Calculating Bessel Functions via the Exp-arc Method

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Outline

1. What are Bessel Functions?
2. Why do we care?
3. Exp-arc explained
4. Results
What are Bessel Functions?

Why do we care?

Exp-arc explained

Results
The second order differential equation

\[ z^2 y'' + zy' + (z^2 - \nu^2)y = 0 \]

is called Bessel’s Equation.

The ordinary Bessel function or order \( \nu \), or the Bessel function of the first kind of order \( \nu \), denoted \( J_\nu(z) \), is a solution to this differential equation.

\( J_\nu(z) \) can be represented as an ascending series

\[ J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k! \Gamma(k + \nu + 1)}. \]
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It is not difficult to show that for $\nu \notin \mathbb{Z}$, $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent. Since Bessel’s Equation is second order, for non-integer $\nu$ this pair generates all the solutions.

When $\nu = n$ is an integer, $J_n(z)$ is also given by the generating function

$$e^{z\frac{t-1}{t}} = \sum_{n=-\infty}^{\infty} J_n(z) t^n.$$  

Replace $t$ by $-t$ and we find that

$$J_{-n}(z) = (-1)^n J_n(z).$$
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So to obtain the second solution to Bessel’s Equation at integer order, define the Bessel function of the second kind \( Y_n(z) \), \( n \in \mathbb{Z} \), by

\[
Y_n(z) := \lim_{\nu \to n} \frac{J_\nu(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}.
\]

For general \( \nu \), \( Y_\nu(z) \) is defined as above without the limit.

We also have the following ascending series representation for \( Y_n(z) \), \( n \in \mathbb{Z} \).

\[
Y_n(z) = \frac{1}{\pi} \left( 2(\gamma + \log(z/2))J_n(z) - \sum_{k=0}^{n-1} \frac{(n-k-1)!(z/2)^{2k-n}}{k!} \right.
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\left. \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{2k+n}(H_k + H_{k+n})}{k!(n+k)!} \right).
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\]
In addition to the $J$ and $Y$ Bessel functions, there are also the *modified Bessel functions* $I_\nu(z)$ and $K_\nu(z)$, which are solutions to the differential equation

$$z^2 y'' + zy' - (z^2 + \nu^2)y = 0.$$ 

$I_\nu(z)$ is usually known as the Bessel function of imaginary argument, and is related to $J_\nu$ by

$$I_\nu(z) = e^{\pi i \nu/2} J_\nu(iz) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k! \Gamma(k + \nu + 1)},$$

and $K_\nu(z)$ is given by

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As \( z \to \infty \), we have the asymptotic expansion

\[
J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \left(\cos(z - \frac{\nu \pi}{2} - \frac{\pi}{4}) \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2} - \nu)_{2k} (\frac{1}{2} + \nu)_{2k}}{k!(2z)^{2k}} \right. \\
- \sin(z - \frac{\nu \pi}{2} - \frac{\pi}{4}) \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2} - \nu)_{2k+1} (\frac{1}{2} + \nu)_{2k+1}}{k!(2z)^{2k+1}} \left. \right),
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as well as similar expressions for \( Y, I, \) and \( K \).

Here the notation \((a)_k\) is the Pochhammer symbol given by

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(a)_k = a(a+1) \ldots (a+k-1).
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Note that this is an asymptotic series, and diverges for fixed \( z \).
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D. Borwein, J. M. Borwein, O-Y. Chan

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Note that this is an asymptotic series, and diverges for fixed $z$. 
There exists, however, a Hadamard expansion that is convergent:

\[
I_\nu(z) = \frac{e^z}{\Gamma(\nu + \frac{1}{2})\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2} - \nu)_k}{k!(2z)^k} \int_0^{2z} t^{\nu+k-\frac{1}{2}} e^{-t} dt.
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Note that the error behaves like \(N^{\nu-1/2}\) when the series is truncated after \(N\) terms.

For more information on the classical theory, see Watson’s book, “A Treatise on the Theory of Bessel Functions.”
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Bessel’s Equation arises as a special case of Laplace’s Equation with cylindrical symmetry.

Thus, Bessel functions occur often in the study of waves in two dimensions.

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Applications to Number Theory

Hardy and the Circle Problem

Let \( r_2(n) \) denote the number of representations of the positive integer \( n \) as a sum of two squares. The “circle problem” is to determine the precise order of magnitude for the “error term" \( P(x) \) defined by

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\sum_{0 \leq n \leq x} 'r_2(n) = \pi x + P(x),
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where the prime ‘ on the summation sign on the left side indicates that if \( x \) is an integer, only \( \frac{1}{2} r_2(x) \) is counted.
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In 1915, Hardy proved that

\[ \sum_{0 \leq n \leq x} \prime r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left( \frac{x}{n} \right)^{1/2} J_1(2\pi \sqrt{nx}). \]

This is equivalent to the following result of Berndt and Zaharescu

\[ \sum_{0 \leq n \leq x} \prime r_2(n) = \pi x \]

\[ + 2\sqrt{x} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ J_1 \left( \frac{4\pi \sqrt{m(n + \frac{1}{4})x}}{\sqrt{m(n + \frac{1}{4})}} \right) - J_1 \left( \frac{4\pi \sqrt{m(n + \frac{3}{4})x}}{\sqrt{m(n + \frac{3}{4})}} \right) \right\}. \]
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The BZ result is a corollary of an entry on page 335 of Ramanujan’s Lost Notebook.

That entry is one of a pair of equations involving a doubly-infinite series of Bessel functions. If we let

\[ F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \]

where, \([x]\) is the greatest integer less than or equal to \(x\), and

\[ Z_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z); \]

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Then the two entries read
For $x > 0$ and $0 < \theta < 1$,

\[
\sum_{n=1}^{\infty} F \left( \frac{x}{n} \right) \sin(2\pi n\theta) = \pi x \left( \frac{1}{2} - \theta \right) - \frac{1}{4} \cot(\pi\theta)
\]

\[
+ \frac{1}{2} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1 \left( 4\pi \sqrt{m(n + \theta)x} \right)}{\sqrt{m(n + \theta)}} - \frac{J_1 \left( 4\pi \sqrt{m(n + 1 - \theta)x} \right)}{\sqrt{m(n + 1 - \theta)}} \right\}.
\]
For $x > 0$ and $0 < \theta < 1$,

$$\sum_{n=1}^{\infty} F \left( \frac{x}{n} \right) \cos(2\pi n\theta) = \frac{1}{4} - x \log(2 \sin(\pi \theta))$$

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I wanted to evaluate a truncation of the right-hand side of the second entry to verify it against the left-hand side. As an example, computing the right-hand side (with one of the sums truncated at 50 terms, the other truncated at 1000 terms, for a total of 50,000 summands) at 28-digit precision in PARI took nearly 4 hours.

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What is “exp-arc”?

In their recent paper to find asymptotics for Laguerre polynomials with effective (explicit) error bounds, D. Borwein, J. M. Borwein, and R. Crandall were led to consider the following integral

$$I(p, q) := \int_{-\pi/2}^{\pi/2} e^{-i\omega} e^{p \cos \omega} d\omega.$$ 

A simple change of variable yields

$$I(p, q) = 4e^p \int_{0}^{1/\sqrt{2}} \frac{\cosh(-2iq \arcsin x) e^{-2px^2}}{\sqrt{1 - x^2}} dx.$$
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This integral, therefore, reduces to an integral involving $e^{\tau \arcsin x}$, or what BBC calls an “exp-arc” integral.

They then exploit the fact that the exp-arc function has a very nice series expansion on $(-1, 1)$, namely

$$e^{\tau \arcsin x} = 1 + \sum_{k=1}^{\infty} \frac{c_k(\tau) x^k}{k!},$$

where

$$c_{2k+1}(\tau) = \tau \prod_{j=1}^{k} \left( \tau^2 + (2j - 1)^2 \right), \quad c_{2k} = \prod_{j=1}^{k} \left( \tau^2 + (2j - 2)^2 \right).$$
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Plugging this into the expression for $I(p, q)$ and interchanging summation and integration we obtain

$$I(p, q) = 4e^p \sum_{k=0}^{\infty} \frac{g_k(-2iq)}{(2k)!} B_k(p),$$

(1)

where

$$g_0 := 1, \quad g_k(\nu) := \prod_{j=1}^{k} \left( (2j - 1)^2 + \nu^2 \right) \text{ for } k \geq 1,$$

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$$B_k(p) := \int_{0}^{1/\sqrt{2}} x^{2k} e^{-2px^2} dx = \frac{1}{2^k \sqrt{2}} \int_{0}^{1} e^{-pu} u^{k-\frac{1}{2}} du.$$
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Note that

- $g_k$ and $B_k$ are rapidly computable via recursion
- The integral $\int_0^1 e^{-pu} u^{k-1/2} du$ is uniformly bounded for all $k > 0$, and so the $B_k$ decrease geometrically as $2^{-k}$.
- $g_k(\nu)/(2k)!$ are bounded for fixed $\nu$.

Therefore, the series for $\mathcal{I}(p, q)$ is geometrically convergent.
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- The integral $\int_0^1 e^{-pu} u^{k-1/2} du$ is uniformly bounded for all $k > 0$, and so the $B_k$ decrease geometrically as $2^{-k}$.
- $g_k(\nu)/(2k)!$ are bounded for fixed $\nu$.

Therefore, the series for $\mathcal{I}(p, q)$ is geometrically convergent.
Outline

1. What are Bessel Functions?
2. Why do we care?
3. Exp-arc explained
4. Results
Integral Representations

\[ J_\nu(z) = \frac{1}{\pi} \int_{0}^{\pi} \cos(\nu t - z \sin t) \, dt - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} e^{-\nu t - z \sinh t} \, dt, \]

\[ Y_\nu(z) = \frac{1}{\pi} \int_{0}^{\pi} \sin(z \sin t - \nu t) \, dt - \frac{1}{\pi} \int_{0}^{\infty} (e^{\nu t} + e^{-\nu t} \cos \nu \pi) e^{-z \sinh t} \, dt, \]

\[ I_\nu(z) = \frac{1}{\pi} \int_{0}^{\pi} e^{z \cos t} \cos \nu t \, dt - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} e^{-z \cosh t - \nu t} \, dt, \]

and

\[ K_\nu(z) = \int_{0}^{\infty} e^{-z \cosh t} \cosh \nu t \, dt = \frac{1}{2} \int_{-\infty}^{\infty} e^{-z \cosh t - \nu t} \, dt. \]
A quick change of variable allows us to express the finite integrals in terms of $\mathcal{I}(p, q)$.

For integral order, the infinite integrals in $J$ and $I$ disappear due to the $\sin \nu \pi$. Thus we have

$$J_n(z) = \frac{1}{2\pi} \left( e^{-\nu \pi/2} \mathcal{I}(iz, n) + e^{\nu \pi/2} \mathcal{I}(-iz, n) \right),$$

and

$$I_n(z) = \frac{1}{2\pi} \left( \mathcal{I}(z, n) + e^{\nu \pi} \mathcal{I}(-z, n) \right).$$
A quick change of variable allows us to express the finite integrals in terms of $I(p, q)$.

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\[
J_n(z) = \frac{1}{2\pi} \left( e^{-in\pi/2} I(iz, n) + e^{in\pi/2} I(-iz, n) \right),
\]

and

\[
I_n(z) = \frac{1}{2\pi} \left( I(z, n) + e^{\pi in} I(-z, n) \right).
\]
To deal with the general case, we need to evaluate the infinite integrals. A change of variables plus integration by parts gives us

\[
\int_0^\infty e^{-\nu t - z \sinh t} \, dt = \frac{1}{\nu} - \frac{z}{\nu} \int_0^\infty e^{-zs} e^{-\nu \text{arcsinh } s} \, ds.
\]

This is an exp-arc integral, but our expansion is only valid on \([0, 1)\). So what should we do?
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Expand about a point other than zero!

For example, the expansion at infinity is

\[ s^\nu e^{-\nu \arcsinh s} = \sum_{n=0}^{\infty} \frac{A_n(\nu)}{s^{2n}}, \]

where \( A_0(\nu) = 2^{-\nu} \) and for \( n \geq 1 \),

\[ A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu} (\nu + n + 1)_{n-1}}{2^{2n} n!}, \]

This expansion is valid for \(|s| > 1\). The coefficients satisfy

\[ A_n = -\frac{(\nu + 2n - 2)(\nu + 2n - 1)}{4n(n + \nu)} A_{n-1}. \]
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How about at other points?

No nice formula for coefficients, but that’s okay.

For fixed $k$, we have the expansion

$$e^{-\nu \arcsinh(k+s)} = \sum_{n=0}^{\infty} \frac{a_n(k, \nu)}{n!} s^n$$

where for $n \geq 0$,

$$a_{n+2} = \frac{1}{k^2 + 1} \left( (\nu^2 - n^2) a_n - k(2n+1) a_{n+1} \right),$$

and

$$a_0 = (k + \sqrt{k^2 + 1})^{-\nu}, \quad a_1 = -\frac{\nu a_0}{\sqrt{k^2 + 1}}.$$
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Thus for any positive integer \( N \), we have

\[
\int_0^\infty e^{-zs} e^{-\nu \text{arcsinh } s} ds = \sum_{n=0}^\infty \left( \frac{a_n(0, \nu)}{n!} \alpha_n(z) + \beta_n(z) \sum_{k=1}^N e^{-kz} \frac{a_n(k, \nu)}{n!} + A_n(\nu) G_n(N + \frac{1}{2}, z, \nu) \right),
\]

where

\[
\alpha_n(z) := \int_0^{1/2} e^{-zs} s^n ds = -\frac{e^{-z/2}}{2^n z} + \frac{n}{z} \alpha_{n-1}(z),
\]

\[
\beta_n(z) := \int_{-1/2}^{1/2} e^{-zs} s^n ds = \frac{e^{z/2}}{(-2)^n z} - \frac{e^{-z/2}}{2^n z} + \frac{n}{z} \beta_{n-1}(z),
\]
What are Bessel Functions?

Why do we care?

Exp-arc explained

Results

\[ G_n(N, z, \nu) := \frac{e^{-Nz}}{N^{2n+\nu-1}} \int_0^\infty e^{-Nzs}(1 + s)^{-2n-\nu} ds \]

\[ = \frac{1}{(\nu + 2n - 1)(\nu + 2n - 2)} \times \left( \frac{e^{-Nz}(2(n - z - 1) + \nu)}{N^{2n+\nu-1}} + z^2 G_{n-1}(N, z, \nu) \right). \]
Key points:

- All the summands are easily computable via recursion
- The recursions only involve elementary operations. The initial conditions $B_0$ (from $I$) and $G_0$ each require one evaluation of incomplete gamma, which can be done via continued fraction.
- Each series converges geometrically, like $2^{-N}$ (as opposed to the Hadamard expansion for $I_\nu$, which is like $N^{-\nu}$)
- Can choose $N$ large to avoid the $A_n G_n$ sum if we want. In this way, we can pre-compute summands involving only $\nu$ and summands involving only $z$ for “one-$z$ many-$\nu$” or “one-$\nu$ many-$z$” evaluations.
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