AMS-ASL Special Session on Logic and Analysis

Exploratory Experimentation and Computation

Friday January 7, 2011, 8:00 a.m.-11:50 a.m. and 1:00 p.m.-3:50 p.m.

Lonely Planet's top 10 cities

10 images in this story
Travel experts Lonely Planet have named the top 10 cities for 2011 in their annual travel bible, *Best in Travel 2011*. The top-listed cities win points for their local cultures, value for money, and overall va-va-voom. So which cities make the cut? Find out here, from 10 to 1...

What do you think of the list? **Tell us here!**

Related links: Lonely Planet destination videos

A weekend in Newcastle

Images: ThinkStock/Getaway

9. Newcastle, Australia
Where I now live

(red)wine

home
"Intuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication."

"In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting."

George Polya (1887-1985)
Abstract: The mathematical research community is facing a great challenge to re-evaluate the role of proof in light of the growing power of current computer systems, of modern mathematical computing packages, and of the growing capacity to data-mine on the Internet. Add to that the enormous complexity of many modern capstone results such as the Poincaré conjecture, Fermat's last theorem, and the Classification of finite simple groups. As the need and prospects for inductive mathematics blossom, the requirement to ensure the role of proof is properly founded remains undiminished. I shall look at the philosophical context with examples and then offer some of five bench-marking examples of the opportunities and challenges we face. (Related paper with DHB, NAMS in press)
I. Working Definitions and Examples of:
- Discovery
- Proof (and of Mathematics)
- Digital-Assistance
- Experimentation (in Maths and in Science)

II. (Some few of) Five Numbers:
- \( p(n) \)
- \( \pi \)
- \( \phi(n) \)
- \( \zeta(3) \)
- \( \frac{1}{\pi} \)

“Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it.”
- Sir Alec Cairncross, 1996

III. A Cautionary Finale

IV. Making Some Tacit Conclusions Explicit

“Mathematical proofs like diamonds should be hard and clear, and will be touched with nothing but strict reasoning.”
- John Locke
Jonathan Borwein
Keith Devlin
with illustrations by Karl H. Hofmann

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AK Peters 2008     Japan & Germany 2010

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Cookbook Mathematics

- State of the art machine translation
- Math magic melting pot
- Full head mathematicians
- No wonder Sergei Brin wants more
“This is the essence of science. Even though I do not understand quantum mechanics or the nerve cell membrane, I trust those who do. Most scientists are quite ignorant about most sciences but all use a shared grammar that allows them to recognize their craft when they see it.

The motto of the Royal Society of London is 'Nullius in verba': trust not in words. Observation and experiment are what count, not opinion and introspection. Few working scientists have much respect for those who try to interpret nature in metaphysical terms. For most wearers of white coats, philosophy is to science as pornography is to sex: it is cheaper, easier, and some people seem, bafflingly, to prefer it. Outside of psychology it plays almost no part in the functions of the research machine.” - Steve Jones

• From his 1997 NYT BR review of Steve Pinker’s *How the Mind Works*. 
“discovering a truth has three components. First, there is the independence requirement, which is just that one comes to believe the proposition concerned by one’s own lights, without reading it or being told. Secondly, there is the requirement that one comes to believe it in a reliable way. Finally, there is the requirement that one’s coming to believe it involves no violation of one’s epistemic state. ... In short, discovering a truth is coming to believe it in an independent, reliable, and rational way.”


“All truths are easy to understand once they are discovered; the point is to discover them.” – Galileo Galilei
Galileo was not alone in this view

“I will send you the proofs of the theorems in this book. Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding proofs of these same theorems.

For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.” - Archimedes (287-212 BCE)

Archimedes to Eratosthenes in the introduction to The Method in Mario Livio's, Is God a Mathematician? Simon and Schuster, 2009
The Archimedes Palimpsest

- 1906 10th-century palimpsest was discovered in Constantinople (Codex C). 1998 bought at auction for $2 million 98-2008 “reconstructed”
- contained works of Archimedes that, sometime before April 14th 1229, were partially erased, cut up, and overwritten by religious text
- after 1929 painted over with gold icons and left in a wet bucket in a garden. It included bits of 7 texts such as On Floating Bodies and of the Method of Mechanical Theorems, thought lost
- Archimedes used knowledge of levers and centres of gravity to envision ways of balancing geometric figures against one another which allowed him to compare their areas or volumes. He then used rigorous geometric argument to prove Method discoveries:

"... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge." (The Method)


Creative commons: http://www.archimedespalimpsest.net
1a. A Recent Discovery (July 2009)
("independent, reliable and rational")

The $n$-dimensional integral

$$W_n(s) := \int_0^1 \int_0^1 \cdots \int_0^1 \left| \sum_{k=1}^{n} e^{2\pi x k i} \right|^s dx_1 dx_2 \cdots dx_n$$

occurs in the study of uniform random walks in the plane. $W_n(1)$ is the expected distance moved after $n$ steps.

$$W_1(1) = 1 \quad W_2(1) = \frac{4}{\pi}$$

$$W_3(1) \approx \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6 \left( \frac{1}{3} \right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6 \left( \frac{2}{3} \right). \quad (1)$$

(1) has been checked to 170 places on 256 cores in about 15 minutes. It originates with our proof (JMM-Nuyens-Straub-Wan) that for $k = 0, 1, 2, 3, \ldots$

We proved the formula below for $2k$ (it counts abelian squares) and numerically observed it was half-true at $k=1/2$. We confirmed (1) to 175 digits well before proof (my seminar)

$$W_3(2k) = \text{$_3F_2$} \left( \frac{1}{2}, -k, -k \mid 4 \right)$$

and $W_3(1) \approx \text{Re} \text{$_3F_2$} \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \mid 4 \right)$
WHAT is MATHEMATICS?

MATHEMATICS, n. a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.

- This definition, from my Collins Dictionary has no mention of proof, nor the means of reasoning to be allowed (vidé Giaquinto). Webster's contrasts:

INDUCTION, n. any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.

and

DEDUCTION, n. a. a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.

b. a conclusion reached by this process.

“If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.” - Kurt Gödel (in his 1951 Gibbs Lecture)
“PROOF, n. a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established. A direct proof proceeds linearly from premises to conclusion; an indirect proof (also called reductio ad absurdum) assumes the falsehood of the desired conclusion and shows that to be impossible. See also induction, deduction, valid.”

Borowski & JB, Collins Dictionary of Mathematics

INDUCTION, n. 3. (Logic) a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them. Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white.

“No. I have been teaching it all my life, and I do not want to have my ideas upset.” - Isaac Todhunter (1820-1884) recording Maxwell's response when asked whether he would like to see an experimental demonstration of conical refraction.
Decide for yourself
WHAT is DIGITAL ASSISTANCE?

- Use of Modern Mathematical Computer Packages
  - Symbolic, Numeric, Geometric, Graphical, ...

- Use of More Specialist Packages or General Purpose Languages
  - Fortran, C++, CPLEX, GAP, PARI, MAGMA, ...

- Use of Web Applications
  - Sloane’s Encyclopedia, Inverse Symbolic Calculator, Fractal Explorer, Euclid in Java, Weeks’ Topological Games, Polymath (Sci. Amer.), ...

- Use of Web Databases

- All entail data-mining [“exploratory experimentation” and “widening technology” as in pharmacology, astrophysics, biotech, ... (Franklin)]
  - Clearly the boundaries are blurred and getting blurrier
  - Judgments of a given source’s quality vary and are context dependent

*Knowing things is very 20th century. You just need to be able to find things.*

Danny Hillis on how Google has already changed how we think in Achenblog, July 1 2008
- changing cognitive styles
Franklin argues that Steinle's "exploratory experimentation" facilitated by "widening technology", as in pharmacology, astrophysics, medicine, and biotechnology, is leading to a reassessment of what legitimates experiment; in that a "local model" is not now prerequisite.

Hendrik Sørenson cogently makes the case that experimental mathematics (as ‘defined’ below) is following similar tracks:

“These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation is also pertinent to mathematics.”

In consequence, boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting more so.
Changing User Experience and Expectations

What is attention? (Stroop test, 1935)

1. Say the color represented by the word.
2. Say the color represented by the font color.

High (young) multitaskers perform #2 very easily. They are great at suppressing information.

http://www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif

Acknowledgements: Cliff Nass, CHIME lab, Stanford (interference and twitter?)
Experimental Methodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically looked to the defining trappings of science: laboratory equipment. Physicists have their particle accelerators, biologists, their electron microscopes, and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their imagination.

New computer programs are starting to give mathematicians the lab instruments that they have been missing. Sophisticated software is enabling researchers to move far more easily and deeper into the mathematical universe. They’re calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the continuities of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discover insights that they might never have reached by traditional means. "Pretty much every mathematical field has been transformed," says Richard Canary, a mathematician at Reed College in Portland, Ore. "Instead of having a number-crunching tool, the computer is becoming more like a gadget that turns over cubes and you find things underneath."

At the same time, the new work is raising unsettling questions about how we regard experimental results.

"I have some of the excitement that Leonardo da Vinci must have felt when he encountered an artistic original. It suddenly made certain calculations seem easy and new," says that's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD: In one sense, math experimenters are nothing new. Despite their field’s reputation as a purely deductive science, the great mathematicians of the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1665, the centrifugal and law of Newton led Isaac Newton to calculate directly the first 35 digits of the number pi, later writing, "I am ashamed to tell you my new numbers I carried these computations, having no other business at the time."

Carl Friedrich Gauss, one of the towering figures of 18th-century mathematics, famously discovered new mathematical results by experimenting with numerical tables. "I have the result, but I do not yet know how to get it," Gauss often said of his discoveries. Experimentalists today, he said, might not even know they have the results. "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Euler later refined the conjecture and recorded figures, but never proved it. It took more than a century for mathematicians to come up with a proof. Like da Vinci’s mathematician, mathematicians can include centuries and decades — but do so with a special facility for calculus.
In 1995 or so Andrew Granville emailed me the number

\[ \alpha := 1.433127426722312 \ldots \]

and challenged me to identify it (our inverse calculator was new in those days).

**Changing representations**, I asked for its continued fraction? It was

\[ [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots ] \quad (1) \]

I reached for a good book on continued fractions and found the answer

\[ \alpha = \frac{I_0(2)}{I_1(2)} \]

where \( I_0 \) and \( I_1 \) are **Bessel functions** of the first kind. (Actually I knew that all arithmetic continued fractions arise in such fashion).

In 2010 there are at least **three** other strategies:

- Given (1), type “arithmetic progression”, “continued fraction” into Google
- Type “1,4,3,3,1,2,7,4,2” into Sloane’s Encyclopaedia of Integer Sequences

I illustrate the outcomes on the next few slides:
Continued Fraction Constant -- from Wolfram MathWorld

- 3 visits - 14/09/07

Perron (1954-57) discusses continued fractions having terms even more general than the arithmetic progression and relates them to various special functions.

mathworld.wolfram.com/ContinuedFractionConstant.html - 31k

HAKMEM -- CONTINUED FRACTIONS -- DRAFT, NOT YET PROOFED

The value of a continued fraction with partial quotients increasing in arithmetic progression is

\[ I \left( \frac{2}{D} \right) \frac{A}{D} \left[ A+D, A+2D, A+3D, \ldots \right] \]

www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

On simple continued fractions with partial quotients in arithmetic ...

0. This means that the sequence of partial quotients of the continued fractions under investigation consists of finitely many arithmetic progressions (with ...


Moreover the MathWorld entry includes

\[ [A + D, A + 2D, A + 3D, \ldots] = \frac{I_{A/D} \left( \frac{2}{D} \right)}{I_{1+A/D} \left( \frac{2}{D} \right)} \]

(Schroeppe, 1972) for real \( A \) and \( D \neq 0 \).
In the Integer Sequence Data Base

Greetings from The On-Line Encyclopedia of Integer Sequences!

1, 4, 3, 3, 1, 2, 7, 4, 2

The Inverse Calculator returns

Best guess:
BesI(0,2)/BesI(1,2)

- We show the ISC on another number next
- Most functionality of ISC is built into “identify” in Maple.
- There’s also Wolfram α

The Inverse Symbolic Calculator (ISC) uses a combination of lookup tables and integer relation algorithms in order to associate with a user-defined, truncated decimal expansion (represented as a floating point expression) a closed form representation for the real number.

Standard lookup results for $12.587886229548403854$

$\exp(1)+\pi^2$

3.146264370 [Try it!]

19.99909998 [Try it!]

**ISC+ now runs at CARMA**

**Less lookup & more algorithms than 1995**
Inverse Symbolic Computation

Inferring mathematical structure from numerical data

- Mixes *large table lookup*, integer relation methods and intelligent preprocessing – needs **micro-parallelism**
- It faces the “curse of exponentiality”
- Implemented as **identify** in **Maple 9.5**
“This is an exceptionally important book. . . . It could be the starting point for many cognitive, social, and educational benefits.”
—From the Foreword by William Higginson, Queen’s University, Canada

“In a time of much sterile math teaching and grimly utilitarian school reform, this elegant and beautiful book brings to life a whole new vision. . . . Nathalie Sinclair makes a brilliant case for rethinking math instruction so that an aesthetically rich learning environment becomes the path to meaning, intellectual journeys, and—dare we say the word?—pleasure.”
—Joseph Featherstone, Michigan State University

In this innovative book, Nathalie Sinclair makes a compelling case for the inclusion of the aesthetic in the teaching and learning of mathematics. Using a provocative set of philosophical, psychological, mathematical, technological, and educational insights, she illuminates how the materials and approaches we use in the mathematics classroom can be enriched for the benefit of all learners. While ranging in scope from the young learner to the professional mathematician, there is a particular focus on middle school, where negative feelings toward mathematics frequently begin. Offering specific recommendations to help teachers evoke and nurture their students’ aesthetic abilities, this book:

- Features powerful episodes from the classroom that show students in the act of developing a sense of mathematical aesthetics.
- Analyzes how aesthetic sensibilities to qualities such as connectedness, fruitfulness, apparent simplicity, visual appeal, and surprise are fundamental to mathematical inquiry.
- Includes examples of mathematical inquiry in computer-based learning environments, revealing some of the roles they play in supporting students’ aesthetic inclinations.

Nathalie Sinclair is an assistant professor in the Department of Mathematics at Michigan State University.

ALSO OF INTEREST—
Improving Access to Mathematics: Diversity and Equity in the Classroom
Na’ilah Suad Nasir and Paul Cobb, Editors
2007/Paper and cloth
In the course of studying multiple zeta values we needed to obtain the closed form partial fraction decomposition for

\[ \frac{1}{x^s(1-x)^t} = \sum_{j \geq 0} \frac{a_{s,t}^j}{x^j} + \sum_{j \geq 0} \frac{b_{s,t}^j}{(1-x)^j} \]

\[ a_{s,t}^j = \begin{pmatrix} s + t - j - 1 \\ s - j \end{pmatrix} \]

This was known to Euler but is easily discovered in Maple. We needed also to show that \( M = A + B - C \) is \textit{invertible} where the \( n \times n \) matrices \( A, B, C \) respectively had entries

\[
(-1)^{k+1} \begin{pmatrix} 2n - j \\ 2n - k \end{pmatrix}, \quad (-1)^{k+1} \begin{pmatrix} 2n - j \\ k - 1 \end{pmatrix}, \quad (-1)^{k+1} \begin{pmatrix} j - 1 \\ k - 1 \end{pmatrix}
\]

Thus, \( A \) and \( C \) are triangular and \( B \) is full.

After messing with many cases I thought to ask for \( M \)'s minimal polynomial

\[
> \text{linalg[minpoly]}(M(12),t); \quad -2 + t + t^2
\]

\[
> \text{linalg[minpoly]}(B(20),t); \quad -1 + t^3
\]

\[
> \text{linalg[minpoly]}(A(20),t); \quad -1 + t^2
\]

\[
> \text{linalg[minpoly]}(C(20),t); \quad -1 + t^2
\]
Once this was discovered proving that for all $n > 2$

\[ A^2 = I, \quad BC = A, \quad C^2 = I, \quad CA = B^2 \]

is a nice combinatorial exercise *(by hand or computer)*. Clearly then

\[
B^3 = B \cdot B^2 = B(CA) = (BC)A = A^2 = I
\]

and the formula

\[
M^{-1} = \frac{M + I}{2}
\]

is again a fun exercise in formal algebra; as is confirming that we have discovered an amusing presentation of the symmetric group $S_3$.

- **characteristic and minimal polynomials** --- which were rather abstract for me as a student --- now become members of a rapidly growing box of symbolic tools, as do many matrix decompositions, etc …

- a **typical** matrix has a full degree minimal polynomial

“Why should I refuse a good dinner simply because I don't understand the digestive processes involved?” - Oliver Heaviside (1850-1925)
2. Phase Reconstruction

**Projectors and Reflectors**: $P_A(x)$ is the metric projection or nearest point and $R_A(x)$ reflects in the tangent: $x$ is red

"All physicists and a good many quite respectable mathematicians are contemptuous about proof."
G. H. Hardy (1877-1947)
Interactive exploration in CINDERELLA

The simplest case is of a line $A$ of height $h$ and the unit circle $B$. With $z_n := (x_n, y_n)$ the iteration becomes

$$x_{n+1} := \cos \theta_n, \quad y_{n+1} := y_n + h - \sin \theta_n, \quad (\theta_n := \arg z_n)$$

A Cinderella picture of two steps from $(4.2, -0.51)$ follows:
Numerical errors in using double precision

Stability using Maple input

Computer Algebra + Interactive Geometry
the Grief is in the GUI
This picture is worth 100,000 ENIACs

The number of ENIACS needed to store the 20Mb TIF file the Smithsonian sold me

Eckert & Mauchly (1946)

The past
Projected Performance
As of early 2011 one will be able to order an Apple desktop machine with appropriate graphics (GPU) cards and software, to achieve on certain problems a teraflop.

Moreover, double-precision floats on GPU will finally be available.

So, again on certain problems, this will be 1000x or so faster than we desk-denizens are.
“The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. 'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation.” - Hermann Weyl

Consider the number of additive partitions, \( p(n) \), of \( n \). Now
\[
5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1
\]
so \( p(5)=7 \). The ordinary generating function discovered by Euler is
\[
\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1 - q^k}^{-1}.
\]
(Use the geometric formula for \( 1/(1-q^k) \) and observe how powers of \( q^n \) occur.)

The famous computation by MacMahon of \( p(200) = 3972999029388 \) done symbolically and entirely naively using (1) on an Apple laptop took 20 min in 1991, and about 0.17 seconds in 2009. Now it took 2 min for \( p(2000) = 47208191756194138886014324067999959512200344166 \).

In 2008, Crandall found \( p(10^9) \) in 3 seconds on a laptop, using the Hardy-Ramanujan-Rademacher „finite“ series for \( p(n) \) with FFT methods. Such fast partition-number evaluation let Crandall find probable primes \( p(1000046356) \) and \( p(1000007396) \). Each has roughly 35,000 digits.

When does easy access to computation discourages innovation: would Hardy and Ramanujan have still discovered their marvellous formula for \( p(n) \)?
"You can't imagine how tight our budget is. We can only work with single-digit numbers."
IIb. The computation of Pi (1986-2010)

These equations specify an algebraic number:

$$1/\pi \sim a_{20}$$

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

and

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3}y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then $1/a_k$ converges quartically to $\pi$. 

BB4: Pi to 2.59 trillion places in 21 steps
Moore’s Law Marches On

1986: It took Bailey 28 hours to compute 29.36 million digits on 1 cpu of the then new CRAY-2 at NASA Ames using (BB4). Confirmation using another BB quadratic algorithm took 40 hours. This uncovered hardware+software errors on the CRAY.

2009 Takahashi on 1024 cores of a 2592 core Appro Xtreme - X3 system 1.649 trillion digits via (Salamin-Brent) took 64 hours 14 minutes with 6732 GB of main memory, and (BB4) took 73 hours 28 minutes with 6348 GB of main memory.

The two computations differed only in the last 139 places.

Fabrice Bellard (Dec 2009) 2.7 trillion places on a 4 core desktop in 133 days after 2.59 trillion by Takahashi (April).

2010: 5 trillion digits (see my Lecture The Life of Pi)

“The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so.” - Leonard Carleson (Lusin’s problem on p.w. convergence of Fourier series in Hilbert space)
IF THERE WERE COMPUTERS IN GALILEO'S TIME
As another measure of what changes over time and what doesn't, consider two conjectures regarding Euler's totient $\phi(n)$ which counts positive numbers less than and relatively prime to $n$.

**Giuga's conjecture (1950)** $n$ is prime if and only if

$$G_n := \sum_{k=1}^{n-1} k^{n-1} \equiv (n - 1) \mod n.$$  

Counterexamples are **Carmichael numbers** (rare birds only proven infinite in 1994) and more: if a number $n = p_1 \cdots p_m$ with $m>1$ prime factors $p_i$ is a counterexample to Giuga's conjecture then the primes are distinct and satisfy

$$\sum_{i=1}^{m} \frac{1}{p_i} > 1$$

and they form a **normal sequence**: $p_i \neq 1 \mod p_j$

(3 rules out 7, 13, 19, 31,... and 5 rules out 11, 31, 41,...)
Guiga’s Conjecture (1951-2009)

With predictive experimentally-discovered heuristics, we built an efficient algorithm to show (in several months in 1995) that any counterexample had \(3459\) prime factors and so exceeded \(10^{13886} \rightarrow 10^{14164}\) in a 5 day desktop 2002 computation.

The method fails after \(8135\) primes—my goal is to exhaust it.

2009 While preparing this talk, I obtained almost as good a bound of \(3050\) primes in under 110 minutes on my notebook and a bound of \(3486\) primes in 14 hours: using Maple not as before C++ which being compiled is faster but in which the coding is much more arduous.

One core of an eight-core MacPro obtained \(3592\) primes and so exceeds \(16673\) digits in 13.5 hrs in Maple. (Now running on 8 cores.)
Lehmer’s Conjecture (1932-2009)

A tougher related conjecture is

Lehmer's conjecture (1932)  $n$ is prime if and only if 
\[ \phi(n) \mid (n - 1) \]

He called this “as hard as the existence of odd perfect numbers.”

Again, prime factors of counterexamples form a normal sequence, but now there is little extra structure.

In a 1997 SFU M.Sc. Erick Wong verified this for 14 primes, using normality and a mix of PARI, C++ and Maple to press the bounds of the "curse of exponentiality."

The related $\phi(n) \mid (n+1)$ is has 8 solutions with at most 7 factors (6 factors is due to Lehmer).

Recall $F_n := 2^{2n} + 1$ the Fermat primes. The solutions are 2, 3, 3.5, 3.5.17, 3.5.17.257, 3.5.17.257.65537 and a rogue pair: 4919055 and 6992962672132095, but 8 factors seems out of sight.

Lehmer “couldn’t” factor $6992962672132097 = 73 \times 95794009207289$. If prime, a 9th would exist: $\phi(n) \mid (n+1)$ and $n+2$ prime $\Rightarrow N := n(n+2)$ satisfies $\phi(N) \mid (N+1)$.
"Vacuums, black holes, antimatter - it's the elusive and intangible which appeals to me."
The following formulas for $\zeta(n)$ have been known for many decades:

(a) $\zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}$, 

(b) $\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \binom{2k}{k}}{k^3}$, 

(c) $\zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}$.

These results have led many to speculate that $Q_5 := \frac{\zeta(5)}{\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \binom{2k}{k}}{k^5}}$ might be some nice rational or algebraic value.

Sadly, PSLQ calculations have established that if $Q_5$ satisfies a polynomial with degree at most 25, then at least one coefficient has 380 digits.

"He (Gauss) is like the fox, who effaces his tracks in the sand with his tail."  -  Niels Abel (1802-1829)
Two more things about $\zeta(5)$

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} = 2\zeta(5) - \frac{4}{3} L^5 + \frac{8}{3} L^3 \zeta(2) + 4L^2 \zeta(3)
\]

\[+80 \sum_{n>0} \left( \frac{1}{(2n)^5} - \frac{L}{(2n)^4} \right) \rho^{2n} \]

Here $\rho := \frac{\sqrt{5} - 1}{2}$ and $L := \log \rho$

(JMB-Broadhurst-Kamnitzer, 2000).

Also, there is a simpler Ramanujan series for $\zeta(4n + 1)$. In particular:

\[
\zeta(5) = \frac{1}{294} \pi^5 + \frac{2}{35} \sum_{k=1}^{\infty} \frac{1}{(1 + e^{2k\pi}) k^5} + \frac{72}{35} \sum_{k=1}^{\infty} \frac{1}{(1 - e^{2k\pi}) k^5},
\]

and $\zeta(5) - \pi^5 / 294 = -0.0039555 \ldots$
Nothing New under the Sun

Margo Kondratieva found a formula of Markov in 1890:

\[
\sum_{n=1}^{\infty} \frac{1}{(n+a)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^6}{(2n+1)!} \times \frac{(5 (n+1)^2 + 6 (a-1) (n+1) + 2 (a-1)^2)}{\prod_{k=0}^{n} (a+k)^4}.
\]

Note: Maple establishes this identity as

\[-1/2 \psi (2, a) = -1/2 \psi (2, a) - \zeta (3) + 5/4 \, _4F_3 ([1, 1, 1, 1], [3/2, 2, 2], -1/4)\]

Hence

\[\zeta (4) = - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\binom{2m}{m} m^4} + \frac{10}{3} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \sum_{k=1}^{m} \frac{1}{k}}{\binom{2m}{m} m^3}.
\]

♦ The case \(a=0\) above is Apéry's formula for \(\zeta(3)\)!

Andrei Andreyevich Markov (1856-1922)
Two Discoveries: 1995 and 2005

♦ Two computer-discovered generating functions
  ▪ (1) was „intuited“ by Paul Erdös (1913-1996)
  ▪ and (2) was a designed experiment
    ▪ was proved by the computer (Wilf-Zeilberger)
    ▪ and then by people (Wilf included)
    ▪ What about 4k+1?

\[
\sum_{k=0}^{\infty} \zeta(4k + 3) x^{4k} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k} (1 - x^4/k^4)} \prod_{m=1}^{k-1} \left( \frac{1 + 4x^4/m^4}{1 - x^4/m^4} \right) \quad (1)
\]

\[
x=0 \text{ gives (b) and (a) respectively}
\]

\[
\sum_{k=0}^{\infty} \zeta(2k + 2) x^{2k} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k} (1 - x^2/k^2)} \prod_{m=1}^{k-1} \left( \frac{1 - 4x^2/m^2}{1 - x^2/m^2} \right) \quad (2)
\]
Apéry summary

1. via PSLQ to 5,000 digits (120 terms)

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots \]

\[ \zeta(x) = 3 \sum_{k=1}^{\infty} \frac{1}{(2k^k)} \frac{1}{(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2} \]

\[ = \sum_{k=0}^{\infty} \zeta(2k + 2)x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2} \]

\[ = \frac{1 - \pi x \cot(\pi x)}{2x^2} \]

2. reduced as hoped

2005 Bailey, Bradley & JMB discovered and proved - in 3Ms - three equivalent binomial identities

\[ 3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\left(\frac{2k}{k}\right) \left(k^2 - n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}} \]

3. was easily computer proven (Wilf-Zeilberger) (now 2 human proofs)

\[ _3F_2 \left( \begin{array}{c} 3n, n + 1, -n \\ 2n + 1, n + 1/2, 4 \end{array} \right) = \frac{\binom{2n}{n}}{\binom{3n}{n}} \]

Euler (1707-73)
II e: Ramanujan-Like Identities

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910. One is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}. \tag{1}$$

Each term of (1) adds 8 correct digits. Gosper used (1) by the computation of a then-record 17 million digits of the c.f. for $\pi$ in 1985—completing the first proof of (1).

A little later David and Gregory Chudnovsky found the following variant, which lies in $\mathbb{Q}(\sqrt{-163})$ rather than $\mathbb{Q}(\sqrt{58})$:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}. \tag{2}$$

Each term of (2) adds 14 correct digits.

The brothers used (2) several times --- culminating in a 1994 calculation to over four billion decimal digits. Their remarkable story was told in a Pulitzer-winning New Yorker article.
New Ramanujan-Like Identities

Guillera has recently found Ramanujan-like identities, including:

\[
\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left( \frac{1}{32} \right)^{2n}
\]

\[
\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left( \frac{1}{2} \right)^{2n}
\]

\[
\frac{32}{\pi^3} = ? \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left( \frac{1}{8} \right)^{2n}
\]

where

\[
r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n - 1)/2}{n!} = \frac{\Gamma(n + 1/2)}{\sqrt{\pi} \Gamma(n + 1)}
\]

Guillera proved the first two using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods. It is true but has no proof.

As far as we can tell there are no higher-order analogues!
Example of Use of Wilf-Zeilberger, I

The first two recent experimentally-discovered identities are

\[
\sum_{n=0}^{\infty} \frac{(4n)(2n)^4}{2^{16n}} \frac{(2n)}{n} \left(120n^2 + 34n + 3\right) = \frac{32}{\pi^2}
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(2n)^5}{2^{20n}} \left(820n^2 + 180n + 13\right) = \frac{128}{\pi^2}
\]

Guillera \textit{cunningly} started by defining

\[
G(n, k) = \frac{(-1)^k}{2^{16n}2^{4k}} \left(120n^2 + 84nk + 34n + 10k + 3\right) \frac{(2n)^4}{\binom{2n}{k}} \frac{(2k)^3}{\binom{2k}{k}} \frac{(4n-2k)}{\binom{2n-k}{2n-k}} \frac{1}{\binom{n+k}{n}^2}
\]

He then used the \textbf{EKHAD} software package to obtain the companion

\[
F(n, k) = \frac{(-1)^k 512}{2^{16n}2^{4k}} \frac{n^3}{4n - 2k - 1} \frac{(2n)^4}{\binom{2n}{k}} \frac{(2k)^3}{\binom{2k}{k}} \frac{(4n-2k)}{\binom{2n-k}{2n-k}} \frac{1}{\binom{n+k}{n}^2}
\]
When we define

\[ H(n, k) = F(n + 1, n + k) + G(n, n + k) \]

Zeilberger’s theorem gives the identity

\[ \sum_{n=0}^{\infty} G(n, 0) = \sum_{n=0}^{\infty} H(n, 0) \]

which when written out is

\[ \sum_{n=0}^{\infty} \frac{(2n)^4 (4n^2)_{2n}}{2^{16n}} \left(120n^2 + 34n + 3 \right) = \sum_{n=0}^{\infty} \frac{(-1)^n (n + 1)^3 (2n+2)^4 (2n)^3 (2n+4)^n}{2^{20n+7} 2n + 3 \binom{2n+2}{n+1}^2 \binom{2n+4}{n+2}^n} \]

\[ + \sum_{n=0}^{\infty} \frac{(-1)^n (204n^2 + 44n + 3)}{2^{20n}} \binom{2n}{n}^5 = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)^5}{2^{20n}} \left(820n^2 + 180n + 13 \right) \]

A limit argument and Carlson’s theorem completes the proof…
Searches for Additional Formulas

We had no PSLQ over number fields so we searched for additional formulas of either the following forms:

\[
\frac{c}{\pi^m} = \sum_{n=0}^{\infty} r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}
\]

\[
\frac{c}{\pi^m} = \sum_{n=0}^{\infty} (-1)^n r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}
\]

where \(c\) is some linear combination of

\[
1, 2^{1/2}, 2^{1/3}, 2^{1/4}, 2^{1/6}, 4^{1/3}, 8^{1/4}, 32^{1/6}, 3^{1/2}, 3^{1/3}, 3^{1/4}, 3^{1/6}, 9^{1/3},
27^{1/4}, 243^{1/6}, 5^{1/2}, 5^{1/4}, 125^{1/4}, 7^{1/2}, 13^{1/2}, 6^{1/2}, 6^{1/3}, 6^{1/4}, 6^{1/6},
7, 36^{1/3}, 216^{1/4}, 7776^{1/6}, 12^{1/4}, 108^{1/4}, 10^{1/2}, 10^{1/4}, 15^{1/2}
\]

where each of the coefficients \(p_i\) is a linear combination of

\[
1, 2^{1/2}, 3^{1/2}, 5^{1/2}, 6^{1/2}, 7^{1/2}, 10^{1/2}, 13^{1/2}, 14^{1/2}, 15^{1/2}, 30^{1/2}
\]

and where \(\alpha\) is chosen as one of the following:

\[
1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, \sqrt{5} - 2, (2 - \sqrt{3})^2,
5\sqrt{13} - 18, (\sqrt{5} - 1)^4/128, (\sqrt{5} - 2)^4, (2^{1/3} - 1)^4/2, 1/(2\sqrt{2}),
(\sqrt{2} - 1)^2, (\sqrt{5} - 2)^2, (\sqrt{3} - \sqrt{2})^4
\]
Relations Found by PSLQ
- Including Guillera’s three we found all known series for \( r(n) \) and no more.
- There are others for other pochhammer symbols

\[
\frac{4}{\pi} = \sum_{n=0}^{\infty} r(n)^3(1 + 6n) \left(\frac{1}{2}\right)^{2n}
\]

\[
\frac{16}{\pi} = \sum_{n=0}^{\infty} r(n)^3(5 + 42n) \left(\frac{1}{8}\right)^{2n}
\]

\[
\frac{12^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-15 + 9\sqrt{3} - 36n + 24\sqrt{3}n) \left(2 - \sqrt{3}\right)^{4n}
\]

\[
\frac{32}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-1 + 5\sqrt{5} + 30n + 42\sqrt{5}n) \left(\frac{(\sqrt{5} - 1)^4}{128}\right)^{2n}
\]

\[
\frac{5^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-525 + 235\sqrt{5} - 1200n + 540\sqrt{5}n) \left(\sqrt{5} - 2\right)^{8n}
\]

\[
\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(1 + 6n) \left(\frac{1}{2\sqrt{2}}\right)^{2n}
\]

\[
\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(-5 + 4\sqrt{2} - 12n + 12\sqrt{2}n) \left(\sqrt{2} - 1\right)^{4n}
\]

\[
\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(23 - 10\sqrt{5} + 60n - 24\sqrt{5}n) \left(\sqrt{5} - 2\right)^{4n}
\]

\[
\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(177 - 72\sqrt{6} + 420n - 168\sqrt{6}n) \left(\sqrt{3} - \sqrt{2}\right)^{8n}
\]

Baruah, Berndt, Chan, “Ramanujan Series for \(1/\pi\). A Survey.” Aug 09, MAA Monthly
"What I appreciate even more than its remarkable speed and accuracy are the words of understanding and compassion I get from it."
III. A Cautionary Example

These constants agree to 42 decimal digits accuracy, but are NOT equal:

$$\int_0^\infty \cos(2x) \prod_{n=0}^\infty \cos(x/n) \, dx =$$

$$0.39269908169872415480783042290993786052464543418723\ldots$$

$$\frac{\pi}{8} =$$

$$0.39269908169872415480783042290993786052464617492189\ldots$$

Computing this integral is (or was) nontrivial, due largely to difficulty in evaluating the integrand function to high precision.

Fourier analysis explains this happens when a hyperplane meets a hypercube (LP) …
IV. Some Conclusions

- We like students of 2010 live in an information-rich, judgement-poor world
- The explosion of information is not going to diminish
  - nor is the desire (need?) to collaborate remotely
- So we have to learn and teach judgement (not obsession with plagiarism)
  - that means mastering the sorts of tools I have illustrated
- We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (few future mathematicians)
  - properly balanced, discovery and proof can live side-by-side and allow for the ordinary and the talented to flourish in their own fashion
- Impediments to the assimilation of the tools I have illustrated are myriad
  - as I am only too aware from recent experiences
- These impediments include our own inertia and
  - organizational and technical bottlenecks (IT - not so much dollars)
  - under-prepared or mis-prepared colleagues
  - the dearth of good modern syllabus material and research tools
  - the lack of a compelling business model (societal goods)

“The plural of 'anecdote' is not 'evidence'.”
- Alan L. Leshner (Science's publisher)
Further Conclusions

New techniques now permit integrals, infinite series sums and other entities to be evaluated to high precision (hundreds or thousands of digits), thus permitting PSLQ-based schemes to discover new identities.

These methods typically do not suggest proofs, but often it is much easier to find a proof (say via WZ) when one “knows” the answer is right.

Full details of all the examples are in *Mathematics by Experiment* or its companion volume *Experimentation in Mathematics* written with Roland Girgensohn. A “Reader’s Digest” version of these is available at [www.experimentalmath.info](http://www.experimentalmath.info) along with much other material.

“Anyone who is not shocked by quantum theory has not understood a single word.” - Niels Bohr