How the World has Changed since 1977

From the preface of my English edition

Let us illustrate the concept of equilibrium by using our single-product example. The model is that of an economy with two parts: holders of stocks (capital owners) and owners of labor resources (workers). By way of interpretation, we may speak of the classes of capitalists and workers, government and people, etc. Back parts, as well as the government and the population, play the double role of consumers and producers. As producers they act in their capacity as "workers" or "farmers," and at consumers they aspire to maximize their utility function \( u \). The relationship between the government and the people is given by the inequality \( u = (K^2, K^1) \), where \( K^2 = 0 \) is the share of profits that goes to the government at time \( t \) and where \( K^1 \) is the government's share of payments to labor at time \( t \).

Let \( p = (n, s) \) be a vector that denotes prices at time \( t \), where \( n \) is the price of a unit of output and where \( s \) is the price of labor (wage rate). A trajectory \((s_n, p_n)\) is said to be an equilibrium trajectory if there exists a nonnegative sequence of prices \( p = (p_n)\) such that:

ABSTRACT

Jonathan M. Borwein
Director
Newcastle Centre for Computer Assisted Research Mathematics and its Applications (CARMA)

"Experimental mathematics is fairly new. It is a way of doing mathematics that has been made possible by fast, powerful, easy-to-use computers, by networks, and by databases. "The use of computers in mathematics for its own sake is a recent phenomenon — much more recent than the computer itself, in fact. (This surprises some outsiders, who assume, incorrectly, that mathematicians led the computer revolution. To be sure, mathematicians invented computers, but then they left it to others to develop them, with very few mathematicians actually using them until relatively recently.)"

I shall describe—with accessible examples—how experimental mathematics works, what it has to offer, and what challenges it presents.

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

—Jacques Hadamard

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As Keith Devlin and I wrote in our 2008 AK Peters book "The Computer as Crucible"...

"Experimental mathematics is the use of a computer to run computations—sometimes no more than trial-and-error tests—to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions, assertions that may themselves arise by computational means, assertions that may themselves arise by computational means, including search. Like contemporary chemists—and before them the alchemists of old—who mix various substances together in a crucible and heat them to a high temperature to see what would happen, today's experimental mathematician puts a hopefully potent mix of numbers, formulas, and algorithms into a computer in the hope that something of interest emerges.

ABSTRACT

Jonathan Borwein
Keith Devlin

with illustrations by Fo Ai-Kee and Michelle

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The inventor of ‘x’ and ‘y’ he did not:
3
i
\begin{equation}
W(0) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}
\end{equation}
occurs in the study of uniform random walks in the plane.

W_k(1) = 1
W_2(1) = \frac{3}{2}
W_3(1) = \frac{9}{2}
\begin{equation}
W_3(2k) = 3^k F_k \left( \frac{1}{3} \right) \quad \text{and} \quad W_3(2k+1) = 3^k \left( R_{2k+3} + \frac{1}{3} F_{2k+2} \right)
\end{equation}

François Vieta (1540-1603): Philosophy Matters

Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by numbers and irrational [magnitudes by irrational [numbers]]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning’s fault but the reckoner’s.

Rather, says Proclus, ARITHMETIC IS MORE EXACT THAN GEOMETRY. To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial, i.e. square root of the difference 2 – \sqrt{3}. Whoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.

* The inventor of ‘x’ and ‘y’ he did not believe in negative numbers

**What the wheel did for transport, the optical fiber did for telecommunications.”**

- Richard Epworth, 1980s co-worker of Kao at the Standard Telecommunications Laboratories; in Harlow, UK

I. DIGITALLY ASSISTED DISCOVERY

Relies on Information and Communication Technology (ICT) not just IT
1946 ENIAC (‘first’ digital computer (5K/sec))
1966 Fiber optics
– Charles Cao (2009 physics Nobel), Hong Kong Uni VC
– 1969 CCDevice (09 physics Nobel) Boyle (Canada) and Smith and the ‘3Ms’ workspaces (open and source counterparts)
1980 Maple
– I have a Maple 1.2 manual
1984 MATLAB
1988 Mathematica


“*All truths are easy to understand once they are discovered; the point is to discover them.”* – Galileo Galilei

OUTLINE

• with background and history of my own activities

Part I. What is Experimental Mathematics?
– Discovery and Digital Assistance

Part II. ISC and Colour Calculator
– Identify and OEIS in action

Part III. Integer Relations (online)
– What they are and what they do
• Elementary and advanced examples
• Advanced examples

“What is a DISCOVERY?”

discovering a truth has three components. First, there is the independence requirement, which is just that one comes to believe the proposition concerned by one’s own lights, without reading it or being told. Secondly, there is the requirement that one comes to believe it in a reliable way. Finally, there is the requirement that one’s coming to believe it involves no violation of one’s epistemic state.

*In short, discovering a truth is coming to believe it in an independent, reliable, and rational way.*


WHAT is a PROOF?

“*PROOF, n. a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established. A direct proof proceeds linearly from premises to conclusion; an indirect proof (also called reductio ad absurdum) assumes the falsehood of the desired conclusion and shows that to be impossible. See also induction, deduction, valid.*”

Borowski & JG, *Collins Dictionary of Mathematics*

INdUCTION. n. 3. (Logic) a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them. Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white.

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“*No. I have been teaching it all my life, and I do not want to have my ideas upset.*”

- Isaac Todhunter (1830-1884) recording Maxwell’s response when asked whether he would like to see an experimental demonstration of conical refraction.
Decide for yourself

WHAT is DIGITAL ASSISTANCE?

- Use of Modern Mathematical Computer Packages
  - Symbolic, Numeric, Geometric, Visual, Graphical, ...
- Use of More Specialist Packages or General Purpose Languages
  - Fortran, C++, CPLEX, GAP, PARI, MAGMA, ...
- Use of Web Applications
  - Sloane's Encyclopedia, Inverse Symbolic Calculator, Fractal Explorer, Euclid in Java, Weeks' Topological Games, ...
- Use of Web Databases
  - Google, MathSciNet, ArXiv, ISTOR, Wikipedia, MathWorld, Planet Math, DLMF, MacTutor, Amazon, ..., Wolfram Alpha (?)

All entail data-mining ["exploratory experimentation" and "widening technology" as in pharmacology, astrophysics, biotech... (Franklin)]
- Clearly the boundaries are blurred and getting blurrier
- Judgments of a given source's quality vary and are context dependent

"Knowing things is very 20th century. You just need to be able to find things." - Danny Hillis on how Google has already changed how we think in Achenblog, July 1, 2008

EXPORATORY EXPERIMENTATION

2004 Laura Franklin argues that Steinle’s “exploratory experimentation” facilitated by “widening technology”, as in pharmacology, astrophysics, medicine, and biotechnology, is leading to a reassessment of what legitimates experiment; in that a “local model” is not now prerequisite.

2008 Hendrik Sørenson cogently argues that experimental mathematics (as ‘defined’ below) is following similar tracks:

“These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation is also pertinent to mathematics.”

- In consequence, boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting more so.

CHANGING USER EXPERIENCE and EXPECTATIONS

What is attention? (Stroop test, 1935)

1. Say the color represented by the word.
2. Say the color represented by the font color.

High (young) multitaskers perform #2 very easily. They are great at suppressing information.

http://www.snie.umich.edu/eptlab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif

Acknowledgements: Cliff Nass, CHIME lab, Stanford (interference and twitter?)

Other Cognitive Shifts

Science Online August 13, 2009

Strategic Reading, Ontologies, and the Future of Scientific Publishing

Allen M. Hewer* and Carole L. Fullmer

The revolution in scientific publishing that has been predicted since the 1950s is about to take place. Scientists have always read strategically, working with many articles simultaneously to search, filter, scan, link, annotate, and analyze fragments of content. An observed recent increase in strategic reading in the online environment will soon be further intensified by two recent trends: (1) the replacement of visual reasoning-reviews, and cognitive resources and (2) the convergence of many scientific disciplines at interoperable ontologies. As a result of these changes, scientists are reconsidering reading tools that take advantage of ontologies, reading practices will become more rapid and effective, and metrics will be revised to reflect the evolution of scientific publishing.

- Potentially hostile to mathematical research patterns

Experimental Methodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results
Numerical errors in using double precision.

Correct picture after calling Maple.

PART II. CHRONOLOGY of ISC and FRIENDS

1973 Sloane’s Handbook of Integer Sequences
1978 Ferguson finds PSLQ Integer Relation Algorithm
1985 Sloane’s Encyclopaedia of Integer Sequences
  – with Plouffe (5,000 entries)
1990 Handbook of Real Numbers (100,000:16Mb)
1995 The Inverse Symbolic Calculator (ISC)
  – binscripts/JAVA (10Gb: wanted by GNU)
1995 The Colour Calculator
1996 Sloane’s Online Encyclopaedia (OEIS) (150,000)
1999 “Identify” added to Maple
2007 ISC2.0 (Python + Cherry Pie) multi-threaded
  – less lookup, more preprocessing and computing

1988-90 A DICTIONARY of REAL NUMBERS

8 pages of preface and 424 of numbers in [0,1]

Very dense!
1989 It took Dalhousie University’s central DEC an afternoon to compress it to 16Mb

8 digits after the decimal point: $1 + \tanh(\sqrt{5}) = 1.9999984175$
COLOR and INVERSE

CALCULATORS (1995)

Inferring mathematical structure from numerical data.
- Mixes table lookup, internal relation methods and intelligent preprocessing – needs micro-parallelism
- It faces the “curse of exponentiality”
- Implemented as identify in Maple.

Smart Lookup gives:
- Table lookup
- Runs on parallelism

Simple Lookup fails.
- Less lookup & more
- Does more for a naive user
- Implemented as a photographic imprinted assisted data recognizer for the naive user

A HOMEWORK CHALLENGE

What are
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 0.26066410498242954414... \\
\sum_{k=1}^{n} \frac{1}{2n^2} = 0.5822405264650125059... ?
\]

The answers are:
- \(\log(2\pi) - 1 - \gamma\)
- \(\frac{\pi^2}{6} - \frac{1}{2}\log(2)^2\).

Here:
\[\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.5772156649015328... \]
is Euler’s mysterious (irrational?) constant.

How about 0.438017879485942412114... ?

Maple 14 runs on Midas.

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Maple 14 runs on Midas.
"Nature laughs at the difficulty of Integration" - Lagrange

"A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself)." - J. E. Littlewood, 1885-1977

End of Part II

Part III on Integer Relation Methods is at
www.carma.newcastle.edu.au/~jb616/papers.html#TALKS

Let \((x_n)\) be a vector of real numbers. An integer relation algorithm finds integers \((a_n)\) such that

\[a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0\]

or provides an exclusion bound.

- At present, the PSLQ algorithm of mathematician-sculptor Helaman Ferguson is the best known integer relation algorithm.
- High precision arithmetic software is required: at least \(d \times n\) digits, where \(d\) is the size (in digits) of the largest of the integers \(a_k\).

INTEGER RELATION ALGORITHMS:

HOW THEY WORK

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PSLQ: INTEGER RELATION ALGORITHMS:

WHAT THEY ARE

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TOP TEN ALGORITHMS


Also: Monte Carlo, Simplex, Krylov Subspace, QR Decomposition, Quicksort, ..., FFT, Fast Multipole Method.

- integer relation detection (PSLQ, 1997) was the most recent of the top ten

HELAMAN FERGUSON
SCULPTOR and MATHEMATICIAN

This polished solid silicon bronze sculpture is inspired by the work of David Borwein, his sons and colleagues, on the conditional series above for salt, Madelung's constant. This series can be summed to uncountably many constants; one is Madelung's constant for electro-chemical stability of sodium chloride.

This constant is a period of an elliptic curve, a real surface in four dimensions. There are uncountably many ways to imagine that surface in three dimensions; one has negative gaussian curvature and is the tangible form of this sculpture. (As described by the artist.)

MADELUNG's CONSTANT
David Borwein CMS Career Award

\[ \sum_{n,m,p} \frac{(-1)^{n+m+p}}{\sqrt{n^2 + m^2 + p^2}} \]

INTEGER RELATION ALGORITHMS:
WHAT THEY DO: ELEMENTARY EXAMPLES

Compute \( \alpha \) to sufficiently high precision \( (O(\alpha^2)) \) and apply LLL to the vector

\( (1, \alpha, \alpha^2, \ldots, \alpha^{n-1}) \).

- Solution integers \( a_i \) are coefficients of a polynomial likely satisfied by \( \alpha \).

An application was to determine explicitly the 4th and 5th bifurcation points of the logistics curve have degrees 256.
FINALIZING FORMULAE

- If we suspect an identity PSLQ is powerful.
- (Machin’s Formula) We try PSLQ on
  \[\arctan(1), \arctan(\frac{1}{2}), \arctan(\frac{1}{3})\]
  and recover \(1, -\frac{1}{4}, 1\). That is,
  \[\pi = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)\]
  [Used on all serious computations of \(\pi\) from 1706 (100 digits) to 1973 (1 million)].

• If we try with \(\arctan(1/238)\) we obtain huge integers.

- (Dase’s ‘Mental’ Formula) We try PSLQ on
  \[\arctan(1), \arctan(\frac{1}{2}), \arctan(\frac{1}{5}), \arctan(\frac{1}{7})\]
  and recover \(-1, 1, 1, \pi\). That is,
  \[\pi = \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{39}\right) + \arctan\left(\frac{1}{82}\right)\]
  [Used by Dase for 205 digits in 1844.]

INTEGRAL RELATIONS in MAPLE

- with(IntegersRelations);Digits:=30;
  \[\text{PSLQ: Digto=16}\]
  \[\{\text{Inhead, Digto}\}\]
  \[\text{PSLQ(\{Pi, arctan(1/2), arctan(1/5), arctan(1/8)\});}\]
  \[\{1, -4, -4, -4\}\]
  \[\text{PSLQ(\{Pi, arctan(1/2), arctan(1/5), arctan(1/9)\});}\]
  \[\{1, 0, 27536, 747556, -3207951\}\]

- Maple also implements the Wilf-Zeilberger algorithm.
- Mathematica can only recognize algebraic numbers.

INTEGRAL RELATION ALGORITHMS: WHAT THEY DO: ADVANCED EXAMPLES

- THE BBP FORMULA FOR PI
- PHYSICAL INTEGRALS
  - ISING AND QUANTUM FIELD THEORY
  - APERY SUMS
  - AND GENERATING FUNCTIONS
- RAMANUJAN SERIES FOR \(1/\pi^N\)

The BBP FORMULA for Pi

In 1996 Bailey, P. Borwein and Plouffe, using PSLQ for months, discovered this formula for \(\pi\):

\[\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left( 
\begin{array}{c}
4 \quad 2 \quad 1 \\
8k + 1 \\
8k + 4 \\
8k + 5 \\
8k + 6
\end{array}
\right)
\]

Indeed, this formula permits one to directly calculate binary or hexadecimal (base-16) digits of \(\pi\) beginning at an arbitrary starting position \(n\), without needing to calculate any of the first \(n-1\) digits.

A finalist for the Edge of Computation Prize, it has been used in compilers, in a record web computation, and in a trillion-digit computation of Pi.

IDENTIFYING THE LIMIT WITH THE ISC (2.0)

We discovered the limit result as follows: We first calculated:

\[C_{1024} \approx 2e^{2\pi}
\]

We then used the Inverse Symbolic Calculator, the online numerical constant recognition facility available at:

http://ddrive.cs.dal.ca/~isc/portal

Output: Mixed constants, 2 with elementary transforms.

\[0.630473503374386796122040192710878964354587...\]

In other words,

\[C_{1024} \approx 2e^{2\pi}\]

The following formulas for $\zeta(s)$ have been known for many decades:

\[
\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \\
\zeta(3) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{31}{15} \zeta(2) - \frac{4}{5 \pi^2}, \\
\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90},
\]

These results have led many to speculate that $\zeta(1)$ might be some nice rational or algebraic value. Sadly (??), PSLQ calculations have established that if $Q_{12}$ satisfies a polynomial with degree at most 25, then at least one coefficient has 380 digits. But positive results exist.

**NEW RAMANUJAN-LIKE IDENTITIES**

Guillera (around 2003) found Ramanujan-like identities, including:

\[
\begin{align*}
\frac{128}{z^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (13 + 18n + 82n^2)}{32} \frac{1}{n^2}, \\
\frac{8}{z^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (1 + 8n + 20n^2)}{32} \frac{1}{n} \\
\frac{32}{z^2} &= \sum_{n=0}^{\infty} \frac{r(n)^2 (1 + 14n + 76n^2 + 168n^2)}{32} \frac{1}{n},
\end{align*}
\]

where

\[
\begin{align*}
\binom{n}{m} &= \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdots (2n-1)/2}{n!} = \frac{f(n+1/2)}{\sqrt{\pi} \Gamma(n+1)},
\end{align*}
\]

Guillera proved the first two using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods. It is true but has no hint of a proof...

As far as we can tell there are no higher-order analogues!

**REFERENCES**

For more information, visit our website: www.experimentalmath.info