Future Prospects for Computer-assisted Mathematics (CMS Notes 12/05)

Effective Computation of Bessel Functions
David Borwein, Jonathan M. Borwein, and O-Yeat Chan
AMS-SIAM Session on Asymptotic Methods in Analysis with Apps, Jan 6th 2008
Talk at www.cs.dal.ca/~jborwein

"Harald Bohr is reported to have remarked ‘Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.’ ”

(D.J.H. Garling)


Harald Bohr 1887-1951

Revised 04/01/2007
The following is a list to useful math tools.

**Utilities**

1. ISC2.0: The Inverse Symbolic Calculator
2. EZ Face: An interface for evaluation of Euler sums and Multiple Zeta Values
3. 3D Function Grapher
4. GraPHedron: Automated and computer assisted conjectures in graph theory
5. Julia and Mandelbrot Set Explorer

**Reference**

6. The On-Line Encyclopedia of Integer Sequences
7. Finch's Mathematical Constants

8. The Digital Library of Mathematical Functions
NIST Digital Library of Mathematical Functions

1. Algebraic & Analytic Methods
2. Asymptotic Approximations
3. Numerical Methods
4. Elementary Functions
5. Gamma Function
6. Exponential, Logarithmic, Sine & Cosine Integrals
7. Error Functions, Dawson's & Fresnel Integrals
8. Incomplete Gamma & Related Functions
9. Airy & Related Functions
10. Bessel Functions
11. Struve & Related Functions
12. Parabolic Cylinder Functions
13. Confluent Hypergeometric Functions
14. Legendre & Associated Legendre Functions
15. Hypergeometric Function
16. Generalized Hypergeometric Functions & Meijer G-Function
17. q-Hypergeometric Functions
18. Orthogonal Polynomials
19. Elliptic Integrals
20. Theta Functions
21. Multidimensional Theta Functions
22. Jacobian Elliptic Functions
23. Weierstrass Elliptic & Modular Functions
24. Bernoulli & Euler Polynomials
25. Zeta & Related Functions
26. Combinatorial Analysis
27. Functions of Number Theory
28. Mathieu Functions & Hill's Equation
29. Lamé Functions
30. Spheroidal Wave Functions
31. Heun Functions
32. Painlevé Transcendents
33. Coulomb Functions
34. 3j, 6j, 9j Symbols
35. Functions of Matrix Argument
36. Integrals with Coalescing Saddles
37. Computer Algebra

Karl Dilcher

Current Content Status
Bessel functions are among the most important functions in mathematical physics and the theory of special functions. The ability to compute their values is equally important.

The standard method of evaluating the Bessel functions has been to use an ascending series for small argument, and the asymptotic (but divergent) series for large argument. In this talk, we describe a new series (based on arc-trig series) that is geometrically convergent in the number of summands, with explicitly computable error estimates for the tails.

Abstract and Outline

- Motivation and Context (JMB)
  - Earlier Talk on Laguerre Asymptotics
- Our New Algorithms (O-YC)
  - Preprint related to Current Talk
This was needed by Crandall to perform high precision computation of Riemann-Zeta, say near $(1+10^{10}i,1+2\cdot10^{10}i)$ to treat primes around $10^{20}$.
Motivation and Context

For example we obtain the large $n$ asymptotic

$$L_n^{(-a)}(-z) \sim S_n(a, z) \left(1 + O(m^{-1/2})\right),$$

where the sub-exponential term $S$ is

$$S_n(a, z) := \frac{e^{-z/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{mz}}}{z^{1/4-a/2} m^{1/4+a/2}}.$$  \hspace{1cm} (4)

In such expressions, $\Re(\sqrt{mz})$ is taken to be $\sqrt{m|z| \cos(\theta/2)}$ where $\theta := \arg(z) \in (-\pi, \pi]$ (we hereby adopt the convention $\arg(-1) := \pi$), and so for $(a, z) \in D$ the expression (4) involves genuinely diverging growth in $n$, due to the sub-exponential $\exp(2\sqrt{mz})$ factor.

What we seek are effective bounds, for example to replace a logical error-bounding statement for an expression $E$ in the following way:

$$\left\{ E = O\left(\frac{1}{\sqrt{m}}\right) \right\} \quad \text{is replaced by} \quad \left\{ E < \frac{C}{\sqrt{m}} \text{ for } m > m' \right\}.$$

We manage this in part by finding the most effective contour for Laguerre polynomials experimentally (C_1 dominates).
Motivation and Context

This is what the new DLMF (A&S) provides (with metadata suppressed)

http://dlmf.nist.gov/X/

§10.2. Definitions

Contents
- §10.2(i) Bessel's Equation
- §10.2(ii) Standard Solutions
- §10.2(iii) Numerically Satisfactory Pairs of Solutions

§10.2(i). Bessel's Equation

10.2.1
\[
 z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0.
\]

This differential equation has a regular singularity at \( z = 0 \) with indices \( \pm \nu \), and an irregular singularity at \( z = \infty \) of rank 1; compare §§2.8(i) and 2.8(ii).

§10.2(ii). Standard Solutions

Bessel Function of the First Kind

10.2.2
\[
 J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\nu + k + 1)}.
\]

This solution of (10.2.1) is an analytic function of \( z \in \mathbb{C} \), except for a branch point at \( z = 0 \) when \( \nu \) is not an integer. The principal branch of \( J_\nu(z) \) corresponds to the principal value of \( \left(\frac{1}{2}z\right)^\nu \) (§4.2(iv)) and is analytic in the \( z \)-plane cut along the interval \( (-\infty, 0] \).

When \( \nu = n \ ( \in \mathbb{Z} ) \), \( J_\nu(z) \) is entire in \( z \).

For fixed \( z \ ( \neq 0 ) \) each branch of \( J_\nu(z) \) is entire in \( \nu \).
Motivation and Context

Inter alia we obtained expressions for $J_\nu$ (and $I_\nu$) at integer order:

$$J_n(z) = \frac{2}{\pi} \sum_{k=0}^{\infty} g_k(-2in) \left( b_k \cos \chi - c_k \sin \chi \right), \quad (66)$$

with angle

$$\chi := z - \pi n/2 - \pi/4,$$

and the coefficients $b_k, c_k$ determined by

$$b_k := B_k(iz)e^{i\pi/4} + B_k(-iz)e^{-i\pi/4},$$
$$i c_k := B_k(iz)e^{i\pi/4} - B_k(-iz)e^{-i\pi/4}.$$

Note that if $z$ is real then each $b_k, c_k$ is real, whence our series here has all real terms. Note that our recursion (45) likewise ignites a recursion amongst the $b_k, c_k$.

$B_k$ is an error-function-class integral

$$B_k(p) := \beta_{2k-1}(p, \pi/4) = \int_0^{1/\sqrt{2}} x^{2k} e^{-2px^2} \, dx. \quad (43)$$

It is both computationally and theoretically important that $B_k$ can be given a closed form (in terms of $\Gamma$- and incomplete $\Gamma$-functions) as well as a recursion relation. Namely, we have

$$B_k(p) = \frac{1}{2} \frac{1}{(2p)^{k+1/2}} \{\Gamma(k + 1/2) - \Gamma(k + 1/2, p)\}, \quad (44)$$
Motivation and Context

This development depended critically on the following \textit{exp-arc} expansions:

For any complex $\tau$ and $x \in [-1, 1]$, one has a remarkable expansion:

$$e^{\tau \arcsin x} = 1 + \sum_{k=1}^{\infty} r_k(\tau) \frac{x^k}{k!},$$

where the coefficients depend on the parity of the index, as

$$r_{2m+1}(\tau) := \tau \prod_{j=1}^{m} \left( \tau^2 + (2j - 1)^2 \right), \quad r_{2m}(\tau) := \prod_{j=1}^{m} \left( \tau^2 + (2j - 2)^2 \right).$$

By differentiating with respect to $x$ we obtain

$$\frac{e^{\tau \arcsin x}}{\sqrt{1 - x^2}} = \frac{1}{\tau} \sum_{k=0}^{\infty} r_{k+1}(\tau) \frac{x^k}{k!},$$

valid for $x \in (-1, 1)$.

I learned (1) from Ramanujan and Berndt while doing number theory.
Motivation and Context

We noted more generally, for $z$ and $\nu$ having positive real part, that

$$J_{\nu}(z) = \frac{1}{\pi} \int_0^{\pi} \cos(\nu t - z \sin t) \, dt - \frac{\sin(\nu \pi)}{\pi} \int_0^{\infty} e^{-\nu t - z \sinh t} \, dt,$$

(68)

with a corresponding representation

$$I_{\nu}(z) = \frac{1}{\pi} \int_0^{\pi} e^{z \cos t} \cos(\nu t) \, dt - \frac{\sin(\nu \pi)}{\pi} \int_0^{\infty} e^{-\nu t - z \cosh t} \, dt,$$

(69)

itself valid for the same cases of $z, \nu$. One wonders whether an exp-arc approach can be used to resolve the integrals here—which contribute when $\nu$ is not an integer—as exp-arc series.

The paper concluded with several open questions: notably

- It would be useful to establish the very most efficient way to calculate $J_n(z)$ with our converging series (66) and to know, for given arguments $n, z$ how many terms of the exp-arc sum yield $b$ good bits in the answer for $J_n(z)$. It should also be possible to extract the classical ascending series for $J_n$ directly from our converging series.
- Can the integral pieces of (68, 69) be resolved as exp-arc series, to provide even more general, universally convergent $I, J$ series (i.e. for noninteger $\nu$)?

And this is what we now consider...
Effective Computation of Bessel Functions, Part II

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6 January 2008
For any complex pair \( (p, q) \) and real numbers \( \alpha, \beta \in (-\pi, \pi) \), let

\[
I(p, q, \alpha, \beta) := \int_{\alpha}^{\beta} e^{-i q \omega} e^{p \cos \omega} d\omega.
\]

Then we have the absolutely convergent representation

\[
I(p, q, \alpha, \beta) = \frac{ie^p}{q} \sum_{k=0}^{\infty} \frac{r_{k+1}(-2i q)}{k!} \int_{\sin \frac{\alpha}{2}}^{\sin \frac{\beta}{2}} x^k e^{-2px^2} dx,
\]

where

\[
r_{2m+1}(\nu) := \nu \prod_{j=1}^{m} \left( \nu^2 + (2j - 1)^2 \right), \quad r_{2m}(\nu) := \prod_{j=1}^{m} \left( \nu^2 + (2j - 2)^2 \right).
\]

These are, you may recall, the coefficients in the series expansion of \( \exp(\arcsin x) \).
For any complex pair \((p, q)\) and real numbers \(\alpha, \beta \in (-\pi, \pi)\), let

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\mathcal{I}(p, q, \alpha, \beta) := \int_{\alpha}^{\beta} e^{-i\omega} e^{p \cos \omega} \, d\omega.
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In particular, for the case where \((\alpha, \beta) = (-\pi/2, \pi/2)\), we have

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\mathcal{I}(p, q) := \mathcal{I}(p, q, -\pi/2, \pi/2) = \frac{2ie^p}{q} \sum_{k=0}^{\infty} \frac{r_{2k+1}(-2iq)}{(2k)!} B_k(p),
\]

with

\[
B_k(p) := \int_{0}^{1/\sqrt{2}} x^{2k} e^{-2px^2} dx = \frac{1}{2^{k+1} \sqrt{2}} \int_{0}^{1} e^{- pu} u^{k-\frac{1}{2}} du
\]

\[
= - \frac{e^{-p}}{p2^{k+1} \sqrt{2}} + \left( k - \frac{1}{2} \right) \frac{B_{k-1}(p)}{2}.
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\]
For integral order, we have from the Laguerre paper

\[ J_n(z) = \frac{1}{2\pi} \left( e^{-in\pi/2} \mathcal{I}(iz, n) + e^{in\pi/2} \mathcal{I}(-iz, n) \right), \]

and

\[ I_n(z) = \frac{1}{2\pi} \left( \mathcal{I}(z, n) + \cos(\pi n) \mathcal{I}(-z, n) \right). \]

As Jon mentioned, we want to use the integral representations to get expressions for general \( \nu \).
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As Jon mentioned, we want to use the integral representations to get expressions for general \( \nu \).
The integral representations are:

\[ J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu t - z \sin t) \, dt - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-\nu t - z \sinh t} \, dt, \]

\[ Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin t - \nu t) \, dt \]

\[ - \frac{1}{\pi} \int_0^\infty (e^\nu t + e^{-\nu t} \cos \nu \pi) e^{-z \sinh t} \, dt, \]

\[ I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos t} \cos \nu t \, dt - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} \, dt, \]

and

\[ K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh \nu t \, dt = \frac{1}{2} \int_{-\infty}^\infty e^{-z \cosh t - \nu t} \, dt. \]
The integrals on \([0, \pi]\) can be expressed in terms of the \(I\) function. Specifically,

\[
J_\nu(z) = \frac{1}{2\pi} \left( e^{-i\nu\pi/2}I(iz, \nu) + e^{i\nu\pi/2}I(-iz, \nu) \right) - \frac{\sin \nu \pi}{\pi} \int_0^\infty \cdots
\]

\[
Y_\nu(z) = \frac{1}{2\pi i} \left( e^{-i\nu\pi/2}I(iz, \nu) - e^{i\nu\pi/2}I(-iz, \nu) \right) - \frac{1}{\pi} \int_0^\infty \cdots
\]
\[ I_\nu(z) = \frac{1}{2} \left( I(z, \nu) + e^{i\nu \pi} I(-z, \nu, 0, \pi/2) + e^{-i\nu \pi} I(-z, -\nu, 0, \pi/2) \right) \]

\[ - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} \ldots \]

\[ = \frac{1}{2\pi} \left( I(z, \nu) + \cos \nu \pi I(-z, \nu) - \sin \nu \pi I^*(z, \nu) \right) \]

\[ - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} \ldots, \]

where

\[ I^*(z, \nu) = \frac{2e^z}{\nu} \sum_{n=0}^{\infty} \frac{r_{2n+2}(2i\nu)}{(2n + 1)!} B_{n+1/2}(z). \]
To get the generalizations we want, we basically just need to evaluate the infinite integrals.

Let us look at the integrals in the $J$ and $Y$ cases. A change of variables plus integration by parts gives us

$$
\int_0^\infty e^{-\nu t-z\sinh t} \, dt = \frac{1}{\nu} - \frac{z}{\nu} \int_0^\infty e^{-zs} e^{-\nu \text{arcsinh } s} \, ds.
$$

The expansion of $e^{-\nu \text{arcsinh } s}$ about $s = 0$, used in the finite case to obtain the series, is only valid on $[0, 1)$. 
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D. Borwein, J. M. Borwein, O-Y. Chan
Effective Computation of Bessel Functions, Part II
For large $s$, it makes sense to expand about infinity!

The series, valid on $(1, \infty)$, is

$$
s^\nu e^{-\nu \text{arcsinh } s} = \sum_{n=0}^{\infty} \frac{A_n(\nu)}{s^{2n}},
$$

where $A_0(\nu) = 2^{-\nu}$ and for $n \geq 1$,

$$
A_n = -\frac{(\nu + 2n - 2)(\nu + 2n - 1)}{4n(n + \nu)} A_{n-1},
$$

from which we easily obtain

$$
A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu} (\nu + n + 1)_{n-1}}{2^{2n} n!}.
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from which we easily obtain

$$A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu}(\nu + n + 1)_{n-1}}{2^{2n} n!}.$$
Note that when $\nu$ is a negative integer, we have problems with the recurrence.

When $n = \lfloor (1 - \nu)/2 \rfloor$, the numerator is 0. When $n = -\nu$, the denominator is zero.

In this case, $A_n(\nu) = (-1)^{\nu+1} A_{n+\nu}(-\nu)$ for $n \geq -\nu$
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If we only used the expansions at 0 and \( \infty \), we could get a series; but there are issues with interchanging summations and integration, since we are integrating up to the boundary of the interval of convergence.

Even after justifying the interchange, the resulting series is very slow due to the “bad” approximation by the series near the boundary.
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Even after justifying the interchange, the resulting series is very slow due to the “bad” approximation by the series near the boundary.
For fixed $k$, $f_k(s) := e^{-\nu \arcsinh(k+s)}$ satisfies the second order differential equation

$$f''_k(s) = \frac{1}{k^2 + 1 + 2ks + s^2} \left( \nu^2 f_k(s) - (k + s)f'_k(s) \right).$$

So if we set

$$e^{-\nu \arcsinh(k+s)} = \sum_{n=0}^{\infty} \frac{a_n(k, \nu)}{n!} s^n,$$

then we have the recurrence relation

$$a_{n+2} = \frac{1}{k^2 + 1} \left( (\nu^2 - n^2) a_n - k(2n + 1)a_{n+1} \right),$$

with

$$a_0 = (k + \sqrt{k^2 + 1})^{-\nu}, \quad a_1 = -\frac{\nu a_0}{\sqrt{k^2 + 1}}.$$
Localize!

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$$a_0 = (k + \sqrt{k^2 + 1})^{-\nu}, \quad a_1 = -\frac{\nu a_0}{\sqrt{k^2 + 1}}.$$
We can subdivide $[0, \infty)$ into the intervals $[0, 1/2], [1/2, 3/2], \ldots, [N - 1/2, N + 1/2], [N + 1/2, \infty)$ and on each interval expand $e^{-\nu \arcsinh s}$ at $k$, the centre of the interval.

Each of these series has radius of convergence $\sqrt{k^2 + 1}$ and so we may interchange summation and integration.

For the infinite interval at the end, we use the expansion about infinity.
We can subdivide $[0, \infty)$ into the intervals $[0, 1/2], [1/2, 3/2], \ldots, [N - 1/2, N + 1/2], [N + 1/2, \infty)$ and on each interval expand $e^{-\nu \text{arcsinh} s}$ at $k$, the centre of the interval.

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For the infinite interval at the end, we use the expansion about infinity.
Thus for any positive integer $N$, we have

\[
\int_{0}^{\infty} e^{-zs} e^{-\nu \text{arcsinh } s} \, ds =
\sum_{n=0}^{\infty} \left( a_n(0, \nu) \alpha_n(z) + \beta_n(z) \sum_{k=1}^{N} e^{-kz} \frac{a_n(k, \nu)}{n!} \right)
+ A_n(\nu) G_n(N + \frac{1}{2}, z, \nu),
\]

where

\[
\alpha_n(z) := \int_{0}^{1/2} e^{-zs} s^n \, ds = -\frac{e^{-z/2}}{2^n z} + \frac{n}{z} \alpha_{n-1}(z),
\]

\[
\beta_n(z) := \int_{-1/2}^{1/2} e^{-zs} s^n \, ds = \frac{e^{z/2}}{(-2)^n z} - \frac{e^{-z/2}}{2^n z} + \frac{n}{z} \beta_{n-1}(z),
\]
and

\[
G_n(\theta, z, \nu) := \frac{e^{-\theta z}}{\theta^{2n+\nu-1}} \int_0^\infty e^{-\theta zs} (1 + s)^{-2n-\nu} ds
\]

\[
= \frac{1}{(\nu + 2n - 1)(\nu + 2n - 2)} \times \left( \frac{e^{-\theta z}(\nu + 2n - 2 - \theta z)}{\theta^{2n+\nu-1}} + z^2 G_{n-1}(\theta, z, \nu) \right).
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So we have found a representation for the Bessel functions in terms of several sums:

Sums involving $\mathcal{I}$ from the integral on $[0, \pi]$, where each summand looks like

$$
\frac{r_{n+1}(2\nu)}{n!} B_{n(+1/2)}(z),
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sums from the subdivisions of the real line on the infinite integral, where a typical summand is

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Let us first look at
\[ \frac{r_{n+1}(2\nu)}{n!}. \]

For simplicity we consider the case \( n \) even, \( n = 2m \). Then this is
\[
\prod_{j=1}^{m} \left( 1 - \frac{1}{2j} - \frac{4\nu^2}{(2j-1)(2j)} \right),
\]
which is bounded and decreasing for \( m > 2|\nu|^2 \). Similarly for odd \( n \).

Also, (for arbitrary \( n \))
\[
B_n(z) = \frac{1}{2^{n+3/2}} \int_0^1 e^{-zu} u^{n-1/2} du
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Thus the terms of type

\[ \frac{r_{n+1}(2\nu)}{n!} B_n(z) = O_{\nu,z}(2^{-n}), \]

where the big-\(O\) constant can be explicitly computed.
For terms of the type

\[
\frac{a_n(k, \nu)}{n!} \beta_n(z) e^{-kz},
\]

note that \(a_n(k, \nu)/n!\) are the Taylor coefficients, and so they are \(O\left(\frac{1}{(k^2+1)^{n/2}}\right)\) from the radius of convergence. We can fairly easily get a weaker but explicit geometric bound using the recurrence relation for \(a_n(k, \nu)\).

\(\beta_n(z)\) is the \(n\)-th moment of the exponential, and can be explicitly computed. A simple estimate yields

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|\beta_n(z) e^{-kz}| \leq e^{-(k-1/2) \text{Re}(z)} \frac{2^n}{2^n}.
\]
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Putting it all together, we see that the (slowest) sums converge like $2^{-n}$, and with explicit big-$O$ constants we may determine how many terms are needed for a specific accuracy.
Other features to note:

- For each type of sum, the summands are all computable via recursion.
- The most difficult computation involved are the computation of $B_0$ and $G_0$, each of which involves an incomplete gamma evaluation. It should be noted that this can be done via continued fractions, so this scheme can be thought of as a continued fraction evaluation scheme for Bessel functions.
- The sum involving $A_n G_n$ is bounded like $O_{\nu}(e^{-z(N+1/2)})$ by estimating the integral of the tail. So one can avoid the computation of $G_0$ altogether by choosing a large enough $N$. 
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Along the same lines, one does not need to compute all of the sums involving $\beta_n$ for large $k$ unless one needs accuracy beyond about $e^{-(k-1/2)\Re(z)}$.

In addition to choosing an optimal $N$, one can also adjust the intervals in dividing the integral on $[0, \infty)$. In particular, the sum arising out of an interval on $(a, b)$ expanded at $k$ converges like

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Our computation scheme has some advantages over the traditional ascending-asymptotic switching scheme:

- Our series are all uniformly geometrically convergent, whereas some asymptotic formulas are divergent series, and some are only algebraically convergent (i.e., like $n^{-\alpha}$ rather than $2^{-n}$).
- Each summand in our series is a product of functions that depend only on $\nu$ or only on $z$, and thus these values can be stored and recycled for one-$\nu$-many-$z$ or one-$z$-many-$\nu$ computations. Note also that each of these functions is eventually decreasing.

The following table compares the performance between the ascending series, the standard divergent asymptotic series, and our series for $J_\nu$ with the choice $N = 1$. 
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Table: Comparison between various series for $J_\nu(z)$.

<table>
<thead>
<tr>
<th>$(\nu, z)$</th>
<th>$M$</th>
<th>Ascending Series</th>
<th>Asymptotic Series</th>
<th>Exp-arc Series</th>
</tr>
</thead>
</table>
| $\nu = 6.2$  
$z = 100$ | 10  | $10^{22}$        | $10^{-32}$        | $10^{-5}$      |
|         | 50  | $10^{41}$        | $10^{-76}$        | $10^{-18}$     |
|         | 100 | $10^{22}$        | $10^{-89}$        | $10^{-33}$     |
|         | 150 | $10^{-19}$       | $10^{-79}$        | $10^{-49}$     |
|         | 200 | $10^{-75}$       | $10^{-55}$        | $10^{-64}$     |
| $\nu = 12.3$  
$z = 50$ | 10  | $10^{18}$        | $10^{-23}$        | $10^{2}$       |
|         | 30  | $10^{17}$        | $10^{-41}$        | $10^{-10}$     |
|         | 50  | $10^{6}$         | $10^{-45}$        | $10^{-17}$     |
|         | 70  | $10^{-11}$       | $10^{-42}$        | $10^{-23}$     |
|         | 100 | $10^{-45}$       | $10^{-28}$        | $10^{-33}$     |
| $\nu = 12.3$  
$z = 75 + 57i$ | 10  | $10^{27}$        | $10^{-4}$         | $10^{13}$      |
|         | 50  | $10^{38}$        | $10^{-48}$        | $10^{-17}$     |
|         | 100 | $10^{14}$        | $10^{-59}$        | $10^{-33}$     |
|         | 120 | $10^{-2}$        | $10^{-56}$        | $10^{-39}$     |
|         | 150 | $10^{-31}$       | $10^{-47}$        | $10^{-48}$     |
|         | 200 | $10^{-89}$       | $10^{-20}$        | $10^{-64}$     |
Thank you for your attention!
The paper is in press in **JMAA**.
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