The Arithmetic of Uniform Random Walks
Workshop on Exploratory Experimentation and Computation in Number Theory

Jonathan M. Borwein FRSC FAAAS FAA
Joint with Dirk Nuyens, Armin Straub, James Wan
(& Wadim Zudilin).
Revised: 01-07-2010

CARMA, the University of Newcastle

July 9th 2010
Abstract

Following Pearson in 1905, we study the expected distance of a two-dimensional walk in the plane with \( n \) unit steps in random directions — what Pearson called a “ramble”.

While the large \( n \) behaviour is well understood, the precise behaviour of the first few steps is quite remarkable and less tractable. Series evaluations and recursions are obtained making it possible to explicitly determine this distance for small number of steps. Hypergeometric and elliptic closed form expressions are given for all the moments of a 2 or 3-step walk and of a 4-step walk.

Heavy use is made of the analytic continuation of the underlying integral (also of modern special functions and computer algebra (CAS)).
Outline

1. Introduction
2. Combinatorics
3. Analysis
4. Probability
5. Open Problems
The random walk integrals

For complex $s$

**Definition**

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x k} \right|^s \, dx$$

Also, let $W_n := W_n(1)$.

**Simplest** case (obvious for geometric reasons):

$$W_1(s) = \int_0^1 |e^{2\pi i x}|^s \, dx = 1.$$
Second simplest case:

\[ W_2 = \int_0^1 \int_0^1 |e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy = ? \]
Second simplest case:

\[ W_2 = \int_0^1 \int_0^1 |e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy = ? \]

Mathematica 7 and Maple 13 think the answer is 0.
Second simplest case:

\[ W_2 = \int_0^1 \int_0^1 \left| e^{2\pi ix} + e^{2\pi iy} \right| \, dx \, dy = ? \]

Mathematica 7 and Maple 13 think the answer is 0.

There is always a 1-dimension reduction

\[ W_n(s) = \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi i x_k} \right|^s \, dx \]

\[ = \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi i x_k} \right|^s \, d(x_1, \ldots, x_{n-1}) \]
Second simplest case:

\[ W_2 = \int_0^1 \int_0^1 |e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy = ? \]

Mathematica 7 and Maple 13 think the answer is 0.

There is always a 1-dimension reduction

\[ W_n(s) = \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, dx \]

\[ = \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi x_k i} \right|^s \, d(x_1, \ldots, x_{n-1}) \]

So \( W_2 = 4 \int_0^{\frac{1}{4}} \cos(\pi x) \, dx = \frac{4}{\pi} \).
Such problems often get much more difficult in five dimensions and above – e.g., Bessel moments, Box integrals, Ising integrals (work with Bailey, Broadhurst, Crandall, ...).
Such problems often get much more difficult in five dimensions and above — e.g., Bessel moments, Box integrals, Ising integrals (work with Bailey, Broadhurst, Crandall, ...).

In fact, $W_5 \approx 2.0081618$ was the best estimate we could compute directly, notwithstanding the availability of 256 cores at the Lawrence Berkeley National Laboratory.
$n \geq 3$ not trivial

- Such problems often get much more difficult in five dimensions and above – e.g., Bessel moments, Box integrals, Ising integrals (work with Bailey, Broadhurst, Crandall, ...).
- In fact, $W_5 \approx 2.0081618$ was the best estimate we could compute directly, notwithstanding the availability of 256 cores at the Lawrence Berkeley National Laboratory.

- Most results now written up (FPSAC 2010, RAMA and Exp. Math). Available at
  
  
  and

$n \geq 3$ not trivial

- Such problems often get much more difficult in \textit{five} dimensions and above – e.g., \textit{Bessel} moments, \textit{Box} integrals, \textit{Ising} integrals (work with Bailey, Broadhurst, Crandall, ...).

- In fact, $W_5 \approx 2.0081618$ was the best estimate we could compute directly, notwithstanding the availability of 256 cores at the \textit{Lawrence Berkeley National Laboratory}.

- Most results now written up (\textit{FPSAC 2010}, \textit{RAMA} and \textit{Exp. Math}). Available at
  
  
  and
  

So what is a Hilbert space? — David Hilbert in \textit{Mathematical Apocrypha} by Steven Krantz
1000 three-step rambles
One 1500-step ramble
1D (and 3D) easy. Expectation of RMS distance is easy ($= \sqrt{n}$).
1D (and 3D) easy. Expectation of RMS distance is easy ($= \sqrt{n}$).
1D or 2D lattice: probability one of returning to the origin.
History

**L:** Pearson posed questions (*Nature*, 1905).

**R:** Rayleigh gave large $n$ asymptotics:

$$p_n(x) \sim \frac{2x}{n} e^{-x^2/n} \quad (*Nature*, 1905).$$

R: Rayleigh gave large $n$ asymptotics:

$$p_n(x) \sim \frac{2x}{n} e^{-x^2/n} \quad (\textit{Nature}, \ 1905).$$

**John William Strutt (Lord Rayleigh) (1842-1919):** discoverer of Argon, explained why sky is blue.
L: Pearson posed questions \((\textit{Nature}, 1905)\).

R: Rayleigh gave large \(n\) asymptotics:
\[
p_n(x) \sim \frac{2x}{n} e^{-x^2/n} \quad (\textit{Nature}, 1905).
\]

John William Strutt (Lord Rayleigh) \((1842-1919)\): discoverer of Argon, explained why sky is \textbf{blue}.

Karl Pearson \((1857-1936)\): founded statistics, eugenicist & socialist, changed name \((C \mapsto K)\), declined knighthood.

R: Rayleigh gave large \( n \) asymptotics:
\[
p_n(x) \sim \frac{2x}{n} e^{-x^2/n}
\]

**John William Strutt (Lord Rayleigh) (1842-1919):** discoverer of Argon, explained why sky is blue.

**Karl Pearson (1857-1936):** founded statistics, eugenicist & socialist, changed name \((C \mapsto K)\), declined knighthood.

- **UNSW:** Donovan and Nuyens, WWII cryptography.
History


R: Rayleigh gave large $n$ asymptotics: $p_n(x) \sim \frac{2x}{n} e^{-x^2/n}$ (*Nature*, 1905).

**John William Strutt (Lord Rayleigh) (1842-1919):** discoverer of Argon, explained why sky is blue.

**Karl Pearson (1857-1936):** founded statistics, eugenicist & socialist, changed name ($C \mapsto K$), declined knighthood.

- **UNSW:** Donovan and Nuyens, WWII cryptography.
- Also appear in quantum chemistry, in quantum physics as hexagonal and diamond lattice integers.
History


R: Rayleigh gave large \( n \) asymptotics:
\[
p_n(x) \sim \frac{2x}{n} e^{-x^2/n} \quad (**Nature**, 1905).
\]

**John William Strutt (Lord Rayleigh) (1842-1919):** discoverer of Argon, explained why sky is blue.

**Karl Pearson (1857-1936):** founded statistics, eugenicist & socialist, changed name \( C \mapsto K \), declined knighthood.

- **UNSW:** Donovan and Nuyens, WWII cryptography.
- Also appear in quantum chemistry, in quantum physics as hexagonal and diamond lattice integers.
- **Sparked interest:** Crandall, Bailey, Broadhurst, Guttmann, Zudilin, et al. Winning posters for AS and JW.
The Arithmetic of Uniform Random Walks

Further studied by Karl Pearson, J. C. Kluyver, and many others; a functional equation of Ramanujan’s master theorem we get integers! For instance, abcd ecbf contributes to $f(3,4)$. Surely: $f(3,1) = 1$. Just a bit harder: $f(3,2) = \frac{2k}{k^2}$ which can be seen from $a b g b b g a b a b$. Summation formulae for $n = 2$ can be obtained from the convolution $f_{n+1}(k) = \sum_{j=1}^{n} f_j(k) / j$. The machinery of combinatorics ensures recurrences for fixed $n$. For instance, for $n = 4$: $(k+2)^2 f_4(k) = 2(4k+1) + 64(k+1)^2 f_4(k) = 0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k = 2$</th>
<th>$k = 4$</th>
<th>$k = 6$</th>
<th>$k = 8$</th>
<th>$k = 10$</th>
<th>$k = 12$</th>
<th>$k = 14$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27324</td>
<td>3.3953</td>
<td>10.8650</td>
<td>37.2514</td>
<td>132.449</td>
<td>438.3889</td>
<td>1453.463</td>
</tr>
<tr>
<td>3</td>
<td>57460</td>
<td>6.45168</td>
<td>36.7052</td>
<td>151.447</td>
<td>514.448</td>
<td>1714.448</td>
<td>5744.448</td>
</tr>
<tr>
<td>5</td>
<td>80099</td>
<td>9.07511</td>
<td>58.9165</td>
<td>211.766</td>
<td>711.766</td>
<td>2311.766</td>
<td>7711.766</td>
</tr>
<tr>
<td>7</td>
<td>103743</td>
<td>11.7083</td>
<td>84.8747</td>
<td>302.8747</td>
<td>1022.8747</td>
<td>3412.8747</td>
<td>11412.8747</td>
</tr>
<tr>
<td>9</td>
<td>129093</td>
<td>14.3415</td>
<td>113.587</td>
<td>418.587</td>
<td>1418.587</td>
<td>4818.587</td>
<td>16218.587</td>
</tr>
<tr>
<td>11</td>
<td>156299</td>
<td>17.9747</td>
<td>145.063</td>
<td>541.063</td>
<td>1841.063</td>
<td>6241.063</td>
<td>21841.063</td>
</tr>
</tbody>
</table>

Sloane’s

Mathematics Department - Tulane University - New Orleans, USA


This work was supported by an IBM Fellowship in Computational Science.
James Wan will talk about the densities

Computer Assisted Mathematical Analysis and Number Theory

Three Minute Thesis

James Wan

Example (Random Walks)

Take \( n \) steps in the plane, each of length 1 and chosen in a random direction. What is the average distance traveled, \( W_n \)?

- \( W_n = \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right| \, dx_1 \cdots dx_k. \)
- For 2 steps, the average distance is \( \frac{4}{\pi} \).
- The probability of returning to the unit disk is \( \frac{1}{n+1} \).

Several 4-step walks

A 500-step walk
$W_n(k)$ at even values

Even values are easier (combinatorial – no square roots).

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_2(k)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>20</td>
<td>70</td>
<td>252</td>
</tr>
<tr>
<td>$W_3(k)$</td>
<td>1</td>
<td>3</td>
<td>15</td>
<td>93</td>
<td>639</td>
<td>4653</td>
</tr>
<tr>
<td>$W_4(k)$</td>
<td>1</td>
<td>4</td>
<td>28</td>
<td>256</td>
<td>2716</td>
<td>31504</td>
</tr>
</tbody>
</table>
Even values are easier (combinatorial – no square roots).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
k & 0 & 2 & 4 & 6 & 8 & 10 \\
\hline
W_2(k) & 1 & 2 & 6 & 20 & 70 & 252 \\
W_3(k) & 1 & 3 & 15 & 93 & 639 & 4653 \\
W_4(k) & 1 & 4 & 28 & 256 & 2716 & 31504 \\
\hline
\end{array}
\]

- Observe that

\[ W_2(s) = \binom{s}{s/2} \]

for \( s > -1 \).

- MathWorld gives \( W_n(2) = n \) (trivial).
### $W_n(k)$ at odd integers

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k = 1$</th>
<th>$k = 3$</th>
<th>$k = 5$</th>
<th>$k = 7$</th>
<th>$k = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.27324</td>
<td>3.39531</td>
<td>10.8650</td>
<td>37.2514</td>
<td>132.449</td>
</tr>
<tr>
<td>3</td>
<td><strong>1.57460</strong></td>
<td>6.45168</td>
<td>36.7052</td>
<td>241.544</td>
<td>1714.62</td>
</tr>
<tr>
<td>4</td>
<td>1.79909</td>
<td>10.1207</td>
<td>82.6515</td>
<td>822.273</td>
<td>9169.62</td>
</tr>
<tr>
<td>5</td>
<td>2.00816</td>
<td>14.2896</td>
<td>152.316</td>
<td>2037.14</td>
<td>31393.1</td>
</tr>
<tr>
<td>6</td>
<td>2.19386</td>
<td>18.9133</td>
<td>248.759</td>
<td>4186.19</td>
<td>82718.9</td>
</tr>
</tbody>
</table>

During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. — Autobiography of Charles Darwin
\[ W_n(k) \] at odd integers

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( k = 1 )</th>
<th>( k = 3 )</th>
<th>( k = 5 )</th>
<th>( k = 7 )</th>
<th>( k = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>1.27324</td>
<td>3.39531</td>
<td>10.8650</td>
<td>37.2514</td>
<td>132.449</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td><strong>1.57460</strong></td>
<td>6.45168</td>
<td>36.7052</td>
<td>241.544</td>
<td>1714.62</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1.79909</td>
<td>10.1207</td>
<td>82.6515</td>
<td>822.273</td>
<td>9169.62</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>2.00816</td>
<td>14.2896</td>
<td>152.316</td>
<td>2037.14</td>
<td>31393.1</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>2.19386</td>
<td>18.9133</td>
<td>248.759</td>
<td>4186.19</td>
<td>82718.9</td>
</tr>
</tbody>
</table>

During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. — Autobiography of Charles Darwin
Resolution at even values

- **General even formula** (proven below, counts abelian squares):

\[ W_n(2k) = \sum_{a_1 + \ldots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2. \]
Resolution at even values

- **General even formula** (proven below, counts abelian squares):

  \[ W_n(2k) = \sum_{a_1 + \ldots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2. \]

- Known to satisfy **convolutions**:

  \[ W_{n_1+n_2}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_{n_1}(2j)W_{n_2}(2(k-j)), \text{ so} \]

  \[ W_5(2k) = \sum_j \binom{k}{j}^2 \binom{2(k-j)}{j}^2 \sum_{\ell} \binom{j}{\ell}^2 \binom{2\ell}{\ell} = \sum_j \binom{k}{j}^2 \sum_{\ell} \binom{2(j-\ell)}{j-\ell} \binom{j}{\ell}^2 \binom{2\ell}{\ell} \]
Resolution at even values

- **General even formula** (proven below, counts abelian squares):

  \[ W_n(2k) = \sum_{a_1 + \ldots + a_n = k} \left( \binom{k}{a_1, \ldots, a_n} \right)^2. \]

- Known to satisfy **convolutions**:

  \[ W_{n_1+n_2}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_{n_1}(2j) W_{n_2}(2(k-j)), \text{ so} \]

  \[ W_5(2k) = \sum_j \binom{k}{j}^2 \left( 2\binom{k-j}{k-j} \right) \sum_\ell \binom{j}{\ell}^2 \left( 2\binom{\ell}{\ell} \right) = \sum_j \binom{k}{j}^2 \sum_\ell \binom{2(j-\ell)}{j-\ell} \binom{j}{\ell}^2 \binom{2\ell}{\ell} \]

- **and recursions**:

  \[
  (k + 2)^2 W_3(2k + 4) - (10k^2 + 30k + 23)W_3(2k + 2) + 9(k + 1)^2 W_3(2k) = 0.
  \]
Recall $W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s dx$. 

Binomial expansion of $W_n(s)$
Binomial expansion of $W_n(s)$

- Recall $W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s d\mathbf{x}.$

- $\left| \sum_{k} e^{2\pi x_k i} \right|^2 = n^2 - 4 \sum_{i<j} \sin^2(\pi (x_j - x_i)).$
Binomial expansion of $W_n(s)$

- Recall $W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, dx$.

- $\left| \sum_{k} e^{2\pi x_k i} \right|^2 = n^2 - 4 \sum_{i<j} \sin^2(\pi(x_j - x_i))$.

- Binomial expansion:

$$W_n(s) = n^s \sum_{m \geq 0} \frac{(-1)^m}{n^{2m}} \binom{s}{2m} \int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right)^m \, dx =: I_{n,m}$$
Binomial expansion of $W_n(s)$

- Recall $W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, dx$.

- $\left| \sum_{k} e^{2\pi x_k i} \right|^2 = n^2 - 4 \sum_{i<j} \sin^2(\pi(x_j - x_i))$.

- Binomial expansion:

  $$W_n(s) = n^s \sum_{m \geq 0} \frac{(-1)^m}{n^{2m}} \binom{s}{2} \int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right)^m \, dx$$

  $$=: I_{n,m}$$

- Experimentally found recursion for $I_{3,m} \ldots$
Conjecture... 

- Looked up $I_{3,m}$ on Sloane’s OEIS: (as on next slide) get

\[1, 6, 42, 312, 2394, 18756, 149136, \ldots\]
Conjecture. . .

- Looked up $I_{3,m}$ on Sloane’s OEIS: (as on next slide) get $1, 6, 42, 312, 2394, 18756, 149136, \ldots$

- Found from A093388 that $I_{3,m}$ is coefficient of $(xyz)^m$ in
  
  $$(8xyz - (x + y)(y + z)(z + x))^m$$

  $$= (3^2xyz - (x + y + z)(xy + yz + zx))^m$$
Conjecture... 

- Looked up $I_{3,m}$ on Sloane’s OEIS: (as on next slide) get $1, 6, 42, 312, 2394, 18756, 149136, \ldots$

- Found from A093388 that $I_{3,m}$ is coefficient of $(xyz)^m$ in 
  
  \[
  (8xyz - (x + y)(y + z)(z + x))^m = (3^2xyz - (x + y + z)(xy + yz + zx))^m 
  \]

- Guessed that $I_{n,m}$ is constant term of 
  
  \[
  \left( n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n) \right)^m 
  \]
Conjecture...

- Looked up $I_{3,m}$ on Sloane’s OEIS: (as on next slide) get
  
  $1, 6, 42, 312, 2394, 18756, 149136, \ldots$

- Found from A093388 that $I_{3,m}$ is coefficient of $(xyz)^m$ in
  
  $$(8xyz - (x + y)(y + z)(z + x))^m = (3^2xyz - (x + y + z)(xy + yz + zx))^m$$

- Guessed that $I_{n,m}$ is constant term of
  
  $$(n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n))^m$$

- Leads to the conjecture:
Conjecture...

- Looked up $I_{3,m}$ on Sloane’s OEIS: (as on next slide) get
  
  $1, 6, 42, 312, 2394, 18756, 149136, \ldots$

- Found from A093388 that $I_{3,m}$ is coefficient of $(xyz)^m$ in

  $$(8xyz − (x + y)(y + z)(z + x))^m$$

  $$= (3^2xyz − (x + y + z)(xy + yz + zx))^m$$

- Guessed that $I_{n,m}$ is constant term of

  $$(n^2 − (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n))^m$$

- Leads to the conjecture:

  $$W_n(s) = n^s \sum_{m \geq 0} (-1)^m \binom{s/2}{m} \sum_{k=0}^m (-1)^k \binom{m}{k} \sum \binom{k}{a_1, \ldots, a_n}^2.$$
The Arithmetic of Uniform Random Walks

OEIS at work

Greetings from The On-Line Encyclopedia of Integer Sequences!

Search: 1, 6, 42, 312, 2394
Displaying 1-1 of 1 results found.

<table>
<thead>
<tr>
<th>A093388</th>
<th>((n+1)^2 a_{n+1} = (17n^2+2+17n+6)a_n - 72n^2 a_{n-1}).</th>
</tr>
</thead>
</table>

1, 6, 42, 312, 2394, 18756, 149136, 1199232, 9729882, 79527084, 654089292, 5408896752, 44941605984, 375002110944, 314110739328, 26402533581312, 222635989516122, 1882882811380284, 15967419789558804, 135752058036988848, 1156869080242393644

*COMMENT*

This is the Taylor expansion of a special point on a curve described by Beauville.

*REFERENCES*


*LINKS*

Matthijs Coster, Sequences

H. Verrill, Some congruences related to modular forms, Section 2.2.

*FORMULA*

\((-1)^n \sum_{k=0}^{n} k \binom{n, k} * (-8)^k \sum_{j=0}^{n-k} \binom{n-k, j} 3\) - Helena Verrill (verrill(AT)mth.math.lsu.edu), Aug 09 2004

*MAPLE*

```maple
f := proc(n) option remember; local m; if n = 0 then RETURN(1); fi; if n = 1 then RETURN(6); fi; m := n - 1; (17*n^2 + 17*n + 6)*f(n-1) - 72*n^2*f(n-2))/n^2; end;
```

*PROGRAM*

(PARI) \(a(n) = \sum_{k=0}^{n} k \binom{n, k} * (-8)^k \sum_{j=0}^{n-k} \binom{n-k, j} 3\)

*CROSSREFS*

This is the seventh sequence in the family beginning A002394, A006077, A083108, A005250, A000172, A002982, Sequence in context: A111886, A091188, A004882 this_sequence A162986, A034171, A185323

Adjacent sequences: A093435, A093436, A093437, this_sequence A093386, A093389, A093392

*KEYWORD*

nonn

*AUTHOR*

Matthijs Coster (matthijs(AT)coster.demon.nl), Apr 29 2004
... and proof

- **Needed** to show

\[
I_{n,m} = \int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(t_j - t_i)) \right)^m \, dt
\]

is the **constant term of**

\[
(n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n))^m = \\
\left( \sum_{i<j} \left( 2 - \frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \right)^m = \left( - \sum_{i<j} \frac{(x_j - x_i)^2}{x_ix_j} \right)^m.
\]
... and proof

- **Needed** to show

\[
I_{n,m} = \int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(t_j - t_i)) \right)^m \, dt
\]

is the **constant term** of

\[
\left( n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n) \right)^m = \\
\left( \sum_{i<j} \left( 2 - \frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \right)^m = \left( -\sum_{i<j} \frac{(x_j - x_i)^2}{x_i x_j} \right)^m.
\]

- To preserve symmetry, we did not use the dimension reduction.
... and proof

- **Needed** to show

  \[ I_{n,m} = \int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(t_j - t_i)) \right)^m \, dt \]

  is the **constant term** of

  \[
  (n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n))^m = \\
  \left( \sum_{i<j} \left( 2 - \frac{x_i}{x_j} - \frac{x_j}{x_i} \right) \right)^m = \left( - \sum_{i<j} \frac{(x_j - x_i)^2}{x_i x_j} \right)^m.
  \]

- To preserve symmetry, we did not use the dimension reduction.
- Now **expanded** the \( m \)-th power on both sides, and amazingly corresponding terms are equal.

  **QED**
So $W_n$ has $\left\lfloor \frac{n+1}{2} \right\rfloor$-term recursion and can be given by $\left\lceil \frac{n+3}{2} \right\rceil$ iterated sums:

For instance

\[
W_3 = 3 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n (-\frac{8}{9})^n \sum_{k=0}^{n} \binom{n}{k} (-\frac{1}{8})^k \sum_{j=0}^{k} \binom{k}{j}^3
\]

\[
= 3 \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{2} \right)^n \sum_{k=0}^{n} \binom{n}{k} (-\frac{1}{9})^k \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}
\]
So $W_n$ has $\lfloor \frac{n+1}{2} \rfloor$-term recursion and can be given by $\lfloor \frac{n+3}{2} \rfloor$ iterated sums:

For instance

$$W_3 = 3 \sum_{n=0}^{\infty} \frac{1}{2^n} \left( -\frac{8}{9} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{8} \right)^k \sum_{j=0}^{k} \binom{k}{j}^3$$

$$= 3 \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{9} \right)^k \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

Recursion gives better approximations than many methods of numerical integration for many values of $s$. 
So $W_n$ has $\left\lfloor \frac{n+1}{2} \right\rfloor$-term recursion and can be given by $\left\lfloor \frac{n+3}{2} \right\rfloor$ iterated sums:

For instance

$$W_3 = 3 \sum_{n=0}^{\infty} \binom{1/2}{n} \left( -\frac{8}{9} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{8} \right)^k \sum_{j=0}^{k} \binom{k}{j}^3$$

$$= 3 \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{9} \right)^k \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

**Recursion** gives better approximations than many methods of numerical integration for many values of $s$.

**Tanh-sinh** (doubly-exponential) quadrature works well for $W_3$ but not so good for $W_4 \approx 1.79909248$. **Quasi-Monte Carlo** was not very accurate.
Theorem (binomial involution)

Given real sequences \((a_n)\) and \((s_n)\), the following are equivalent:

\[
s_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k,
\]

\[
a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_k.
\]
Theorem (binomial involution)

Given real sequences \((a_n)\) and \((s_n)\), the following are equivalent:

\[
s_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k, \]

\[
a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_k. \]

We can now give a proof of the even formula. Recall

\[
W_n(2i) = n^{2i} \sum_{m \geq 0} (-1)^m \binom{2i}{m} \sum_{k=0}^{m} \frac{(-1)^k}{n^{2k}} \binom{m}{k} \sum_{\sum a_i = k} \binom{k}{a_1, \ldots, a_n}^2.
\]

and appeal to the involution. QED
Carlson’s theorem: from discrete to continuous

**Theorem (Carlson)**

If $f(z)$ is analytic for $\Re(z) \geq 0$, its growth on the imaginary axis is bounded by $e^{cy}, |c| < \pi$, and

$$0 = f(0) = f(1) = f(2) = \ldots$$

then $f(z) = 0$ identically.

- $\sin(\pi z)$ does not satisfy the conditions of the theorem, as it grows like $e^{\pi y}$ on the imaginary axis.
Carlson’s theorem: from discrete to continuous

Theorem (Carlson)

If \( f(z) \) is analytic for \( \Re(z) \geq 0 \), its growth on the imaginary axis is bounded by \( e^{cy} \), \(|c| < \pi \), and

\[
0 = f(0) = f(1) = f(2) = \ldots
\]

then \( f(z) = 0 \) identically.

- \( \sin(\pi z) \) does not satisfy the conditions of the theorem, as it grows like \( e^{\pi y} \) on the imaginary axis.
- \( W_n(s) \) satisfies the conditions of the theorem (and is in fact analytic for \( \Re(s) > -2 \) when \( n > 2 \)).
So integer recurrences yield complex functional equations. Viz

\[(s+4)^2 W_3(s+4) - 2(5s^2 + 30s + 46) W_3(s+2) + 9(s+2)^2 W_3(s) = 0.\]
Analytic continuation

- So integer recurrences yield complex functional equations. Viz

\[(s+4)^2 W_3(s+4) - 2(5s^2 + 30s + 46) W_3(s+2) + 9(s+2)^2 W_3(s) = 0.\]

- This gives analytic continuations of the ramble integrals to the complex plane, with poles at certain negative integers (likewise for all \( n \)).
Analytic continuation

- So integer recurrences yield complex functional equations. Viz

\[(s+4)^2W_3(s+4) - 2(5s^2+30s+46)W_3(s+2) + 9(s+2)^2W_3(s) = 0.\]

- This gives analytic continuations of the ramble integrals to the complex plane, with poles at certain negative integers (likewise for all \(n\)).

- \(W_3(s)\) has a simple pole at \(-2\) with residue \(\frac{2}{\sqrt{3}\pi}\), and other simple poles at \(-2k\) with residues a rational multiple of \(\text{Res}_{-2}\).
So integer recurrences yield complex functional equations. Viz

\[(s+4)^2W_3(s+4)-2(5s^2+30s+46)W_3(s+2)+9(s+2)^2W_3(s) = 0.\]

This gives analytic continuations of the ramble integrals to the complex plane, with poles at certain negative integers (likewise for all \(n\)).

\(W_3(s)\) has a simple pole at \(-2\) with residue \(\frac{2}{\sqrt{3}\pi}\), and other simple poles at \(-2k\) with residues a rational multiple of \(\text{Res}_{-2}\).

We have found a “fractional binomial transform”, that is, the transform gives us back a sequence that satisfies the same functional equation:

\[W_n(s) = n^s \sum_{m \geq 0} (-1)^m \binom{s}{m} \sum_{k=0}^{m} \frac{(-1)^k}{n^{2k}} \binom{m}{k} \sum \binom{k}{a_1, \ldots, a_n}^2.\]
Odd dimensions look like 3

\[ W_3(s) \text{ on } [-6, \frac{5}{2}] \]
Odd dimensions look like 3

\[ W_3(s) \text{ on } [-6, \frac{5}{2}] \]

Prove the zeroes are converging to but not at integers.
Some even dimensions look more like 4

\[ W_4(s) \text{ on } [-6, 1/2]. \quad W_5 \text{ on } [-6, 2] \text{ (T)}, \quad W_6 \text{ on } [-6, 2] \text{ (B)}. \]
Some even dimensions look more like 4

\[ L: W_4(s) \text{ on } [-6, 1/2]. \quad R: W_5 \text{ on } [-6, 2] (T), W_6 \text{ on } [-6, 2] (B). \]

- The functional equation for \( n = 4 \) is

\[
(s + 4)^3W_4(s + 4) - 4(s + 3)(5s^2 + 30s + 48)W_4(s + 2) + 64(s + 2)^3W_4(s) = 0
\]

(with double poles). Why is \( W_4 \) positive on \( R \)
Some even dimensions look more like 4

**L:** $W_4(s)$ on $[-6, 1/2]$.  **R:** $W_5$ on $[-6, 2]$ (T), $W_6$ on $[-6, 2]$ (B).

- The *functional equation* for $n = 4$ is

$$
(s + 4)^3W_4(s + 4) - 4(s + 3)(5s^2 + 30s + 48)W_4(s + 2)
+ 64(s + 2)^3W_4(s) = 0
$$

(with double poles).  *Why is $W_4$ positive on R*

- There are (infinitely many) multiple poles iff $4|n$.  

---

**JMB/JW**  
The Arithmetic of Uniform Random Walks
In particular, we have now shown that

\[
W_3(2k) = \sum_{a_1+a_2+a_3=k} \binom{k}{a_1, a_2, a_3}^2 = 3F_2\left(\begin{array}{c}
\frac{1}{2}, -k, -k \\
1, 1
\end{array} \bigg| 4\right) = V_3(2k)
\]

where \(pF_q\) is the generalized hypergeometric function. We discovered numerically that: \(V_3(1) = 1.57459 - .12602652i\)

**Theorem (Real part)**

*For all integers \(k\) we have \(W_3(k) = \Re(V_3(k)).\)*
A discovery demystified

In particular, we have now shown that

\[ W_3(2k) = \sum_{a_1+a_2+a_3=k} \left( \binom{k}{a_1, a_2, a_3} \right)^2 = 3F_2 \left( \begin{array}{c} 1/2, -k, -k \\ 1, 1 \end{array} \right| 4 \right) =: V_3(2k) \]

where \( F \) is the generalized hypergeometric function. We discovered numerically that: \( V_3(1) = 1.57459 - .12602652i \)

**Theorem (Real part)**

For all integers \( k \) we have \( W_3(k) = \Re(V_3(k)) \).

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first. . . . So there isn’t any place to publish, in a dignified manner, what you actually did in order to get to do the work. — Richard Feynman (Nobel acceptance 1966)
Proof with hindsight

\[ k = 1. \text{ From the dimension reduction, and elementary manipulations,} \]

\[
W_3(1) = \int_0^1 \int_0^1 |1 + e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 \sqrt{4 \sin(2\pi t) \sin(2\pi (s + t/2)) - 2 \cos(2\pi t) + 3} \, ds \, dt.
\]
Proof with hindsight

$k = 1$. From the dimension reduction, and elementary manipulations,

\[
W_3(1) = \int_0^1 \int_0^1 \left| 1 + e^{2\pi ix} + e^{2\pi iy} \right| \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 \sqrt{4 \sin(2\pi t) \sin(2\pi(s + t/2)) - 2 \cos(2\pi t) + 3} \, ds \, dt.
\]

Let \( s + t/2 \rightarrow s \), and use periodicity of the integrand, to end up with

\[
W_3 = \int_0^1 \left\{ \int_0^1 \sqrt{4 \cos(2\pi s) \sin(\pi t) - 2 \cos(2\pi t) + 3} \, ds \right\} \, dt.
\]

The inner integral can now be computed because

\[
\int_0^\pi \sqrt{a + b \cos(s)} \, ds = 2\sqrt{a + b} \ E \left( \sqrt{\frac{2b}{a + b}} \right).
\]
Here $E(x)$ is the **elliptic integral** of the second kind:

$$E(x) := \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2(t)} \, dx.$$ 

After simplification,

$$W_3 = \frac{4}{\pi^2} \int_0^{\pi/2} (2 \sin(t) + 1)E \left( \frac{2\sqrt{2\sin(t)}}{1 + 2\sin(t)} \right) \, dt.$$
Here $E(x)$ is the elliptic integral of the second kind:

$$E(x) := \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2(t)} \, dx.$$ 

After simplification,

$$W_3 = \frac{4}{\pi^2} \int_0^{\pi/2} (2 \sin(t) + 1) E \left( \frac{2 \sqrt{2 \sin(t)}}{1 + 2 \sin(t)} \right) \, dt.$$ 

Now we recall Jacobi’s imaginary transform,

$$(x + 1) E \left( \frac{2 \sqrt{x}}{x + 1} \right) = \Re(2E(x) - (1 - x^2)K(x))$$

and substitute. Here $K(x)$ is the elliptic integral of the first kind.
Here $E(x)$ is the elliptic integral of the second kind:

$$E(x) := \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2(t)} \, dx.$$ 

After simplification,

$$W_3 = \frac{4}{\pi^2} \int_0^{\pi/2} (2 \sin(t) + 1) E \left( \frac{2\sqrt{2 \sin(t)}}{1 + 2 \sin(t)} \right) \, dt.$$ 

Now we recall Jacobi’s imaginary transform,

$$(x + 1) E \left( \frac{2\sqrt{x}}{x + 1} \right) = \Re(2E(x) - (1 - x^2)K(x))$$

and substitute. Here $K(x)$ is the elliptic integral of the first kind. 

- This is where $\Re$ originates: $(V_3(-1) = 0.896441 - 0.517560i)$.
Using the integral definition of $K$ and $E$, we write $W_3$ as a double integral involving only $\sin$:

$$W_3 = \Re \left( \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1 + a^2 \sin^2(t) - 2a^2 \sin^2(t) \sin^2(r)}{\sqrt{1 - a^2 \sin^2(t) \sin^2(r)}} \, dt \, dr \right),$$

where $a = 2$. 

Using the integral definition of $K$ and $E$, we write $W_3$ as a double integral involving only $\sin$:

$$W_3 = \Re \left( \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1 + a^2 \sin^2(t) - 2 a^2 \sin^2(t) \sin^2(r)}{\sqrt{1 - a^2 \sin^2(t) \sin^2(r)}} \, dt \, dr \right),$$

where $a = 2$.

- Expand using the binomial theorem, evaluate the integral term by term for small $a$, and use analytic continuation to deduce equality with $\Re(V_3(1))$. 
Using the integral definition of $K$ and $E$, we write $W_3$ as a double integral involving only $\sin$:

$$W_3 = \Re \left( \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1 + a^2 \sin^2(t) - 2 a^2 \sin^2(t) \sin^2(r)}{\sqrt{1 - a^2 \sin^2(t) \sin^2(r)}} \, dt \, dr \right),$$

where $a = 2$.

- Expand using the binomial theorem, evaluate the integral term by term for small $a$, and use analytic continuation to deduce equality with $\Re(V_3(1))$.
- A similar (and easier) proof obtains for $W_3(-1)$. 

Using the integral definition of $K$ and $E$, we write $W_3$ as a double integral involving only $\sin$:

$$W_3 = \Re \left( \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1 + a^2 \sin^2(t) - 2 a^2 \sin^2(t) \sin^2(r)}{\sqrt{1 - a^2 \sin^2(t) \sin^2(r)}} \, dt \, dr \right),$$

where $a = 2$.

- Expand using the binomial theorem, evaluate the integral term by term for small $a$, and use analytic continuation to deduce equality with $\Re(V_3(1))$.
- A similar (and easier) proof obtains for $W_3(-1)$.
- As both sides satisfy the same 2-term recursion (computer provable), we are done. QED
A pictorial proof

\[ W_3(s) - V_3(s) \text{ on } [0, 12] \]
A pictorial proof

\[ W_3(s) - V_3(s) \text{ on } [0, 12] \]

This was hard to draw when we discovered it as we had no good closed form for \( W_3 \).
We then confirmed 175 digits of

\[ W_3(1) \approx 1.57459723755189365749 \ldots \]
Closed forms

- We then confirmed 175 digits of

\[ W_3(1) \approx 1.57459723755189365749 \ldots \]

- Armed with knowledge of elliptic integrals:

\[
W_3(1) = \frac{16\sqrt[3]{4\pi^2}}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4\pi^4}} = W_3(-1) + \frac{6/\pi^2}{W_3(-1)},
\]

\[
W_3(-1) = \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4\pi^4}} = \frac{2^{\frac{1}{3}}}{4\pi^2} \beta^2 \left( \frac{1}{3} \right).
\]

Here \( \beta(s) := B(s, s) = \frac{\Gamma(s)^2}{\Gamma(2s)} \).
Closed forms

- We then confirmed 175 digits of

\[ W_3(1) \approx 1.57459723755189365749 \ldots \]

- Armed with knowledge of elliptic integrals:

\[ W_3(1) = \frac{16 \sqrt[3]{4\pi^2}}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8 \sqrt[3]{4\pi^4}} = W_3(-1) + \frac{6/\pi^2}{W_3(-1)}, \]

\[ W_3(-1) = \frac{3\Gamma(\frac{1}{3})^6}{8 \sqrt[3]{4\pi^4}} = \frac{2^{\frac{1}{3}}}{4\pi^2} \beta^2 \left( \frac{1}{3} \right). \]

Here \( \beta(s) := B(s, s) = \frac{\Gamma(s)^2}{\Gamma(2s)}. \)

- Obtained via singular values of the elliptic integral and Legendre’s identity.
Meijer-G functions (1936–)

**Definition**

\[
G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \mid x \right) := \frac{1}{2\pi i} \times \\
\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=n+1}^p \Gamma(a_j - s) \prod_{j=m+1}^q \Gamma(1 - b_j + s)} x^s ds.
\]

The contour \( L \) lies between poles of \( \Gamma(1 - a_i - s) \) and \( \Gamma(b_i + s) \).

A broad generalization of hypergeometric functions—capturing Bessel \( Y, K \) and much more. Important in CAS.

\( W_3(s) \) is among first non-trivial higher order Meijer-G function in closed form:

\[
W_3(s) = \Gamma(1 + s^2)^{\frac{1}{2}} \sqrt{\pi} \Gamma(-s^2) G_{2,1}^{1,3} \left( \begin{array}{c} 1, 1, 1 \\ 1, 2, -s^2, -s^2 \end{array} \mid 1/4 \right).
\]
Meijer-G functions (1936–)

Definition

\[ G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right) := \frac{1}{2\pi i} \times \]

\[ \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=n+1}^p \Gamma(a_j - s) \prod_{j=m+1}^q \Gamma(1 - b_j + s)} x^s ds. \]

- The contour \( L \) lies between poles of \( \Gamma(1 - a_i - s) \) and of \( \Gamma(b_i + s) \).
Meijer-G functions (1936–)

**Definition**

\[
G_{p,q}^{m,n} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| x \right) := \frac{1}{2\pi i} \times \\
\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=n+1}^p \Gamma(a_j - s) \prod_{j=m+1}^q \Gamma(1 - b_j + s)} x^s ds.
\]

- **The contour** \( L \) lies between **poles** of \( \Gamma(1 - a_i - s) \) and of \( \Gamma(b_i + s) \).
- **A broad generalization** of hypergeometric functions—capturing Bessel \( Y, K \) and much more. **Important in CAS.**
Meijer-G functions (1936–)

Definition

\[
G_{p,q}^{m,n} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| x \right) := \frac{1}{2\pi i} \times \\
\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=n+1}^p \Gamma(a_j - s) \prod_{j=m+1}^q \Gamma(1 - b_j + s)} x^s ds.
\]

- The contour \( L \) lies between poles of \( \Gamma(1 - a_i - s) \) and of \( \Gamma(b_i + s) \).
- A broad generalization of hypergeometric functions—capturing Bessel \( Y, K \) and much more. Important in CAS.
- \( W_3(s) \) is among first non-trivial higher order Meijer-G function in closed form:

\[
W_3(s) = \frac{\Gamma(1 + \frac{s}{2})}{\sqrt{\pi} \Gamma(-\frac{s}{2})} G_{33}^{21} \left( \begin{array}{c} 1, 1, 1 \\ \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2} \end{array} \bigg| \frac{1}{4} \right).
\]
Visualizing $W_3$ in the complex plane
Visualizing $W_3$ in the complex plane

- Easily drawn now from the Meijer-G representation in Mathematica.
Visualizing $W_3$ in the complex plane

- Easily drawn now from the Meijer-G representation in Mathematica.
- Each point is coloured differently (black is zero and white infinity). Note the poles and zeros.
Likewise for $W_4(s)$ we found (first Crandall via CAS) that:

\[ W_4(s) = 2s\pi\Gamma(1 + s/2)\Gamma(-s/2)G_{22}^{44}(1, 1 - s/2, 1, 1 - s/2, 1/2 - s/2, -s/2, -s/2, \cdots 1/2) \]

Provable by residue calculus methods. But not helpful for odd integer values. We look elsewhere...
Likewise for $W_4(s)$ we found (first Crandall via CAS) that:

$$W_4(s) = \frac{2^s \Gamma(1 + s/2)}{\pi \Gamma(-s/2)} G_{44}^{22} \left( \begin{array}{c} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{array} \right).$$
Meijer form for $W_4$

Likewise for $W_4(s)$ we found (first Crandall via CAS) that:

$$W_4(s) = \frac{2^s \Gamma(1 + s/2)}{\pi \Gamma(-s/2)} G_{44}^{22} \left( \begin{array}{c} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{array} \right).$$

Provable by residue calculus methods.
Likewise for $W_4(s)$ we found (first Crandall via CAS) that:

$$W_4(s) = \frac{2^s \Gamma(1 + s/2)}{\pi \Gamma(-s/2)} G_{44}^{22} \left( \begin{array}{c} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{array} \right) .$$

- Provable by residue calculus methods.
- But not helpful for odd integer values. We look elsewhere ...
Meijer form for $W_4$

Likewise for $W_4(s)$ we found (first Crandall via CAS) that:

$$W_4(s) = \frac{2^s \Gamma(1 + s/2)}{\pi \Gamma(-s/2)} G_{22}^{24} \left( \begin{array}{c} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{array} \right| 1 \right).$$

- Provable by *residue calculus* methods.
- But not helpful for odd integer values. We look elsewhere ...

*He [Gauss (or Mma)] is like the fox, who effaces his tracks in the sand with his tail.* — Niels Abel (1802-1829)
The Dutch mathematician J.C. Kluyver (1906) found a Bessel representation for the cumulative radial distribution function \( P_n \) and density \( p_n \) of the distance after \( n \)-steps:

\[
P_n(t) = t \int_0^\infty J_1(xt) J_0^n(x) \, dx
\]

\[
p_n(t) = t \int_0^\infty J_0(xt) J_0^n(x)x \, dx \quad (n \geq 4)
\]

where \( J_n(x) \) is the Bessel J function of the first kind (also Watson (1932, §49); 3-dim walks are elementary).
The Dutch mathematician J.C. Kluyver (1906) found a Bessel representation for the cumulative radial distribution function \( P_n \) and density \( p_n \) of the distance after \( n \)-steps:

\[
P_n(t) = t \int_0^\infty J_1(xt) J_0^n(x) \, dx
\]

\[
p_n(t) = t \int_0^\infty J_0(xt) J_0^n(x)x \, dx \quad (n \geq 4)
\]

where \( J_n(x) \) is the Bessel J function of the first kind (also Watson (1932, §49); 3-dim walks are elementary).

So \( P_n(1) = \frac{1}{n+1} J_0(0) = \frac{1}{n+1} \) (Pearson’s original question).
Alternative representations

The Dutch mathematician J.C. Kluyver (1906) found a Bessel representation for the cumulative radial distribution function \( P_n \) and density \( p_n \) of the distance after \( n \)-steps:

\[
P_n(t) = t \int_0^\infty J_1(xt) J_0^n(x) \, dx
\]

\[
p_n(t) = t \int_0^\infty J_0(xt) J_0^n(x) x \, dx \quad (n \geq 4)
\]

where \( J_n(x) \) is the Bessel J function of the first kind (also Watson (1932, §49); 3-dim walks are elementary).

- So \( P_n(1) = \frac{1}{n+1} J_0(0) = \frac{1}{n+1} \) (Pearson’s original question).
- From this we obtain for \( 2k > s > -\frac{n}{2} \) that

\[
W_n(s) = 2^{s+1-k} \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(k - \frac{s}{2})} \int_0^\infty x^{2k-s-1} \left(-\frac{1}{x} \frac{d}{dx}\right)^k J_0^n(x) \, dx.
\]

(1)

This is a very useful oscillatory 1-dimensional integral!
The densities for $n = 3, 4$ are “modular” (JW’s talk)

$\sigma(x) := \frac{3-x}{1+x}$. Then on $[0, 3]$, $\sigma^2(x) = x$ and $\sigma$ is an involution that sends $[0, 1]$ to $[1, 3]$ and

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)).$$  \hspace{1cm} (2)

So $\frac{3}{4}p'_3(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}$ and $p(1) = \infty$. Indeed,

$$p_3(\alpha) = \frac{2 \sqrt{3}\alpha}{\pi (3 + \alpha^2)} 2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; \frac{\alpha^2 (9 - \alpha^2)^2}{(3 + \alpha^2)^3} \right).$$  \hspace{1cm} (3)

The densities $p_3 (L)$ and $p_4 (R)$
The densities for $5 \leq n \leq 8$ (and large $n$ approximation)
The densities for $5 \leq n \leq 8$ (and large $n$ approximation)

$p_{2n+4}, p_{2n+5}$ both $n$-times continuously differentiable for $x > 0$.

$(p_n(x) \sim \frac{2x}{n} e^{-x^2/n})$
We (humans and computers) now obtained:

Corollary (Hypergeometric forms for noninteger $s > -2$)

$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \left(\frac{s}{s-1}\right)^2 \binom{s}{s-1} + \left(\frac{s}{s-1}\right)^2 \binom{s}{s-1}$

and

$W_4(s) = \frac{1}{2^{2s}} \tan\left(\frac{\pi s}{2}\right) \left(\frac{s}{s-2}\right)^3 \binom{s}{s-2} + \left(\frac{s}{s-2}\right)^3 \binom{s}{s-2}$
Simplifying the integral

We (humans and computers) now obtained:

**Corollary (Hypergeometric forms for noninteger \( s > -2 \))**

\[
W_3(s) = \frac{1}{2^{2s+1}} \tan \left( \frac{\pi s}{2} \right) \left( \frac{s}{s-1} \right)^2 \frac{3}{2} F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4} \right) + \left( \frac{s}{s-1} \right) \frac{3}{2} F_2 \left( \frac{-s}{2}, \frac{-s}{2}, \frac{-s}{2} \left| \frac{1}{4} \right. \right),
\]

and

\[
W_4(s) = \frac{1}{2^{2s}} \tan \left( \frac{\pi s}{2} \right) \left( \frac{s}{s-1} \right)^3 \frac{4}{3} F_3 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s+1}{s+3} \left| 1 \right. \right) + \left( \frac{s}{s-1} \right) \frac{4}{3} F_3 \left( \frac{1}{2}, \frac{-s}{2}, \frac{-s}{2}, \frac{-s}{2} \left| 1 \right. \right).
\]
We (humans and computers) now obtained:

Corollary (Hypergeometric forms for noninteger $s > -2$)

\[
W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \left(\frac{s}{s-1}\right)^2 3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| \frac{1}{4} \right. \right) + \left(\frac{s}{s-1}\right) 3F_2\left(\frac{-s}{2}, \frac{-s}{2}, \frac{-s}{2} \left| \frac{1}{4} \right. \right),
\]

and

\[
W_4(s) = \frac{1}{2^{2s}} \tan\left(\frac{\pi s}{2}\right) \left(\frac{s}{s-1}\right)^3 4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s}{s-1} + 1 \left| \frac{1}{4} \right. \right) + \left(\frac{s}{s-1}\right) 4F_3\left(\frac{1}{2}, \frac{-s}{2}, \frac{-s}{2}, \frac{-s}{2} \left| \frac{1}{4} \right. \right).
\]

We (humans) were able to provably take the limit:

\[
W_4(-1) = \frac{\pi}{4} 7F_6\left(\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| \frac{1}{4}, 1, 1, 1, 1, 1 \right. \right)
\]

\[
= \frac{\pi}{4} 6F_5\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| 1, 1, 1, 1, 1 \right. \right) + \frac{\pi}{64} 6F_5\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \left| 2, 2, 2, 2, 2 \right. \right).
\]
Simplifying the integral

- We (humans and computers) now obtained:

**Corollary (Hypergeometric forms for noninteger \( s > -2 \))**

\[
W_3(s) = \frac{1}{2^{2s+1}} \tan \left( \frac{\pi s}{2} \right) \left( \frac{s}{2} \right)^2 {}_3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| \frac{1}{4} \right. \right) + \left( \frac{s}{2} \right) {}_3F_2 \left( \frac{-s}{2}, -\frac{s}{2}, -\frac{s}{2} \left| \frac{1}{4} \right. \right),
\]

and

\[
W_4(s) = \frac{1}{2^{2s}} \tan \left( \frac{\pi s}{2} \right) \left( \frac{s}{2} \right)^3 {}_4F_3 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s+1}{2} \left| 1 \right. \right) + \left( \frac{s}{2} \right) {}_4F_3 \left( \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \left| 1 \right. \right).
\]

- We (humans) were able to provably take the limit:

\[
W_4(-1) = \frac{\pi}{4} {}_7F_6 \left( \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| 1 \right. \right)
\]

\[
= \frac{\pi}{4} {}_6F_5 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| 1 \right. \right) + \frac{\pi}{64} {}_6F_5 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \left| 1 \right. \right).
\]

- We have proven the corresponding result for \( W_4(1) \) ....
An elliptic integral harvest

Indeed, we now have various representations including:

\[ W_4(1) = \frac{9\pi}{4} \sum_{k=1}^{m} \frac{3^n}{3^n} \left( \begin{array}{c} 7/4, 3/2, 3/2, 1/2, 1/2, 1/2 \\ 3/4, 2, 2, 2, 1, 1 \end{array} \right) - 2\pi \sum_{k=1}^{n} \frac{3^n}{3^n} \left( \begin{array}{c} 5/4, 1/2, 1/2, 1/2, 1/2, 1/2 \\ 1/4, 1, 1, 1, 1, 1 \end{array} \right) \]
An elliptic integral harvest

Indeed, we now have various representations including:

\[ W_4(1) = \frac{9\pi}{4} 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{array} \right| 1 \right) - 2\pi 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} \right| 1 \right). \]

Discovered by PSLQ, these rely on results by Nesterenko and by Zudilin.
An elliptic integral harvest

Indeed, we now have various representations including:

\[ W_4(1) = \frac{9\pi}{4} 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{array} \bigg| 1 \right) - 2\pi 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} \bigg| 1 \right). \]

- Discovered by PSLQ, these rely on results by Nesterenko and by Zudilin.
- Inter alia we prove relations such as:

\[ 2 \int_0^1 K(k)^2 \, dk = \int_0^1 K'(k)^2 \, dk = \left( \frac{\pi}{2} \right)^4 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} \bigg| 1 \right). \]
An elliptic integral harvest

Indeed, we now have various representations including:

\[ W_4(1) = \frac{9\pi}{4} 7F_6 \left( \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \left| 1 \right. \right) - 2\pi 7F_6 \left( \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| 1 \right. \right). \]

- Discovered by PSLQ, these rely on results by Nesterenko and by Zudilin.
- Inter alia we prove relations such as:

\[ 2 \int_0^1 K(k)^2 dk = \int_0^1 K'(k)^2 dk = \left( \frac{\pi}{2} \right)^4 7F_6 \left( \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| 1 \right. \right). \]

- We also deduce that

\[ W_4(-1) = \frac{8}{\pi^3} \int_0^1 K^2(k) dk \]
\[ W_4(1) = \frac{96}{\pi^3} \int_0^1 E'(k) K'(k) dk - 8 W_4(-1). \]
Open problems

We have already proven:

\[
W_4(2k) = \sum_{a_1 + \ldots + a_4 = k} \left( \begin{array}{c} k \\ a_1, \ldots, a_4 \end{array} \right)^2 \\
= \sum_{j \geq 0} \binom{k}{j}^2 {}_3F_2 \left( \begin{array}{c} 1/2, -k+j, -k+j \\ 1, 1 \end{array} \right| 4 \right) =: V_4(2k)
\]
We have already proven:

\[
W_4(2k) = \sum_{a_1 + \ldots + a_4 = k} \binom{k}{a_1, \ldots, a_4}^2
= \sum_{j \geq 0} \binom{k}{j}^2 \ _3F_2\left(\begin{array}{c}
1/2, -k + j, -k + j \\
1, 1
\end{array} \left| 4 \right. \right).
\]

\[=: V_4(2k)\]

"Any time you are stuck on a problem, introduce more notation" — Chris Skinner
Open problems

We have already proven:

\[ W_4(2k) = \sum_{a_1 + \cdots + a_4 = k} \binom{k}{a_1, \ldots, a_4}^2 \]
\[ = \sum_{j \geq 0} \binom{k}{j}^2 \, _3F_2 \left( \begin{array}{c} 1/2, -k + j, -k + j \\ 1, 1 \end{array} \right| 4 \right). \]

\[ =: V_4(2k) \]

“Any time you are stuck on a problem, introduce more notation” — Chris Skinner

Conjecture

For all integers \( k \) we have,

\[ W_4(k) = \Re(V_4(k)). \]
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).

But what about: for complex $s$ and positive integer $n$,

$$W_{2n}(s) = \sum_{j \geq 0} \left( \frac{s}{2} \right)^2 W_{2n-1}(s - 2j).$$

Now has very strong numerical evidence (Broadhurst). (Convergence is very rapid, using (1) to compute $W_n$ values).

Proven for $n = 2, 3$ assuming (G); and for all $n$ assuming also all odd poles are simple.

And what is a closed form for $W_5(1)$? We do know that $p(1) = \text{Res}_{-2}(W_5) = \sqrt{\frac{15}{3\pi}} F_2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2}\mid -4\right)$.

We know $\text{Res}_{-4}(W_5)$ only numerically—but to 500 digits (by Bailey in about 5.5hrs on 1 MacPro core). All others residues are linear combinations of these two.
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).

But what about: for complex $s$ and positive integer $n$,

$$W_{2n}(s) = \sum_{j \geq 0} \binom{s/2}{j}^2 W_{2n-1}(s - 2j).$$

True for all $n$ and even integer $s$ or for $n = 1$. 
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).

But what about: for complex \( s \) and positive integer \( n \),

\[
W_{2n}(s) = \sum_{j \geq 0} \left( \frac{s}{2j} \right)^2 W_{2n-1}(s - 2j).
\]

- True for all \( n \) and even integer \( s \) or for \( n = 1 \).
- Now has very strong numerical evidence (Broadhurst). (convergence is very rapid, using (1) to compute \( W_n \) values).
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).

But what about: for complex $s$ and positive integer $n$,

$$W_{2n}(s) = \sum_{j \geq 0} \left( \frac{s}{2j} \right)^2 W_{2n-1}(s - 2j).$$

True for all $n$ and even integer $s$ or for $n = 1$.

Now has very strong numerical evidence (Broadhurst).
(convergence is very rapid, using (1) to compute $W_n$ values).

Proven for $n = 2, 3$ assuming (G); and for all $n$ assuming also all odd poles are simple.
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).

But what about: for complex $s$ and positive integer $n$,

$$W_{2n}(s) = \sum_{j \geq 0} \left( \frac{s}{2j} \right)^2 W_{2n-1}(s - 2j).$$

True for all $n$ and even integer $s$ or for $n = 1$.

Now has very strong numerical evidence (Broadhurst).
(convergence is very rapid, using (1) to compute $W_n$ values).

Proven for $n = 2, 3$ assuming (G); and for all $n$ assuming also all odd poles are simple.

And what is a closed form for $W_5(1)$? We do know that

$$p_4(1) = \text{Res}_{-2}(W_5) = \frac{\sqrt{15}}{3\pi} \, _{3}F_{2} \left( \begin{array}{c} \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \\ 1, 1 \end{array} \middle| -4 \right).$$
Conjecture is “almost” proved—via residue calculus from Meijer-G form—modulo a technical growth estimate (G).

But what about: for complex $s$ and positive integer $n$,

$$W_{2n}(s) = \sum_{j \geq 0} \binom{s/2}{j}^2 W_{2n-1}(s - 2j).$$

True for all $n$ and even integer $s$ or for $n = 1$.

Now has very strong numerical evidence (Broadhurst). (convergence is very rapid, using (1) to compute $W_n$ values).

Proven for $n = 2, 3$ assuming (G); and for all $n$ assuming also all odd poles are simple.

And what is a closed form for $W_5(1)$? We do know that

$$p_4(1) = \text{Res}_{-2}(W_5) = \frac{\sqrt{15}}{3\pi} {}_3F_2 \left( \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \left| \frac{-4}{1}, 1 \right. \right).$$

We know $\text{Res}_{-4}(W_5)$ only numerically—but to 500 digits (by Bailey in about 5.5hrs on 1 MacPro core). All others residues are linear combinations of these two.
Thank you ...

Two ramblers at ANZIAM 2010
Thank you ...

**Conclusion.** We continue to be fascinated by this blend of combinatorics, analysis, probability, and differential equations, all tied together with experimental mathematics.

Two ramblers at ANZIAM 2010