slowly. Pictorially we see significantly more:

$\sqrt{n} t_n$ is bounded iff $\mathcal{R}_1(a, b)$ diverges.

* $\sqrt{n} t_n$ is bounded iff $\mathcal{R}_1(a, b)$ diverges.
• Scaling by $\sqrt{n}$, and coloring odd and even iterates, fine structure appears.

The attractors for various $|a| = |b| = 1$.

★ This is now fully explained, especially the original rate of convergence, which follows by a fine singular-value argument.
11. The 'Chaotic' Case

Jacobsen-Masson theory used in Theorem 9.1 shows, unlike $\mathcal{R}_1(1, i)$, even/odd fractions for $\mathcal{R}_1(i, i)$ behave “chaotically,” neither converge.

When $a = b = i$, $(t_n)$ exhibit a fourfold quasi-oscillation, as $n$ runs through values mod 4. Plotted versus $n$, the (real) sequence $t_n(1, 1)$ exhibits the “serpentine oscillation” of four separate “necklaces.”

For $a = i$, the detailed asymptotic is

$$t_n(1, 1) = \sqrt{\frac{2}{\pi}} \cosh \frac{\pi}{2} \frac{1}{\sqrt{n}} \left(1 + O \left(\frac{1}{n}\right)\right) \times$$

$$\begin{cases} (-1)^{n/2} \cos(\theta - \log(2n)/2) & n \text{ is even} \\ (-1)^{(n+1)/2} \sin(\theta - \log(2n)/2) & n \text{ odd} \end{cases}$$

where $\theta := \arg \Gamma((1 + i)/2)$. 

71
The subtle four fold serpent.

This behavior seems very difficult to infer directly from the recurrence.
Analysis is based on a striking hypergeometric parametrization which was both experimentally discovered and computer proved!

It is

\[ t_n(1, 1) = \frac{1}{2} F_n(a) + \frac{1}{2} F_n(-a), \]

where

\[ F_n(a) := -\frac{a^n 2^{1-\omega}}{\omega \beta(n + \omega, -\omega)} 2F_1 \left( \omega, \omega; n + 1 + \omega; \frac{1}{2} \right), \]

while

\[ \beta(n + 1 + \omega, -\omega) = \frac{\Gamma(n + 1)}{\Gamma(n + 1 + \omega) \Gamma(-\omega)}, \]

and

\[ \omega = \frac{1 - 1/a}{2}. \]
12. More General Fractions

Study of \( R \) devolved to hard but compelling conjectures on complex dynamics, with many interesting proven and unproven generalizations (e.g., Borwein-Luke, 2004).

For any sequence \( a \equiv (a_n)_{n=1}^\infty \), we consider continued fractions like

\[
S_1(a) = \frac{1^2 a_1^2}{1 + \frac{2^2 a_2^2}{1 + \frac{3^2 a_3^2}{1 + \cdots}}}
\]

- We studied convergence properties for deterministic and random sequences \((a_n)\).

- For the deterministic case the best results are for periodic sequences, satisfying \( a_j = a_{j+c} \) for all \( j \) and some finite \( c \).
A period three dynamical system
(odd and even iterates)

- The cases (i) \( a_n = \text{Const} \in \mathbb{C} \), (ii) \( a_n = -a_{n+1} \in \mathbb{C} \), (iii) \(|a_{2n}| = 1, a_{2n+1} = i\), and (iv) \( a_{2n} = a_{2m}, a_{2n+1} = a_{2m+1} \) with \(|a_n| = |a_m| \ \forall \ m, n \in \mathbb{N}\), were already covered.
13. Final Open Problems

- Again on the basis of numerical experiments, we acknowledge that some “deeper” AGM identity might hold.

\[ \frac{R_1(a, b) + R_1(b, a)}{2} \neq R_1 \left( \frac{a + b}{2}, \sqrt{ab} \right) \]

but the LHS agrees numerically with some variant, call it \( S_1((a+b)/2, \sqrt{ab}) \), naively chosen as one of (3.1) or (3.2).

- Such coincidences are remarkable, and difficult so far to predict.

- We maintain hope that there should ultimately be a comprehensive theory under which the lovely AGM relation—suitably modified—holds for all complex values.
1. What precisely is the domain of pairs for which $R_1(a, b)$ converges, and some AGM holds?

2. Relatedly, when does the fraction depart from its various analytic representations?

3. What is the precise domain of validity of the sech formulae (3.1) and (3.2)? Is Conjecture 9.0 right?

4. While $R(i) := R_1(i, i)$ does not converge, the $\psi$-function representation of Section 4 has a definite value at $a = i$. Does some limit such as $\lim_{\epsilon \to 0} R_1(i + \epsilon, i)$ exist and coincide?

5. Despite a host of closed forms for $R(a) := R_1(a, a)$, we know no nontrivial closed form for $R_1(a, b)$ with $a \neq b$. 

77
All physicists and a good many quite respectable mathematicians are contemptuous about proof.

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Beauty is the first test. There is no permanent place in the world for ugly mathematics.

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14. Other References


