Meetings with Computer Algebra and Special Functions
A Ramanujan Style Talk

Jonathan M. Borwein FRSC FAA FAAAS
Laureate Professor & Director of CARMA, Univ. of Newcastle
This talk: http://carma.newcastle.edu.au/jon/msf.pdf
Prepared for JonFest DownUnder, Nov 29, 30 and Dec 1
Revised: April 12, 2012
Contents. We will cover some of the following:

1. Introduction and Three Elementary Examples
   - 8. Archimedes and Pi
   - 14. A 21st Century postscript
   - 24. Sinc functions

2. Three Intermediate Examples
   - 33. What is that number?
   - 39. Lambert W
   - 44. What is that continued fraction?

3. More Advanced Examples
   - 51. What is that probability?
   - 57. What is that limit, II?
   - 62. What is that transition value?

4. Current Research and Conclusions
   - 64. What is that expectation?
   - 67. What is that density?
   - 68. Part II and Conclusions?
Abstract

*It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.* When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I intend to display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
- In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic and numeric computations. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as gfun.

My intention is to show off the interplay between numeric and symbolic computing while exploring the various topics in my title.
Abstract

*It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.* When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I intend to display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.

- Much of this work requires extensive symbolic and numeric computations. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as *gfun*.

My intention is to show off the interplay between numeric and symbolic computing while exploring the various topics in my title.
Abstract

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I intend to display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
- In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic and numeric computations. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as gfun.

My intention is to show off the interplay between numeric and symbolic computing while exploring the various topics in my title.
Abstract

*It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.* When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I intend to display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
- In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic and numeric computations. It makes frequent use of the new NIST *Handbook of Mathematical Functions* and related tools such as *gfun*.

My intention is to show off the interplay between numeric and symbolic computing while exploring the various topics in my title.
Abstract

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I intend to display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
- In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic and numeric computations. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as gfun.

My intention is to show off the interplay between numeric and symbolic computing while exploring the various topics in my title.
Congratulations to NIST

DLMF: NIST is still a 19C handbook in 21C dress.
DDMF: INRIA’s way of the future?

Special Functions in the 21st Century: Theory & Applications
April 6–8, 2011
Washington, DC

Objectives. The conference will provide a forum for the exchange of expertise, experience and insights among world leaders in the subject of special functions. Participants will include expert authors, editors and validators of the recently published NIST Handbook of Mathematical Functions and Digital Library of Mathematical Functions (DLMF). It will also provide an opportunity for DLMF users to interact with its creators and to explore potential areas of fruitful future developments.

Special Recognition of Professor Frank W. J. Olver. This conference is dedicated to Professor Olver in light of his seminal contributions to the advancement of special functions, especially in the area of asymptotic analysis and as Mathematics Editor of the DLMF.

Plenary Speakers
Richard Askey, University of Wisconsin
Michael Berry, University of Bristol
Nalini Joshi, University of Sydney, Australia
Leonard Maximon, George Washington University
William Reinhardt, University of Washington
Roderick Wong, City University of Hong Kong

Call for Contributed Talks (25 Minutes)
Abstracts may be submitted to Daniel.Lozier@nist.gov until March 15, 2011.

Registration and Financial Assistance. Registration fee: $120. Courtesy of SIAM, limited travel support is available for US-based postdoc and early career researchers. Courtesy of City University of Hong Kong and NIST, partial support is available for others in cases of need. Submit all requests for financial assistance to Daniel.Lozier@nist.gov.


Organizing Committee. Daniel Lozier, NIST, Gaithersburg, Maryland; Adri Olde Daalhuis, University of Edinburgh; Nico Temme, CWI, Amsterdam; Roderick Wong, City University of Hong Kong

To register online for the conference, and reserve a room at the conference hotel, see http://math.nist.gov/~DLozier/SP21
Congratulations to NIST

DLMF: NIST is still a 19C handbook in 21C dress.

DDMF: INRIA’s way of the future?

Special Functions in the 21st Century: Theory & Applications
April 6–8, 2011
Washington, DC

Objectives. The conference will provide a forum for the exchange of expertise, experience and insights among world leaders in the subject of special functions. Participants will include expert authors, editors and validators of the recently published NIST Handbook of Mathematical Functions and Digital Library of Mathematical Functions (DLMF). It will also provide an opportunity for DLMF users to interact with its creators and to explore potential areas of fruitful future developments.

Special Recognition of Professor Frank W. J. Olver. This conference is dedicated to Professor Olver in light of his seminal contributions to the advancement of special functions, especially in the area of asymptotic analysis and as Mathematics Editor of the DLMF.

Plenary Speakers
Richard Askey, University of Wisconsin
Michael Berry, University of Bristol
Nalini Joshi, University of Sydney, Australia
Leonard Maximon, George Washington University
William Reinhardt, University of Washington
Roderick Wong, City University of Hong Kong

Call for Contributed Talks (25 Minutes)
Abstracts may be submitted to Daniel.Lozier@nist.gov until March 15, 2011.

Registration and Financial Assistance. Registration fee: $120. Courtesy of SIAM, limited travel support is available for US-based postdoc and early career researchers. Courtesy of City University of Hong Kong and NIST, partial support is available for others in cases of need. Submit all requests for financial assistance to Daniel.Lozier@nist.gov.


Organizing Committee. Daniel Lozier, NIST, Gaithersburg, Maryland; Adrij Olde Daalhuis, University of Edinburgh; Nico Temme, CWI, Amsterdam; Roderick Wong, City University of Hong Kong

To register online for the conference, and reserve a room at the conference hotel, see http://math.nist.gov/~DLozier/SF21
2. Introduction and Three Elementary Examples
32. Three Intermediate Examples
50. More Advanced Examples
64. Current Research and Conclusions

9. Archimedes and Pi
15. A 21st Century postscript
25. Sinc functions

DLMF and DDMF

http://ddmf.msr-inria.inria.fr/

Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

This is release 1.6.1 of DDMF. Select a mathematical rendering to enable access to the contents.

What's New? The main change in this release 1.6.1, dated June 2011, is:

- Display DDMF source code online.

Release History.

More on the project:
- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project
- Article on the project at ICMS2010
- Source code used to generate these pages
- List of related projects.

The DDMF project (2008–2011) is hosted and supported by the Microsoft Research – INRIA Joint Centre.
This describes joint research with many collaborators over many years – especially DHB and REC.

Earlier results are to be found in the books:

- *Mathematics by Experiment* with DHB (2004-08)
- *Experimentation in Mathematics* with DHB & RG (2005)

Recent results are surveyed in:

- [http://carma.newcastle.edu.au/~jb616/papers.html#BOOKS](http://carma.newcastle.edu.au/~jb616/papers.html#BOOKS)

Exploratory experimentation: with DHB, AMS Notices Nov11

What are closed forms: with REC, AMS Notices in press

This talk and related talks are housed at [www.carma.newcastle.edu.au/~jb616/papers.html#TALKS](http://www.carma.newcastle.edu.au/~jb616/papers.html#TALKS)
Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.

2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and
   - *Experimentation in Mathematics* with DHB & RG (2005)
   

3. Recent results are surveyed in:

4. Exploratory experimentation: with DHB, AMS Notices Nov11
   What are closed forms: with REC, AMS Notices in press

5. This talk and related talks are housed at [www.carma.newcastle.edu.au/~jb616/papers.html#TALKS](http://www.carma.newcastle.edu.au/~jb616/papers.html#TALKS).
Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.

2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and *Experimentation in Mathematics* with DHB & RG (2005)

3. Recent results are surveyed in:
   - [http://carma.newcastle.edu.au/~jb616/papers.html#BOOKS](http://carma.newcastle.edu.au/~jb616/papers.html#BOOKS)

4. Exploratory experimentation: with DHB, *AMS Notices* Nov11

   What are closed forms: with REC, *AMS Notices* in press

5. This talk and related talks are housed at [www.carma.newcastle.edu.au/~jb616/papers.html#TALKS](http://www.carma.newcastle.edu.au/~jb616/papers.html#TALKS)
Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.

2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and *Experimentation in Mathematics* with DHB & RG (2005)

3. Recent results are surveyed in:

4. Exploratory experimentation: with DHB, AMS Notices Nov11
   - What are closed forms: with REC, AMS Notices in press

5. This talk and related talks are housed at [www.carma.newcastle.edu.au/~jb616/papers.html#TALKS](http://www.carma.newcastle.edu.au/~jb616/papers.html#TALKS).
Some of my **Current Collaborators** (Straub, Borwein and Wan)
La plus ça change, l

Meetings with Special Functions
1. What is that Integral?

\[ \int_0^1 \frac{(1 - x)^4 x^4}{1 + x^2} \, dx = ??? \quad (1) \]

Remark (Kondo-Yee, 2011.)

1. What is that Integral? (Bailey and Crandall)

Question

\[ \int_{0}^{1} \frac{(1 - x)^4 x^4}{1 + x^2} \, dx = ??? \] (1)

Remark (Kondo-Yee, 2011.)

Question

\[ \int_0^1 \frac{(1-x)^4 x^4}{1 + x^2} \, dx = ??? \]

Remark (Kondo-Yee, 2011.)

Let’s be Clear: $\pi$ Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* ‘knows’ this since

$$0 < \int_0^1 \frac{(1 - x)^4 x^4}{1 + x^2} \, dx = \frac{22}{7} - \pi, \quad (2)$$

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

**Assume we trust it.** Then the integrand is strictly positive on $(0,1)$, and the answer in (2) is an area and so strictly positive, despite millennia of claims that $\pi$ is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to $\pi$. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$
Let’s be Clear: $\pi$ Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* ‘knows’ this since

\[ 0 < \int_{0}^{1} \frac{(1 - x)^4 x^4}{1 + x^2} \, dx = \frac{22}{7} - \pi, \quad (2) \]

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

**Assume we trust it.** Then the integrand is strictly positive on $(0, 1)$, and the answer in $(2)$ is an area and so strictly positive, despite millennia of claims that $\pi$ is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to $\pi$. These commence:

\[ 3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots \]
Let’s be Clear: $\pi$ Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* ‘knows’ this since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2} \, dx = \frac{22}{7} - \pi,$$

(2)

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

**Assume we trust it.** Then the integrand is strictly positive on $(0, 1)$, and the answer in (2) is an area and so strictly positive, despite millennia of claims that $\pi$ is $22/7$.

- **Accidentally**, $22/7$ is one of the early continued fraction approximation to $\pi$. These commence:

  \[
  3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots
  \]
As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

**Algorithm (Archimedes)**

Set $a_0 := 2\sqrt{3}$, $b_0 := 3$. Compute

\[
\begin{align*}
a_{n+1} &= \frac{2a_nb_n}{a_n + b_n} \quad (H) \\
b_{n+1} &= \sqrt{a_{n+1}b_n} \quad (G)
\end{align*}
\]

These tend to $\pi$, error decreasing by a factor of four at each step.

- The greatest mathematician (scientist) to live before the Enlightenment. To compute $\pi$ Archimedes had to invent many subjects — including numerical and interval analysis.
Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \quad (H)$$

$$b_{n+1} = \sqrt{a_{n+1}b_n} \quad (G)$$

These tend to $\pi$, error decreasing by a factor of four at each step.

• The greatest mathematician (scientist) to live before the Enlightenment. To compute $\pi$ Archimedes had to invent many subjects — including numerical and interval analysis.
Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \quad (H)$$

$$b_{n+1} = \sqrt{a_{n+1}b_n} \quad (G)$$

These tend to $\pi$, error decreasing by a factor of four at each step.

- The greatest mathematician (scientist) to live before the Enlightenment. To compute $\pi$ Archimedes had to invent many subjects — including numerical and interval analysis.
Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \quad (H)$$

$$b_{n+1} = \sqrt{a_{n+1} b_n} \quad (G)$$

These tend to $\pi$, error decreasing by a factor of four at each step.

- The greatest mathematician (scientist) to live before the Enlightenment. To compute $\pi$ Archimedes had to invent many subjects — including numerical and interval analysis.
Proving $\pi$ is not $\frac{22}{7}$

In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4 (1 - x)^4}{1 + x^2} \, dx = \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4 t - 4 \arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus proves (2).

QED

One can take this idea a bit further. Note that

$$\int_0^1 x^4 (1 - x)^4 \, dx = \frac{1}{630}.$$  (3)
Proving $\pi$ is not $\frac{22}{7}$

In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4 (1-x)^4}{1+x^2} \, dx = \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4 t - 4 \arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus proves (2). QED

One can take this idea a bit further. Note that

$$\int_0^1 x^4 (1-x)^4 \, dx = \frac{1}{630}. \quad (3)$$
Hence
\[
\frac{1}{2} \int_0^1 x^4 (1 - x)^4 \, dx < \int_0^1 \frac{(1 - x)^4 x^4}{1 + x^2} \, dx < \int_0^1 x^4 (1 - x)^4 \, dx.
\]

Combine this with (2) and (3) to derive:

\[
\frac{223}{71} < \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260} < \frac{22}{7}
\]

and so re-obtain Archimedes’ famous

\[
\frac{3}{10} \frac{10}{71} < \pi < \frac{3}{10} \frac{10}{70}.
\]
Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60’s Sydney exam, and traces back to 1944 (Dalziel).

Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

---

*I have no satisfaction in formulas unless I feel their arithmetical magnitude.*—Baron William Thomson Kelvin

In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.
Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60’s Sydney exam, and traces back to 1944 (Dalziel).

Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

*I have no satisfaction in formulas unless I feel their arithmetical magnitude.*—Baron William Thomson Kelvin

In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.
Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60’s Sydney exam, and traces back to 1944 (Dalziel).

Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

*I have no satisfaction in formulas unless I feel their arithmetical magnitude.* — Baron William Thomson Kelvin

In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.
Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60’s Sydney exam, and traces back to 1944 (Dalziel).

---

*I have no satisfaction in formulas unless I feel their arithmetical magnitude.* — Baron William Thomson Kelvin

In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

---

Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)
2. Introduction and Three Elementary Examples
32. Three Intermediate Examples
50. More Advanced Examples
64. Current Research and Conclusions

2. BBP Digit Extraction Formulas

Algorithm (What We Did, January to March 2011)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I obtained and confirmed on a 4-rack BlueGene/P system at IBM's Benchmarking Centre in Rochester, Minn, USA:

1. 106 digits of $\pi^2$ base 2 at the ten trillionth place base 64
2. 94 digits of $\pi^2$ base 3 at the ten trillionth place base 729
3. 141 digits of $G$ base 2 at the ten trillionth place base 4096

- $G$ is Catalan's constant. The full computation suite took about 1500 cpu years.
2. BBP Digit Extraction Formulas

Algorithm (What We Did, January to March 2011)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I obtained and confirmed on a 4-rack BlueGene/P system at IBM’s Benchmarking Centre in Rochester, Minn, USA:

1. **106** digits of \( \pi^2 \) base 2 at the **ten trillion**th place base **64**
2. **94** digits of \( \pi^2 \) base 3 at the **ten trillion**th place base **729**
3. **141** digits of \( G \) base 2 at the **ten trillion**th place base **4096**

- \( G \) is **Catalan's constant**. The full computation suite took about **1500** cpu years.
Prior to **1996**, most folks thought to compute the $d$-th digit of $\pi$, you had to generate the (order of) the entire first $d$ digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of $\pi$. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of $\pi$. It produces:

  - a modest-length string hex or binary digits of $\pi$, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.
What BBP Does?

Prior to 1996, most folks thought to compute the $d$-th digit of $\pi$, you had to generate the (order of) the entire first $d$ digits.

• **This is not true**, at least for hex (base 16) or binary (base 2) digits of $\pi$. In 1996, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of $\pi$. It produces:

  a modest-length string hex or binary digits of $\pi$, beginning at an any position, using no prior bits;

  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.
Prior to 1996, most folks thought to compute the $d$-th digit of $\pi$, you had to generate the (order of) the entire first $d$ digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of $\pi$. In 1996, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of $\pi$. It produces:
  
  - a modest-length string hex or binary digits of $\pi$, beginning at any position, *using no prior bits*;
  - is implementable on any modern computer;
  - requires no multiple precision software;
  - requires very little memory; and has
  - a computational cost growing only slightly faster than the digit position.
Prior to **1996**, most folks thought to compute the \( d \)-th digit of \( \pi \), you had to generate the (order of) the entire first \( d \) digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of \( \pi \). In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of \( \pi \). It produces:

- a modest-length string hex or binary digits of \( \pi \), beginning at an any position, *using no prior bits*;

  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.
What BBP Does?

Prior to 1996, most folks thought to compute the $d$-th digit of $\pi$, you had to generate the (order of) the entire first $d$ digits.

- This is not true, at least for hex (base 16) or binary (base 2) digits of $\pi$. In 1996, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of $\pi$. It produces:
  - a modest-length string hex or binary digits of $\pi$, beginning at an any position, using no prior bits;
    1. is implementable on any modern computer;
    2. requires no multiple precision software;
    3. requires very little memory; and has
    4. a computational cost growing only slightly faster than the digit position.
What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

(5)

- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$\pi = 4 \, _2F_1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5$$

where $\, _2F_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.
What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

(5)

- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$\pi = 4 \, _2\!F_1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5$$

where $_2\!F_1(1, 1/4; 5/4, -1/4) = 0.955933837 \ldots$ is a Gauss hypergeometric function.
What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right)$$

(5)

- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$\pi = 4 \, _2F_1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5$$

where $_2F_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.
What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right)$$

(5)

- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$\pi = 4 \ _2F_1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5$$

where $\ _2F_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.
BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**
- Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...

Won by David Deutsch — discoverer of **Quantum Computing**
• BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**
  – Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...
• Won by David Deutsch — discoverer of Quantum Computing
The Edge of Computation Prize Finalist

BBP was the only mathematical finalist (of about 40) for the first Edge of Computation Science Prize

- Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...

- Won by David Deutsch — discoverer of Quantum Computing.

THE $100,000 EDGE OF COMPUTATION SCIENCE PRIZE

For individual scientific work, extending the computational idea, performed, published, or newly applied within the past ten years.

The Edge of Computation Science Prize, established by Edge Foundation, Inc., is a $100,000 prize initiated and funded by science philanthropist Jeffrey Epstein.
BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**

- Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...

- Won by David Deutsch — discoverer of Quantum Computing
Remarkably, both formulas below have the needed digit-extraction properties:

\[
\pi^2 = \frac{9}{8} \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \times \left\{ \frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right\}
\]

\[
\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \times \left\{ \frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} \right\}
\]

\[
- \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} + \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right\}
\]
Remarkably, both formulas below have the needed digit-extraction properties:

\[
\pi^2 = \frac{9}{8} \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \times \left\{ \frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right\}
\]

\[
\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \times \left\{ \frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} \\
- \frac{27}{(12k+5)^2} - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} \\
- \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right\}
\]
Base-64 digits of $\pi^2$ beginning at position 10 trillion.

The first run produced base-64 digits from position $10^{12} - 1$. It required an average of 253,529 secs per thread, divided into seven partitions of 2048 threads. The total cost was

$$7 \cdot 2048 \cdot 253529 = 3.6 \times 10^9$$

CPU-secs.

Each IBM Blue Gene P system rack features 4096 cores, so the total cost is 10.3 “rack-days.” The second run, producing digits starting from position $10^{12}$, took the same time (within a few minutes).

The two resulting base-8 digit strings are

$75|60114505303236475724500005743262754530363052416350634|573227604$

$xx|60114505303236475724500005743262754530363052416350634|220210566$

(each pair of base-8 digits corresponds to a base-64 digit).

Digits in agreement are delimited by |. Note that 53 consecutive base-8 digits (159 binary digits) agree.
$\pi^2$ base 2 (with DHB & IBM, 2011)

Base-64 digits of $\pi^2$ beginning at position 10 trillion.

The first run produced base-64 digits from position $10^{12} - 1$. It required an average of 253,529 secs per thread, divided into seven partitions of 2048 threads. The total cost was

$$7 \cdot 2048 \cdot 253529 = 3.6 \times 10^9 \text{ CPU-secs.}$$

Each IBM Blue Gene P system rack features 4096 cores, so the total cost is 10.3 "rack-days." The second run, producing digits starting from position $10^{12}$, took the same time (within a few minutes). The two resulting base-8 digit strings are

$$75|601145053032364757245000005743262754530363052416350634|573227604$$

$$xx|601145053032364757245000005743262754530363052416350634|220210566$$

(each pair of base-8 digits corresponds to a base-64 digit). Digits in agreement are delimited by |. Note that 53 consecutive base-8 digits (159 binary digits) agree.
Base-729 digits of $\pi^2$ beginning at position 10 trillion.

Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was

$$6.5 \times 10^9 \text{CPU-secs}$$

or 18.4 “rack-days” for each run.

- Each rack-day is approximately 11.25 years of serial computing time on one core.

The two resulting base-9 digit strings are

001|12264485064548583177111135210162856048323453468|10565567635862

xxx|12264485064548583177111135210162856048323453468|04744867134524

(each triplet of base-9 digits corresponds to one base-729 digit).

Note that 47 consecutive base-9 digits (94 base-3 digits) agree.
Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was

\[ 6.5 \times 10^9 \text{CPU-secs} \]

or \textbf{18.4} “rack-days” for each run.

- Each rack-day is approximately 11.25 years of serial computing time on one core.

The two resulting base-9 digit strings are

\[
\begin{align*}
001 & | 12264485064548583177111135210162856048323453468 | 10565567635862 \\
xxx & | 12264485064548583177111135210162856048323453468 | 04744867134524
\end{align*}
\]

(each triplet of base-9 digits corresponds to one base-729 digit).

Note that 47 consecutive base-9 digits (94 base-3 digits) agree.
Base-729 digits of $\pi^2$ beginning at position 10 trillion. Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was

$$6.5 \times 10^9 \text{CPU-secs}$$

or **18.4** “rack-days” for each run.

- Each rack-day is approximately 11.25 years of serial computing time on one core.

The two resulting base-9 digit strings are

$$001|122644850645485831771111135210162856048323453468|10565567635862$$

$$xxx|122644850645485831771111135210162856048323453468|04744867134524$$

(each triplet of base-9 digits corresponds to one base-729 digit). Note that 47 consecutive base-9 digits (94 base-3 digits) agree.
But not $\pi^2$ base 10 or $\pi$ base 3:

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} = \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6n)!(532n^2 + 126n + 9)}{(n!)^6 10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^{12n} \times \left\{ \frac{177147}{24n + 1} + \frac{118098}{24n + 2} + \frac{78732}{24n + 5} + \frac{104976}{24n + 6} + \frac{52488}{24n + 7} + \frac{23328}{24n + 10} + \frac{23328}{24n + 11} - \frac{15552}{24n + 13} - \frac{10368}{24n + 14} - \frac{6912}{24n + 17} - \frac{9216}{24n + 18} - \frac{4608}{24n + 19} - \frac{2048}{24n + 22} - \frac{2048}{4n + 23} \right\}.$$

- It will not work base-3 because of the 2.
Be skeptical. Almqvist-Guillera (2011) discovered:

\[
\frac{1}{\pi^2} = \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6} \frac{(532n^2 + 126n + 9)}{10^{6n+3}}.
\]

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

\[
\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^{12n} \left( \frac{177147}{24n + 1} + \frac{118098}{24n + 2} + \frac{78732}{24n + 5} + \frac{104976}{24n + 6} + \frac{52488}{24n + 7} + \frac{23328}{24n + 10} + \frac{23328}{24n + 11} - \frac{15552}{24n + 13} - \frac{10368}{24n + 14} - \frac{6912}{24n + 17} \right).
\]

- It will not work base-3 because of the 2.
But not $\pi^2$ base 10 or $\pi$ base 3:

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} \neq \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6} \frac{(532n^2 + 126n + 9)}{10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{12n} \times \left\{ \frac{177147}{24n + 1} + \frac{118098}{24n + 2} + \frac{78732}{24n + 5} + \frac{104976}{24n + 6} + \frac{52488}{24n + 7} + \frac{23328}{24n + 10} + \frac{23328}{24n + 11} + \frac{15552}{24n + 13} + \frac{10368}{24n + 14} - \frac{6912}{24n + 17} - \frac{9216}{24n + 18} - \frac{4608}{24n + 19} - \frac{2048}{24n + 22} - \frac{2048}{4n + 23} \right\}.$$

- It will not work base-3 because of the $2$. 
But not $\pi^2$ base 10 or $\pi$ base 3:

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} = \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6} \frac{(532n^2 + 126n + 9)}{10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^{12n} \left( \frac{177147}{24n+1} + \frac{118098}{24n+2} + \frac{78732}{24n+5} + \frac{104976}{24n+6} + \frac{52488}{24n+7} 
+ \frac{23328}{24n+10} + \frac{23328}{24n+11} - \frac{15552}{24n+13} - \frac{10368}{24n+14} - \frac{6912}{24n+17} 
- \frac{9216}{24n+18} - \frac{4608}{24n+19} - \frac{2048}{24n+22} - \frac{2048}{4n+23} \right).$$

- It will not work base-3 because of the 2.
Two Sporadic Rational Gems

**Gourevich 2001**

\[
\frac{2^5}{\pi^3} \pi^2 \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^7}{(1)_n^7} (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{2}\right)^{6n}
\]

where \( a_n := a(a + 1) \cdots (a + n - 1) \) so that \((1)_n = n!\)

**Cullen 2010**

\[
\frac{2^{11}}{\pi^4} = \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^7 (\frac{3}{4})_n}{(1)_n^9} (21 + 466n + 4340n^2 + 20632n^3 + 43680n^4) \left(\frac{1}{2}\right)^{12n}
\]

I rediscovered and confirmed both to 10,000 digits while preparing the slide! As follows....
Two Sporadic Rational Gems

Gourevich 2001

\[ \frac{2^5}{\pi^3} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^7}{(1)_n^7} (1 + 14n + 76n^2 + 168n^3) \left( \frac{1}{2} \right)^{6n} \]

where \( a_n := a(a + 1) \cdots (a + n - 1) \) so that \( (1)_n = n! \)

Cullen 2010

\[ \frac{2^{11}}{\pi^4} = \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n^7(\frac{3}{4})_n^9}{(1)_n^9} (21 + 466n + 4340n^2 + 20632n^3 + 43680n^4) \left( \frac{1}{2} \right)^{12n} \]

I rediscovered and confirmed both to 10,000 digits while preparing the slide! As follows....
Discovering and validating Cullen’s formula in *Maple*:

\[
\frac{1}{\pi^4} = \sum_{n=0}^{\infty} \frac{p\left(\frac{1}{4}, n\right)^7 p\left(\frac{3}{4}, n\right) p\left(\frac{1}{2}, n\right)^7}{n!^9} 2^{-12n} \left(466 n + 4340 n^2 + 20632 n^3 + 43680 n^4 + 21\right)
\]

- Confirming the value of the sum to **10,000** places is near instant and **100,000** places took **21.35** secs.
Discovering and validating Cullen’s formula in *Maple*:

\[
\frac{1}{\pi^4} = \sum_{n=0}^{\infty} \frac{1}{2048} \left( \frac{1}{4}, n \right) p \left( \frac{3}{4}, n \right) p \left( \frac{1}{2}, n \right)^7 n!^9 2^{-12n} (466n + 4340n^2 + 20632n^3 + 43680n^4 + 21) 
\]

- Confirming the value of the sum to **10,000** places is near instant and **100,000** places took **21.35** secs.
3. What is that **Sequence**?

For \( n = 0, 1, 2, \ldots \) set

\[
J_n := \int_{-\infty}^{\infty} \text{sinc} \ x \cdot \text{sinc} \left( \frac{x}{3} \right) \cdots \text{sinc} \left( \frac{x}{2n+1} \right) \ dx.
\]

Then — as **Maple** and **Mathematica** confirm — we have:

\[
J_0 = \int_{-\infty}^{\infty} \text{sinc} \ x \ dx = \pi,
\]

\[
J_1 = \int_{-\infty}^{\infty} \text{sinc} \ x \cdot \text{sinc} \left( \frac{x}{3} \right) \ dx = \pi,
\]

\[
\vdots
\]

\[
J_6 = \int_{-\infty}^{\infty} \text{sinc} \ x \cdot \text{sinc} \left( \frac{x}{3} \right) \cdots \text{sinc} \left( \frac{x}{13} \right) \ dx = \pi.
\]
\[ J_7 = \int_{-\infty}^{\infty} \text{sinc} x \cdot \text{sinc} \left( \frac{x}{3} \right) \cdots \text{sinc} \left( \frac{x}{15} \right) \, dx \]

\[ = \frac{467807924713440738696537864469}{467807924720320453655260875000} \pi < \pi, \]

where the fraction is approximately 0.99999999998529 \ldots .

1912 G. Pólya showed that given the slab

\[ S_k(\theta) := \{ x \in \mathbb{R}^n : |\langle k, x \rangle| \leq \theta / 2, x \in C^n \} \]

inside the hypercube \( C^n = [-\frac{1}{2}, \frac{1}{2}]^n \) cut off by the hyperplanes \( \langle k, x \rangle = \pm \theta / 2 \), then

\[ \text{Vol}_n(S_k(\theta)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\theta x)}{x} \prod_{j=1}^{n} \frac{\sin(k_j x)}{k_j x} \, dx. \]
The really obvious pattern — see Corollary below — is confounded by

\[
J_7 = \int_{-\infty}^{\infty} \text{sinc } x \cdot \text{sinc } \left( \frac{x}{3} \right) \cdots \text{sinc } \left( \frac{x}{15} \right) \, dx
\]

\[
= \frac{46780792471344073864537864469}{467807924720320453655260875000} \pi < \pi,
\]

where the fraction is approximately 0.99999999998529\ldots.

1912 G. Pólya showed that given the slab

\[ S_k(\theta) := \{ x \in R^n : |\langle k, x \rangle| \leq \theta/2, \, x \in C^n \} \]

inside the hypercube \( C^n = [-\frac{1}{2}, \frac{1}{2}]^n \) cut off by the hyperplanes \( \langle k, x \rangle = \pm \theta/2 \), then

\[
\text{Vol}_n(S_k(\theta)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\theta x)}{x} \prod_{j=1}^{n} \frac{\sin(k_j x)}{k_j x} \, dx.
\]
$\pi, \pi, \pi, \pi, \pi, \pi, ?$ has gone viral

- Also [http://www.tumblr.com/tagged/the-borwein-integral-is-the-troll-of-calculus](http://www.tumblr.com/tagged/the-borwein-integral-is-the-troll-of-calculus)
- There is even a movie: [http://www.qwiki.com/embed/Borwein_integral](http://www.qwiki.com/embed/Borwein_integral)
Mathematics is becoming Hybrid:

1968 A ‘solved’ MAA problem.
1971 Withdrawn.
May 2011 Seemed still ‘open’? (JSTOR).
Oct 2011 (MAA, Aug-Sept 2012): a fine symbolic/numeric/graphic (SNaG) chal-
lenge:
http://carma.newcastle.edu.au/jon/sink.pdf and below:
Mathematics is becoming Hybrid:

and none to soon

5529 [1967, 1015; 1968; 914]. Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia

Evaluate

\[ \int_{-\infty}^{\infty} \prod_{j=1}^{n} \sin \frac{k_j(x - a_j)}{x - a_j} \, dx, \]

with \( k_j, a_j, j = 1, 2, \ldots, n \) real numbers.

Note. The published solution for this problem is in error. Murray S. Klamkin remarks that it is to be expected that the given integral depend on all the \( k's \) and be symmetric in \( k, a_j \). The formula obtained in the solution

\[ I = \pi \prod_{j=2}^{n} \frac{\sin k(a_{j-1} - a_j)}{a_j - a_{j-1}} \]

does not involve \( k_1 \) and is not symmetric as required. (\( k_1 = 0 \) must imply \( I = 0 \).) Accordingly the solution is withdrawn and we urge our readers to reconsider the problem.

1968 A ‘solved’ MAA problem.
1971 Withdrawn.
May 2011 Seemed still ‘open’? (JSTOR).

Oct 2011 (MAA, Aug-Sept 2012): a fine symbolic/numeric/graphic (SNaG) challenge:
http://carma.newcastle.edu.au/jon/sink.pdf and below:
Mathematics is becoming Hybrid:

5529 [1967, 1015; 1968, 914]. Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia

Evaluate

\[ \int_{-\infty}^{\infty} \prod_{j=1}^{\infty} \frac{\sin k_j(x - a_j)}{x - a_j} \, dx, \]

with \( k_j, a_j, j = 1, 2, \ldots, \infty \) real numbers.

Note. The published solution for this problem is in error. Murray S. Klamkin remarks that it is to be expected that the given integral depend on all the \( k's \) and be symmetric in \( k, a \). The formula obtained in the solution

\[ I = \pi \prod_{j=2}^{\infty} \frac{\sin k(a_{j-1} - a_j)}{a_{j-1} - a_j} \]

does not involve \( k_1 \) and is not symmetric as required. (\( k_1 = 0 \) must imply \( I = 0 \).)

Accordingly the solution is withdrawn and we urge our readers to reconsider the problem.

1968 A 'solved' MAA problem.
1971 Withdrawn.
May 2011 Seemed still 'open'? (JSTOR).
Oct 2011 (MAA, Aug-Sept 2012): a fine symbolic/numeric/graphic (SNaG) challenge:
http://carma.newcastle.edu.au/jon/sink.pdf and below:
What has happened to $J_7$?

The fact that $J_0 = J_1 = \cdots = J_6 = \pi$ follows from:

**Corollary (Simplest Case)**

Suppose $k_1, k_2, \ldots, k_n > 0$ and there is an index $\ell$ such that

$$k_\ell > \frac{1}{2} \sum k_i.$$

Then, the original solution to the Monthly problem is valid:

$$I_n = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{\sin(k_i(x - a_i))}{x - a_i} \, dx = \pi \prod_{i \neq \ell} \frac{\sin(k_i(a_\ell - a_i))}{a_\ell - a_i}.$$
What has happened to $J_7$?

**Theorem (First bite, DB-JB 1999)**

Denote $K_m = k_0 + k_1 + l, \ldots + k_m$. If $2k_j \geq k_n > 0$ for $j = 0, 1, \ldots, n - 1$ and $K_n > 2k_0 \geq K_{n-1}$ then

$$
\int_{-\infty}^{\infty} \prod_{j=0}^{n} \frac{\sin(k_jx)}{x} \, dx = \pi k_1 k_2 \cdots k_n - \frac{\pi}{2^{n-1}n!} (K_n - 2k_0)^n. \quad (7)
$$

But if $2k_0 > K_n$ the integral evaluates to $\pi k_1 k_2 \cdots k_n$.

The theorem makes it clear that the pattern that $J_n = \pi$ for $n = 0, 1, \ldots, 6$ breaks for $J_7$ because

$$
\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{15} > 1
$$

whereas all earlier partial sums are less than 1.
What has happened to $J_7$?

**Theorem (First bite, DB-JB 1999)**

Denote $K_m = k_0 + k_1 + l, \ldots + k_m$. If $2k_j \geq k_n > 0$ for $j = 0, 1, \ldots, n - 1$ and $K_n > 2k_0 \geq K_{n-1}$ then

$$
\int_{-\infty}^{\infty} \prod_{j=0}^{n} \sin\left(\frac{k_j x}{x}\right) \, dx = \pi k_1 k_2 \cdots k_n - \frac{\pi}{2^n - 1} (K_n - 2k_0)^n. \quad (7)
$$

But if $2k_0 > K_n$ the integral evaluates to $\pi k_1 k_2 \cdots k_n$.

The theorem makes it clear that the pattern that $J_n = \pi$ for $n = 0, 1, \ldots, 6$ breaks for $J_7$ because

$$
\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{15} > 1
$$

whereas all earlier partial sums are less than 1.
Theorem (Baillie-Borwein-Borwein, MAA 2008)

Suppose that $k_1, k_2, \ldots, k_n > 0$. If $k_1 + k_2 + \ldots + k_n < 2\pi$ then

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} \text{sinc}(k_j x) \, dx = \sum_{m=-\infty}^{\infty} \prod_{j=1}^{n} \text{sinc}(k_j m).$$

As a consequence, with $k_j = \frac{1}{2j+1}$:

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \text{sinc}\left(\frac{x}{2j+1}\right) \, dx \geq \sum_{m=-\infty}^{\infty} \prod_{j=0}^{n} \text{sinc}\left(\frac{m}{2j+1}\right)$$

with equality iff $n = 1, 2, \ldots, 7, 8, \ldots, 40248$. 

J.M. Borwein
Meetings with Special Functions
Theorem (Baillie-Borwein-Borwein, MAA 2008)

Suppose that \( k_1, k_2, \ldots, k_n > 0 \). If \( k_1 + k_2 + \ldots + k_n < 2\pi \) then

\[
\int_{-\infty}^{\infty} \prod_{j=1}^{n} \text{sinc}(k_j x) \, dx = \sum_{m=-\infty}^{\infty} \prod_{j=1}^{n} \text{sinc}(k_j m) \tag{8}
\]

As a consequence, with \( k_j = \frac{1}{2j+1} \):

\[
\int_{-\infty}^{\infty} \prod_{j=0}^{n} \text{sinc} \left( \frac{x}{2j+1} \right) \, dx \geq \sum_{m=-\infty}^{\infty} \prod_{j=0}^{n} \text{sinc} \left( \frac{m}{2j+1} \right) \tag{9}
\]

with equality iff \( n = 1, 2, \ldots, 7, 8, \ldots, 40248 \).
Other Surprises

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds. (John Maynard Keynes, 1883-1946)

Example (What is equality?)

- An entertaining example takes the reciprocals of primes $2, 3, 5, \ldots$: using the Prime Number theorem one estimates that the sinc integrals equal the sinc sums until the number of products is about $10^{176}$.

- That of course makes it rather unlikely to find by mere testing an example where the two are unequal.

- Even worse for the naive tester is the fact that the discrepancy between integral and sum is always less than $10^{-10^86}$ — smaller if the Riemann hypothesis is true.
Other Surprises

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds. (John Maynard Keynes, 1883-1946)

Example (What is equality?)

- An entertaining example takes the reciprocals of primes $2, 3, 5, \ldots$: using the **Prime Number theorem** one estimates that the sinc integrals equal the sinc sums until the number of products is about $10^{176}$.

- That of course makes it rather unlikely to find by mere testing an example where the two are unequal.

- Even worse for the naive tester is the fact that the discrepancy between integral and sum is always less than $10^{-10^{86}}$ — smaller if the **Riemann hypothesis** is true.
How to Judge a new Scientific Claim

Was the problem and solution the ‘GPS’

1995: Andrew Granville emailed and challenged me to identify:

\[ \alpha := 1.4331274267223 \ldots \]  

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

\[ \alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \]  

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

\[ \alpha = \frac{I_1(2)}{I_0(2)} \]  

where \( I_0 \) and \( I_1 \) are *Bessel functions* of the first kind.
4. What is that Number?

1995: Andrew Granville emailed and challenged me to identify:

\[ \alpha := 1.4331274267223 \ldots \]  

(10)

I think this was a test I could have failed.

- I asked \textit{Maple} for its continued fraction.
- In conventional concise notation I was rewarded with

\[ \alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots] \]  

(11)

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

\[ \alpha = \frac{I_1(2)}{I_0(2)} \]  

(12)

where \( I_0 \) and \( I_1 \) are \textit{Bessel functions} of the first kind.
4. What is that **Number?**

1995: Andrew Granville emailed and challenged me to identify:

\[ \alpha := 1.4331274267223 \ldots \]  \hspace{1cm} (10)

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

\[ \alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \]  \hspace{1cm} (11)

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

\[ \alpha = \frac{I_1(2)}{I_0(2)} \]  \hspace{1cm} (12)

where \( I_0 \) and \( I_1 \) are *Bessel functions* of the first kind.
1995: Andrew Granville emailed and challenged me to identify:

\[
\alpha := 1.4331274267223 \ldots \tag{10}
\]

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

\[
\alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \tag{11}
\]

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

\[
\alpha = \frac{I_1(2)}{I_0(2)} \tag{12}
\]

where \( I_0 \) and \( I_1 \) are *Bessel functions* of the first kind.
4. What is that **Number?**

1995: Andrew Granville emailed and challenged me to identify:

\[ \alpha := 1.4331274267223 \ldots \quad (10) \]

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

\[ \alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \quad (11) \]

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

\[ \alpha = \frac{I_1(2)}{I_0(2)} \quad (12) \]

where \( I_0 \) and \( I_1 \) are *Bessel functions* of the first kind.
Actually, I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In 2011 there are at least three “zero-knowledge” strategies:

1. Given (11), type “arithmetic progression”, “continued fraction” into Google.

2. Type “1, 4, 3, 3, 1, 2, 7, 4, 2” into Sloane’s Encyclopedia of Integer Sequences.¹

3. Type the decimal digits of $\alpha$ into the Inverse Symbolic Calculator.²

I illustrate the results of each strategy.

²The Inverse Symbolic Calculator http://isc.carma.newcastle.edu.au/ was newly web-accessible in the same year, 1995.
What is that Number?

Actually, I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In 2011 there are at least three “zero-knowledge” strategies:

1. Given (11), type “arithmetic progression”, “continued fraction” into Google.

2. Type “1, 4, 3, 3, 1, 2, 7, 4, 2” into Sloane’s Encyclopedia of Integer Sequences.¹

3. Type the decimal digits of $\alpha$ into the Inverse Symbolic Calculator.²

I illustrate the results of each strategy.

---

²The Inverse Symbolic Calculator http://isc.carma.newcastle.edu.au/ was newly web-accessible in the same year, 1995.
Actually, I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In 2011 there are at least three “zero-knowledge” strategies:

1. Given (11), type “arithmetic progression”, “continued fraction” into Google.

2. Type “1, 4, 3, 3, 1, 2, 7, 4, 2” into Sloane’s Encyclopedia of Integer Sequences.¹

3. Type the decimal digits of $\alpha$ into the Inverse Symbolic Calculator.²

I illustrate the results of each strategy.

²The Inverse Symbolic Calculator http://isc.carma.newcastle.edu.au/ was newly web-accessible in the same year, 1995.
What is that Number?

1. On Oct 15, 2008, on typing “arithmetic progression”, “continued fraction” into Google, the first 3 hits were:

---

Continued Fraction Constant -- from Wolfram MathWorld

- 3 visits - 14/09/07

Perron (1954-57) discusses continued fractions having terms even more general than the arithmetic progression and relates them to various special functions. ...  
mathworld.wolfram.com/ContinuedFractionConstant.html - 31k

HAKMEM -- CONTINUED FRACTIONS -- DRAFT, NOT YET PROOFED

The value of a continued fraction with partial quotients increasing in arithmetic progression is $I(2/D) A/D \ [A+D, A+2D, A+3D, \ldots]$  
www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k

On simple continued fractions with partial quotients in arithmetic ...

0. This means that the sequence of partial quotients of the continued fractions under investigation consists of finitely many arithmetic progressions (with ...


Moreover the MathWorld entry includes

$$[A + D, A + 2D, A + 3D, \ldots] = \frac{I_{A/D}(\frac{2}{D})}{I_{1 + A/D}(\frac{2}{D})}$$  
(Schroeppe 1972) for real $A$ and $D \neq 0$
2. Typing the first few digits into Sloane’s interface results in the response shown in the Figure on the next slide.

- In this case we are even told what the series representations of the requisite Bessel functions are.
- We are given sample code (in this entry in Mathematica), and we are lead to many links and references.
- The site is well moderated.
- Note also that this strategy only became viable after May 14th 2001 when the sequence was added to the database which now contains in excess of 158,000 entries.
Sloane’s Online Encyclopedia (OEIS)

Greetings from The On-line Encyclopedia of Integer Sequences!

Search: 1, 4, 3, 3, 1, 2, 7, 4, 2
Displaying 1-1 of 1 results found.

AO60397
Decimal representation of continued fraction 1, 2, 3, 4, 5, 6, 7, ...

OFFSET
1,2

COMMENT
The value of this continued fraction is the ratio of two Bessel functions: BesselI(0,2)/BesselI(1,2) = A070910/A086782. Or, equivalently, to the ratio of the sums: sum_{n=0..inf} 1/(n!n!) and sum_{n=0..inf} n/(n!n!). — Mark Hudson (mrmarkhudson(AT)hotmail.com), Jan 31 2003

FORMULA
1/A052119.

EXAMPLE
c=1.433127426722311758317183455775 ...

MATHEMATICA
RealDigits[ FromContinuedFraction[ Range[ 44]], 10, 110] [[1]]
(* Or *) RealDigits[ BesselI[0, 2] / BesselI[1, 2], 10, 110] [[1]]
(* Or *) RealDigits[ Sum[1/(n!!), (n, 0, Infinity)] / Sum[n/(n!!), (n, 0, Infinity)], 10, 110] [[1]]

CROSSREFS
cf. A052119, A001055.

AUTHOR
Robert G. Wilson v (rgwv(AT)rgwv.com), May 14 2001
What is that number? Strategy 3

3. If one types the decimal representation of $\alpha$ into the Inverse Symbolic Calculator (ISC) it returns:

Best guess: $\text{BesI}(0,2)/\text{BesI}(1,2)$

• Most of the functionality of the ISC is built into the `identify` function in versions of Maple starting with version 9.5.
• For example,

  > identify(4.45033263602792)

  returns

  $\sqrt{3} + e$.

• As always, the experienced user will be able to extract more from this tool than the novice for whom the ISC will often produce more.
5. What is that Limit?

MAA Problem 10832, 2000 (Donald E. Knuth): Evaluate

\[ S = \sum_{k=1}^{\infty} \left( \frac{k^k}{k!e^k} - \frac{1}{\sqrt{2\pi k}} \right) . \]

**Solution:** Using *Maple*, we easily produced the approximation

\[ S \approx -0.08406950872765599646 . \]

“Smart Lookup” in the Inverse Symbolic Calculator, yielded

\[ S \approx -\frac{2}{3} - \frac{1}{\sqrt{2\pi}} \zeta \left( \frac{1}{2} \right) . \quad (13) \]

- Calculations to higher precision (50 decimal digits) confirmed this approximation. Thus within a few minutes we “knew” the answer.
Why should such an identity hold and be provable?

• One clue was provided by the surprising speed with which \textit{Maple} was able to calculate a high-precision value of this slowly convergent infinite sum.

• Evidently, the \textit{Maple} software knew something that we did not. Peering under the covers, we found that \textit{Maple} was using the \textit{Lambert W} function, which is the functional inverse of $w(z) = ze^z$.

• Another clue was the appearance of $\zeta(1/2)$ in the discovered identity, together with an obvious allusion to Stirling’s formula in the problem.
This led us to

**Conjecture**

\[
\sum_{k=1}^{\infty} \left( \frac{1}{\sqrt{2\pi k}} - \frac{(1/2)^{k-1}}{(k-1)!\sqrt{2}} \right) = \frac{1}{\sqrt{2\pi}} \zeta \left( \frac{1}{2} \right),
\]

where \((x)_n := x(x + 1)\cdots(x + n - 1)\).

- **Maple** successfully evaluated this summation, to the RHS.

We now needed to establish that

\[
\sum_{k=1}^{\infty} \left( \frac{k^k}{k!e^k} - \frac{(1/2)^{k-1}}{(k-1)!\sqrt{2}} \right) = -\frac{2}{3}.
\]
We noted the presence of the Lambert $W$ function,

$$W(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!}. \quad (15)$$

Since

$$\sum_{k=1}^{\infty} \frac{(1/2)_{k-1} z^{k-1}}{(k-1)!} = \frac{1}{\sqrt{1-z}}$$

an appeal to Abel’s limit theorem showed it sufficed to prove:

**Conjecture**

$$\lim_{z \to 1} \left( \frac{dW(-z/e)}{dz} + \frac{1}{\sqrt{2-2z}} \right) = \frac{2}{3}.$$

- Again, Maple can be coaxed to establish the identity.
We noted the presence of the Lambert $W$ function,

$$W(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!}.$$  \hspace{1cm} (15)

Since

$$\sum_{k=1}^{\infty} \frac{(1/2)^{k-1} z^{k-1}}{(k - 1)!} = \frac{1}{\sqrt{1 - z}},$$

an appeal to Abel’s limit theorem showed it sufficed to prove:

Conjecture

$$\lim_{z \to 1} \left( \frac{dW(-z/e)}{dz} + \frac{1}{\sqrt{2 - 2z}} \right) = \frac{2}{3}.$$

- Again, Maple can be coaxed to establish the identity.
The above manipulations took considerable human ingenuity, in addition to symbolic manipulation and numerical discovery.

A challenge for the next generation of mathematical computing software, is to more completely automate this class of operations.

E.g., *Maple* does not recognize $W$ from its Maclaurin series (15).

**Figure:** $W$ on the real line
6. What is that Continued fraction?

The Ramanujan AGM continued fraction

\[ R_\eta(a, b) = \frac{a}{\eta + \frac{b^2}{\eta + \frac{4a^2}{\eta + \frac{9b^2}{\eta + \ldots}}} } \]

enjoys attractive algebraic properties such as a striking arithmetic-geometric mean relation & elegant links with elliptic-function theory.

- The fraction presented a serious computational challenge, which we could not resist.
5. What is that Continued fraction?  

The AG fraction.

Figure: Yellow cardioid in which everything works

**Theorem (AG continued fraction)**

For \( \eta > 0 \) and complex \( a, b \) the fraction \( R_\eta \) converges and satisfies:

\[
R_\eta \left( \frac{a + b}{2}, \sqrt{ab} \right) = \frac{R_\eta(a, b) + R_\eta(b, a)}{2}
\]

if and only if \( a/b \in \mathcal{H} \) the cardioid given by

\[
\mathcal{H} := \{ z \in \mathbb{C} : \left| \frac{2\sqrt{z}}{1 + z} \right| < 1 \}.
\]
What is that Continued fraction?

**Theorem (For \( a > 0 \))**

\[
\mathcal{R}_1(a, a) = \int_0^\infty \frac{\sech\left(\frac{\pi x}{2a}\right)}{1 + x^2} \, dx
\]

\[= 2a \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1 + (2k - 1)a}
\]

\[= \frac{1}{2} \left( \psi \left( \frac{3}{4} + \frac{1}{4a} \right) - \psi \left( \frac{1}{4} + \frac{1}{4a} \right) \right)
\]

\[= \frac{2a}{1 + a} F \left( \frac{1}{2a} + \frac{1}{2}, 1; \frac{1}{2a} + \frac{3}{2}; -1 \right) \quad (Gauss c.f.)
\]

\[= 2 \int_0^1 \frac{t^{1/a}}{1 + t^2} \, dt
\]

\[= \int_0^\infty e^{-x/a} \sech(x) \, dx.
\]
What is that Continued fraction?  

• This is deduced from a Riemann sum via an elliptic integral/theta-function formula.

• For $a = p/q$ rational we obtain an explicit closed form. Special cases include

$$\mathcal{R}(1) = \log 2 \quad \text{and} \quad \mathcal{R} \left( \frac{1}{2} \right) = 2 - \frac{\pi}{2}.$$ 

– Originally, we could not compute 4 digits of these values! Now have fast methods in all of $C^2$.

• For $a$ with strictly positive (or negative) real part $\mathcal{R}(a) := \mathcal{R}_1(a)$ exists and is holomorphic.

• $\mathcal{R}(r i) \ (r \neq 0)$ behaves chaotically with 4-fold bifurcation.

• Find a closed form for $\mathcal{R}(a, b)$ for some $a \neq b$?

J.M. Borwein  
Meetings with Special Functions
What is that **Continued fraction**?

- This is deduced from a Riemann sum via an elliptic integral/theta-function formula.
- For $a = p/q$ rational we obtain an explicit closed form. Special cases include
  
  $$\mathcal{R}(1) = \log 2 \quad \text{and} \quad \mathcal{R}\left(\frac{1}{2}\right) = 2 - \frac{\pi}{2}.$$  

  - Originally, we could not compute 4 digits of these values! Now have fast methods in all of $\mathbb{C}^2$.

- For $a$ with strictly positive (or negative) real part $\mathcal{R}(a) := \mathcal{R}_1(a)$ exists and is holomorphic.
- $\mathcal{R}(ri)$ ($r \neq 0$) behaves chaotically with 4-fold bifurcation.
- Find a closed form for $\mathcal{R}(a, b)$ for some $a \neq b$?
What is that **Continued fraction**?  

The first sech-integral for $\mathcal{R}(a)$ and the even Euler numbers $E_{2n} := (-1)^n \int_0^\infty \text{sech}(\pi x/2) x^{2n} \, dx$

yield

$$
\mathcal{R}(a) \sim \sum_{n \geq 0} E_{2n} a^{2n+1}
$$

giving an **asymptotic series of zero radius** of convergence. Here the $E_{2n}$ commence $1, -1, 5, -61, 1385, -50521, 2702765 \ldots$

Moreover, for the **asymptotic error**, we have:

$$
\left| \mathcal{R}(a) - \sum_{n=1}^{N-1} E_{2n} a^{2n+1} \right| \leq |E_{2N}| a^{2N+1},
$$

- It is a classic theorem of Borel that for every real sequence $(a_n)$ there is a $C^\infty$ function $f$ on $\mathbb{R}$ with $f^{(n)}(0) = a_n$.
- Who knew they could be so explicit?
What is that Continued fraction?

The first sech-integral for $\mathcal{R}(a)$ and the even Euler numbers

$$E_{2n} := (-1)^n \int_0^\infty \text{sech}(\pi x/2) x^{2n} \, dx$$

give

$$\mathcal{R}(a) \sim \sum_{n \geq 0} E_{2n} a^{2n+1},$$

giving an asymptotic series of zero radius of convergence. Here the $E_{2n}$ commence $1, -1, 5, -61, 1385, -50521, 2702765 \ldots$

Moreover, for the asymptotic error, we have:

$$\left| \mathcal{R}(a) - \sum_{n=1}^{N-1} E_{2n} a^{2n+1} \right| \leq |E_{2N}| a^{2N+1},$$

- It is a classic theorem of Borel that for every real sequence $(a_n)$ there is a $C^\infty$ function $f$ on $\mathbb{R}$ with $f^{(n)}(0) = a_n$.
- Who knew they could be so explicit?
Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

**Theorem (Divergence of $\mathcal{R}$)**

Consider the linearised dynamical system $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for $n$ even, odd resp. (or is more general).

Then $\sqrt{n} t_n$ is bounded $\iff \mathcal{R}_1(a, b)$ diverges.

Numerically all we learned is that $t_n \to 0$ slowly.

Pictorially we saw more (in *Cinderella*):


J.M. Borwein
Meetings with Special Functions
Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

**Theorem (Divergence of $\mathcal{R}$)**

Consider the linearised dynamical system $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for $n$ even, odd resp. (or is more general). Then $\sqrt{n} t_n$ is bounded $\Leftrightarrow \mathcal{R}_1(a, b)$ diverges.

Numerically all we learned is that $t_n \to 0$ slowly. Pictorially we saw more (in *Cinderella*):

What is that **Continued fraction?**

Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

**Theorem (Divergence of \( \mathcal{R} \))**

Consider the linearised dynamical system \( t_0 := t_1 := 1 \):

\[
t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left( 1 - \frac{1}{n} \right) t_{n-2},
\]

where \( \omega_n = a^2, b^2 \) for \( n \) even, odd resp. (or is more general). Then \( \sqrt{n} t_n \) is bounded \( \Leftrightarrow \mathcal{R}_1(a, b) \) diverges.

Numerically all we learned is that \( t_n \to 0 \) slowly.

Pictorially we saw more (in *Cinderella*):

Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

**Theorem (Divergence of $\mathcal{R}$)**

Consider the linearised dynamical system $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for $n$ even, odd resp. (or is more general).

Then $\sqrt{n} t_n$ is bounded $\iff \mathcal{R}_1(a, b)$ diverges.

Numerically all we learned is that $t_n \to 0$ slowly.

Pictorially we saw more (in *Cinderella*):


J.M. Borwein
Meetings with Special Functions
La plus ça change, II

YOU WANT YOUR COUSIN TO SEND YOU A FILE? EASY.
HE CAN EMAIL IT TO— ... OH, IT'S 25 MB? HMM...

DO EITHER OF YOU HAVE AN FTP SERVER? NO, RIGHT.
IF YOU HAD WEB HOSTING, YOU COULD UPLOAD IT...

HMM. WE COULD TRY ONE OF THOSE MEGASHARE UPLOAD SITES,
BUT THEY'RE FLAKY AND FULL OF DELAYS AND PORN POPUPS.

HOW ABOUT AIM DIRECT CONNECT? ANYONE STILL USE THAT?

OH, WAIT, DROPBOX! IT'S THIS RECENT STARTUP FROM A FEW
YEARS BACK THAT SYNCs FOLDERS BETWEEN COMPUTERS.
YOU JUST NEED TO MAKE AN ACCOUNT, INSTALL THE—

OH, HE JUST DROVE OVER TO YOUR HOUSE
WITH A USB DRIVE?

UH, COOL, THAT WORKS, TOO.

I LIKE HOW WE'VE HAD THE INTERNET FOR DECADES,
YET "SENDING FILES" IS SOMETHING EARLY
ADOPTERS ARE STILL FIGURING OUT HOW TO DO.
7. What is that **Probability**?

**Question (SIAM 100 digit challenge, 2003)**

#10. A particle at the center of a $10 \times 1$ rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

- See also: [http://www-m3.ma.tum.de/m3old/bornemann/challengebook/index.html](http://www-m3.ma.tum.de/m3old/bornemann/challengebook/index.html).
- Image is a walk on the first two billion bits of Pi: see [http://carma.newcastle.edu.au/piwalk.shtml](http://carma.newcastle.edu.au/piwalk.shtml).
7. What is that Probability?

**Question (SIAM 100 digit challenge, 2003)**

A particle at the center of a $10 \times 1$ rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

- See also: http://www-m3.ma.tum.de/m3old/bornemann/challengebook/index.html.
- Image is a walk on the first two billion bits of Pi: see http://carma.newcastle.edu.au/piwalk.shtml.
Problem #10: **Hitting the Ends.**

1. Monte-Carlo methods are impracticable.

2. Reformulate *deterministically* as the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends, arising from a 5-point discretization of Laplace’s equation with Dirichlet boundary conditions.

3. Solved with a well chosen *sparse Cholesky* solver.

4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.
Problem #10: **Hitting the Ends.**

1. Monte-Carlo methods are impracticable.

2. Reformulate *deterministically* as the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends, arising from a 5-point discretization of Laplace’s equation with Dirichlet boundary conditions.

3. Solved with a well chosen *sparse Cholesky* solver.

4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.
Problem #10: **Hitting the Ends.**

1. **Monte-Carlo methods** are impracticable.
2. Reformulate *deterministically as the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of Laplace’s equation with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.
Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.

2. Reformulate *deterministically as the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of Laplace’s equation with Dirichlet boundary conditions.

3. Solved with a well chosen *sparse Cholesky* solver.

4. A reliable numerical value of

\[ 3.837587979 \cdot 10^{-7} \]

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.
Problem #10: **Hitting the Ends.**

1. Monte-Carlo methods are impracticable.

2. Reformulate *deterministically as the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of Laplace’s equation with Dirichlet boundary conditions.

3. Solved with a well chosen *sparse Cholesky* solver.

4. A reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.
What is that Probability?

Problem #10: **Hitting the Ends.**

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically as the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of Laplace’s equation with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.
We develop two analytic solutions — which must agree — on a general $2a \times 2b$ rectangle:

1. Via separation of variables on the underlying PDE

\[
p(a, b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \text{sech} \left( \frac{\pi(2n + 1)}{2} \rho \right)
\]

where $\rho := a/b$.

2. Using conformal mappings, yields

\[
\arccot \rho = p(a, b) \frac{\pi}{2} + \arg K \left( e^{i p(a,b) \pi} \right)
\]

where $K$ is the complete elliptic integral of the first kind.
What is that Probability?

We develop two analytic solutions — which must agree — on a general $2a \times 2b$ rectangle:

1. Via separation of variables on the underlying PDE

$$p(a, b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \text{sech} \left( \frac{\pi(2n + 1)}{2} \rho \right)$$

where $\rho := a/b$.

2. Using conformal mappings, yields

$$\arccot \rho = p(a, b) \frac{\pi}{2} + \text{arg} K \left( e^{ip(a,b)\pi} \right)$$

where $K$ is the complete elliptic integral of the first kind.
We develop two analytic solutions — which must agree — on a general \(2a \times 2b\) rectangle:

1. Via *separation of variables* on the underlying PDE

\[
p(a, b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{sech} \left( \frac{\pi(2n+1)}{2 \rho} \right)
\]  

(16)

where \(\rho := a/b\).

2. Using *conformal mappings*, yields

\[
\text{arccot } \rho = p(a, b) \frac{\pi}{2} + \arg K \left( e^{ip(a,b)\pi} \right)
\]  

(17)

where \(K\) is the *complete elliptic integral* of the first kind.
Now (3.2.29) in Pi&AGM shows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{sech} \left( \frac{\pi(2n+1)}{2} \rho \right) = \frac{1}{2} \arcsin k_\rho$$

(18)

exactly when $k_\rho^2$ is parameterized by \textit{theta functions} as follows.

- As Jacobi discovered via the \textit{nome}, $q = \exp(-\pi \rho)$:

$$k_\rho^2 = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}$$

$q := e^{-\pi \rho}$.

- Comparing (18) and (16) we see that the solution is

$$p = \frac{2}{\pi} \arcsin (k_{100}),$$

$$k_{100} = 6.02806910155971082882540712292 \ldots \cdot 10^{-7}.$$
What is that Probability? Bornemann’s solution, 3.

Now \((3.2.29)\) in \textit{Pi\&AGM} shows that

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{sech} \left( \frac{\pi(2n+1)}{2} \rho \right) = \frac{1}{2} \arcsin k_\rho
\]  

(18)

exactly when \(k_\rho^2\) is parameterized by \textit{theta functions} as follows.

- As Jacobi discovered via the \textit{nome}, \(q = \exp(-\pi \rho)\):

\[
k_\rho^2 = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \quad q := e^{-\pi \rho}.
\]

- Comparing (18) and (16) we see that the solution is

\[
p = \frac{2}{\pi} \arcsin (k_{100}), \quad k_{100} = 6.02806910155971082882540712292 \ldots \cdot 10^{-7}.
\]
Now (3.2.29)] in Pi&AGM shows that

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{sech} \left( \frac{\pi(2n+1)}{2} \rho \right) = \frac{1}{2} \arcsin k_{\rho^2}
\]

(18)

exactly when \( k_{\rho^2} \) is parameterized by \textit{theta functions} as follows.

- As Jacobi discovered via the \textit{nome}, \( q = \exp(-\pi \rho) \):

\[
k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \quad q := e^{-\pi \rho}.
\]

- Comparing (18) and (16) we see that the solution is

\[
p = \frac{2}{\pi} \arcsin (k_{100}),
\]

\[
k_{100} = 6.02806910155971082882540712292 \ldots \cdot 10^{-7}.
\]
What is that Probability?

- Classical nineteenth century modular function theory tells us all rational singular values $k_n$ are algebraic (solvable).

- Now, we can hunt in books or obtain the solution automatically in Maple: Thence

$$k_{100} := \left( (3 - 2\sqrt{2}) (2 + \sqrt{5}) (-3 + \sqrt{10}) (-\sqrt{2} + \frac{4}{5}) \right)^2$$

- No one anticipated a closed form like this, except perhaps a few harmonic analysts.
  
  - For what boundaries can one emulate this?

- In fact $k_{210}$ was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.
What is that Probability?

- Classical nineteenth century modular function theory tells us all rational singular values $k_n$ are algebraic (solvable).

- Now, we can hunt in books or obtain the solution automatically in Maple: Thence

$$k_{100} := \left( (3 - 2 \sqrt{2}) (2 + \sqrt{5}) (-3 + \sqrt{10}) (-\sqrt{2} + \frac{4}{5}) \right)^2$$

- No one anticipated a closed form like this, except perhaps a few harmonic analysts.
  - For what boundaries can one emulate this?

- In fact $k_{210}$ was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.
Classical nineteenth century modular function theory tells us all rational singular values $k_n$ are algebraic (solvable).

Now, we can hunt in books or obtain the solution automatically in Maple: Thence

$$k_{100} := \left( (3 - 2 \sqrt{2}) \left( 2 + \sqrt{5} \right) \left( -3 + \sqrt{10} \right) \left( -\sqrt{2} + \frac{4}{\sqrt{5}} \right)^2 \right)^2$$

No one anticipated a closed form like this, except perhaps a few harmonic analysts.

- For what boundaries can one emulate this?

In fact $k_{210}$ was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.
Classical nineteenth century modular function theory tells us all rational singular values $k_n$ are algebraic (solvable).

Now, we can hunt in books or obtain the solution *automatically* in *Maple*:

\[
k_{100} := \left( (3 - 2\sqrt{2}) \left(2 + \sqrt{5}\right) (-3 + \sqrt{10}) \left(-\sqrt{2} + \frac{4}{\sqrt{5}}\right)^2 \right)^2
\]

No one anticipated a closed form like this, except perhaps a few harmonic analysts.

For what boundaries can one emulate this?

In fact $k_{210}$ was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.
What is that Probability?

A taste of Ramanujan

A modular function is a function, $\lambda(q)$, that can be related through an algebraic expression called a modular equation to the same function expressed in terms of the same variable, $q$, raised to an integral power: $\lambda(q^n)$. The integral power, $p$, determines the “order” of the modular equation. An example of a modular function is

$$\lambda(q) = 16q \prod_{n=1}^{\infty} \left( \frac{1 + q^{2n}}{1 + q^{n}} \right)^4.$$

Its associated seventh-order modular equation, which relates $\lambda(q)$ to $\lambda(q^7)$, is given by

$$\Phi(\lambda(q)\lambda(q^7)) + \Phi([1 - \lambda(q)][1 - \lambda(q^7)]) = 1.$$

Singular values are solutions of modular equations that must also satisfy additional conditions. One class of singular values corresponds to computing a sequence of values, $k_p$, where

$$k_p = \sqrt[p]{\left(\frac{e^{\pi i/3}}{\phi}\right)}$$

and $p$ takes integer values. These values have the curious property that the logarithmic expression

$$\frac{2}{\sqrt[p]{p}} \log(k_p^4)$$

coincides with many of the first digits of pi. The number of digits the expression has in common with pi increases with larger values of $p$.

Ramanujan was unparalleled in his ability to calculate these singular values. One of his most famous is the value when $p$ equals 210, which was included in his original letter to G. H. Hardy. It is

$$k_{210} = (\sqrt{2} - 1)^2(2 - \sqrt{3})(\sqrt{7} - \sqrt{6})(8 - 3\sqrt{7})(\sqrt{10} - 3)(\sqrt{15} - \sqrt{14})(4 - \sqrt{15})(6 - \sqrt{35})$$

This number, when plugged into the logarithmic expression, agrees with pi through the first 20 decimal places. In comparison, $k_{440}$ yields a number that agrees with pi through more than one million digits.

Applying this general approach, Ramanujan constructed a number of remarkable series for pi, including the one shown in the illustration on the preceding page. The general approach also underlies the two-step, iterative algorithms in the top illustration on the opposite page. In each iteration the first step (calculating $y_x$) corresponds to computing one of a sequence of singular values by solving a modular equation of the appropriate order; the second step (calculating $y_x$) is tantamount to taking the logarithm of the singular value.
8. What is that Limit, II?

Consider:

\[
C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

\[
D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} (u_i - u_j)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

\[
E_n := 2 \int_0^1 \cdots \int_0^1 \left(\prod_{1 \leq j < k \leq n} \frac{u_k - u_j}{u_k + u_j}\right)^2 dt_2 \cdots dt_n,
\]

where (in the last line) \( u_k = \prod_{i=1}^k t_i \).

- The \( D_n \) integrals arise in the Ising model (showing ferromagnetic temperature driven phase shifts).
- The \( C_n \) have tight connections to quantum field theory. Also \( E_n \leq D_n \leq C_n \) and \( E_n \sim D_n \).
8. What is that Limit, II?

Consider:

\[
C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

\[
D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} \left(\frac{u_i - u_j}{u_i + u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

\[
E_n := 2 \int_0^1 \cdots \int_0^1 \left(\prod_{1 \leq j < k \leq n} \frac{u_k - u_j}{u_k + u_j}\right)^2 dt_2 dt_3 \cdots dt_n,
\]

where (in the last line) \( u_k = \prod_{i=1}^k t_i \).

- The \( D_n \) integrals arise in the Ising model (showing ferromagnetic temperature driven phase shifts)
- The \( C_n \) have tight connections to quantum field theory. Also \( E_n \leq D_n \leq C_n \) and \( E_n \sim D_n \).
8. What is that Limit, II?

Consider:

\[
C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

\[
D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} (u_i - u_j)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

\[
E_n := 2 \int_0^1 \cdots \int_0^1 \left(\prod_{1 \leq j < k \leq n} \left(\frac{u_k - u_j}{u_k + u_j}\right)\right)^2 dt_2 \cdots dt_n,
\]

where (in the last line) \( u_k = \prod_{i=1}^k t_i \).

- The \( D_n \) integrals arise in the Ising model (showing ferromagnetic temperature driven phase shifts)
- The \( C_n \) have tight connections to quantum field theory. Also \( E_n \leq D_n \leq C_n \) and \( E_n \sim D_n \).
What is that Limit, II?

- Fortunately, the $C_n$ can be written as one-dim integrals:

$$C_n = \frac{2^n}{n!} \int_0^\infty p K_0^n(p) \, dp,$$

where $K_0$ is the *modified Bessel function*.

- Computing $C_n$ to 1000-digit (overkill) accuracy, we identified

$$C_3 = L_{-3}(2) := \sum_{n \geq 0} \left( \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right), \quad C_4 = \frac{7}{12} \zeta(3),$$

- Here $\zeta$ is Riemann zeta. In particular

$$C_{1024} = 0.63047350337438679612204019271087890435458707871273 \ldots,$$

is the limit value to that precision. The ISC returned

$$\lim_{n \to \infty} C_n = 2e^{-2\gamma},$$

where $\gamma$ is *Euler’s constant*. (Now proven.)
What is that Limit, II?

- Fortunately, the $C_n$ can be written as one-dim integrals:

$$C_n = \frac{2^n}{n!} \int_0^\infty pK_0^n(p) \, dp,$$

where $K_0$ is the modified Bessel function.

- Computing $C_n$ to 1000-digit (overkill) accuracy, we identified

$$C_3 = \text{L}_{-3}(2) := \sum_{n \geq 0} \left( \frac{1}{(3n + 1)^2} - \frac{1}{(3n + 2)^2} \right), \quad C_4 = \frac{7}{12} \zeta(3),$$

- Here $\zeta$ is Riemann zeta. In particular

$$C_{1024} = 0.63047350337438679612204019271087890435458707871273 \ldots,$$

is the limit value to that precision. The ISC returned

$$\lim_{n \to \infty} C_n = 2e^{-2\gamma},$$

where $\gamma$ is Euler’s constant. (Now proven.)
For $D_5$, $E_5$, we could integrate one variable symbolically.
What is that Limit, II?

- Nonetheless, we obtained 240-digits or more on a highly parallel computer system — impossible without a dimension reduction, and needed for reliable $D_5, E_5$ hunts.
  - We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations:

\[
\begin{align*}
D_2 &= 1/3, \quad D_3 = 8 + 4\pi^2/3 - 27L_{-3}(2), \quad D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2, \\
E_2 &= 6 - 8\log 2, \quad E_3 = 10 - 2\pi^2 - 8\log 2 + 32\log^2 2, \\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 \\
&\quad - 22\pi^2/3.
\end{align*}
\]

For $D_2, D_3, D_4$, these confirmed known analytic (physics) results. Also:

\[
E_5 = 42 - 1984\text{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) \log 2 - 40\log 2 + 40\pi^2\log^2 2 - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2, \quad (19)
\]

where $\text{Li}_4$ denotes the quadra-logarithm.
What is that Limit, II?

Nonetheless, we obtained 240-digits or more on a highly parallel computer system — impossible without a dimension reduction, and needed for reliable $D_5, E_5$ hunts.

- We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations:

\[
\begin{align*}
D_2 &= 1/3, \\
D_3 &= 8 + 4\pi^2/3 - 27 L_{-3}(2), \\
D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2, \\
E_2 &= 6 - 8 \log 2, \\
E_3 &= 10 - 2\pi^2 - 8 \log 2 + 32 \log^2 2, \\
E_4 &= 22 - 82\zeta(3) - 24 \log 2 + 176 (\log^3 2)/3 + 16\pi^2 \log 2 - 22\pi^2/3.
\end{align*}
\]

For $D_2, D_3, D_4$, these confirmed known analytic (physics) results. Also:

\[
E_5 = 42 - 1984 \text{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) \log 2 - 40 \log 2 + 40\pi^2 \log^2 2 - 62\pi^2/3 + 40(\pi^2 \log 2)/3 + 88 \log^4 2 + 464 \log^2 2, \quad (19)
\]

where $\text{Li}_4$ denotes the quadra-logarithm.
Nonetheless, we obtained 240-digits or more on a highly parallel computer system — impossible without a dimension reduction, and needed for reliable $D_5, E_5$ hunts.

- We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations:

\[
\begin{align*}
D_2 & = 1/3, & D_3 & = 8 + 4\pi^2/3 - 27 \log_3(2), & D_4 & = 4\pi^2/9 - 1/6 - 7\zeta(3)/2, \\
E_2 & = 6 - 8 \log 2, & E_3 & = 10 - 2\pi^2 - 8 \log 2 + 32 \log^2 2, \\
E_4 & = 22 - 82\zeta(3) - 24 \log 2 + 176 (\log^3 2)/3 + 16\pi^2 \log 2 - 22\pi^2/3.
\end{align*}
\]

For $D_2, D_3, D_4$, these confirmed known analytic (physics) results. Also:

\[
\begin{align*}
E_5 & = 42 - 1984 \text{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) \log 2 - 40 \log 2 + 40\pi^2 \log^2 2 - 62\pi^2/3 + 40(\pi^2 \log 2)/3 + 88 \log^4 2 + 464 \log^2 2,
\end{align*}
\]

where $\text{Li}_4$ denotes the quadra-logarithm.
What is that **Limit, II?**

I only understand things through examples and then gradually make them more abstract. I don’t think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It’s just very strange. That’s the way his mind worked. (David Mumford, 2004)

1. The form in (19) for $E_5$ was confirmed to 240-digit accuracy.
2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.
3. We tried but failed to recognize $D_5$ in terms of similar constants as described in the paper.
4. The 500-digit numerical value is accessible\(^3\) if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

\(^3\)http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.
I only understand things through examples and then gradually make them more abstract. I don’t think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It’s just very strange. That’s the way his mind worked. (David Mumford, 2004)

The form in (19) for $E_5$ was confirmed to 240-digit accuracy.

This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.

We tried but failed to recognize $D_5$ in terms of similar constants as described in the paper.

The 500-digit numerical value is accessible\(^3\) if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

\[^3\text{http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.}\]
I only understand things through examples and then gradually make them more abstract. I don’t think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It’s just very strange. That’s the way his mind worked. (David Mumford, 2004)

1. The form in (19) for $E_5$ was confirmed to 240-digit accuracy.

2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.

3. We tried but failed to recognize $D_5$ in terms of similar constants as described in the paper.

4. The 500-digit numerical value is accessible\(^3\) if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

\(^3\)http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.
I only understand things through examples and then gradually make them more abstract. I don’t think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It’s just very strange. That’s the way his mind worked.

(David Mumford, 2004)

1. The form in (19) for $E_5$ was confirmed to 240-digit accuracy.

2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.

3. We tried but failed to recognize $D_5$ in terms of similar constants as described in the paper.

4. The 500-digit numerical value is accessible\(^3\) if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

\(^3\)http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.
9. What is that **Transition value**?

Example (Weakly coupling oscillators)

In an important analysis of coupled *Winfree oscillators*, Quinn, Rand, and Strogatz looked at an $N$-oscillator scenario whose bifurcation phase offset $\phi$ is implicitly defined, with a conjectured asymptotic behavior: $\sin \phi \sim 1 - c_1/N$,; and with experimental estimate $c_1 = 0.605443657 \ldots$. We derived the exact value of this “QRS constant’:

$c_1$ is the *unique zero* of the Hurwitz zeta $\zeta(1/2, z/2)$ for $z \in (0, 2)$.

- We were able to prove the conjectured behavior. Moreover, we sketched the higher-order asymptotic behavior; something that would have been impossible without discovery of an analytic formula.
9. What is that Transition value?

Example (Weakly coupling oscillators)

In an important analysis of coupled Winfree oscillators, Quinn, Rand, and Strogatz looked at an $N$-oscillator scenario whose bifurcation phase offset $\phi$ is implicitly defined, with a conjectured asymptotic behavior: $\sin \phi \sim 1 - c_1/N$; and with experimental estimate $c_1 = 0.605443657\ldots$. We derived the exact value of this “QRS constant’:

$c_1$ is the unique zero of the Hurwitz zeta $\zeta(1/2, z/2)$ for $z \in (0, 2)$.

- We were able to prove the conjectured behavior. Moreover, we sketched the higher-order asymptotic behavior; something that would have been impossible without discovery of an analytic formula.
What is that Transition value?

- Does this deserve to be called a closed form?
  - Resoundingly ‘yes’, unless all inverse functions such as that in Bornemann’s probability are to be eschewed.
- Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on chimera — coupled systems which self-organize in part and remain disorganized elsewhere.
  - Now numerical limits still need a closed form.
- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.
What is that Transition value?

- Does this deserve to be called a closed form?
- Resoundingly ‘yes’, unless all inverse functions such as that in Bornemann’s probability are to be eschewed.
- Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on chimera — coupled systems which self-organize in part and remain disorganized elsewhere.
- Now numerical limits still need a closed form.
- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.
What is that Transition value?

**Does this deserve to be called a closed form?**

- Resoundingly ‘yes’, unless all inverse functions such as that in Bornemann’s probability are to be eschewed.

**Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on chimera — coupled systems which self-organize in part and remain disorganized elsewhere.**

- Now numerical limits still need a closed form.

- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.
What is that **Transition value?**

**Does this deserve to be called a closed form?**

- Resoundingly ‘yes’, unless all inverse functions such as that in Bornemann’s probability are to be eschewed.

**Such QRS constants** are especially interesting in light of recent work by Strogatz, Lang et al on *chimera* — coupled systems which self-organize in part and remain disorganized elsewhere.

- Now numerical limits still need a closed form.

**Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.**
10. What is that Expectation?

- There is much recent research on calculation of expected distances of points inside a hypercube to the hypercube — or expected distances between points in a hypercube, etc.
- Some expectations $\langle |\vec{r}| \rangle$ for random $\vec{r} \in [0, 1]^n$ are

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
</tr>
<tr>
<td>$n = 3$</td>
</tr>
<tr>
<td>$n = 4$</td>
</tr>
</tbody>
</table>

- Box integrals are not just a mathematician’s curiosity — they are being used to assess randomness of (rat) brain synapses positioned within a parallelepiped. But now we (B-Crandall-Rose) wish to use Cantor Boxes.
There is much recent research on calculation of expected distances of points inside a hypercube to the hypercube — or expected distances between points in a hypercube, etc. Some expectations $\langle |\vec{r}| \rangle$ for random $\vec{r} \in [0, 1]^n$ are

**Example**

- $n = 2 \quad \frac{\sqrt{2}}{3} + \frac{1}{3} \log (1 + \sqrt{2})$.
- $n = 3 \quad \frac{1}{4} \sqrt{3} - \frac{1}{24} \pi + \frac{1}{2} \log (2 + \sqrt{3})$.
- $n = 4 \quad \frac{2}{5} - \frac{G}{10} + \frac{3}{10} \text{Ti}_2 \left(3 - 2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10} \tan\left(\frac{1}{\sqrt{8}}\right)$.

Box integrals are not just a mathematician’s curiosity — they are being used to assess randomness of (rat) brain synapses positioned within a parallelepiped. But now we (B-Crandall-Rose) wish to use Cantor Boxes.
10. What is that **Expectation?**  

Box integrals

- There is much recent research on calculation of **expected distances** of points inside a hypercube to the hypercube — or expected distances between points in a hypercube, etc.
- Some expectations $\langle |\vec{r}| \rangle$ for random $\vec{r} \in [0, 1]^n$ are

\[
\begin{align*}
  n = 2 & \quad \frac{\sqrt{2}}{3} + \frac{1}{3} \log (1 + \sqrt{2}). \\
  n = 3 & \quad \frac{1}{4} \sqrt{3} - \frac{1}{24} \pi + \frac{1}{2} \log (2 + \sqrt{3}). \\
  n = 4 & \quad \frac{2}{5} - \frac{G}{10} + \frac{3}{10} \text{Ti}_2 \left(3 - 2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10} \arctan \left(\frac{1}{\sqrt{8}}\right).
\end{align*}
\]

- Box integrals are not just a mathematician’s curiosity — they are being used to assess randomness of **(rat) brain synapses** positioned within a parallelepiped. But now we (B-Crandall-Rose) wish to use Cantor Boxes.
A very recent result is that every box integral $\langle |\vec{r}|^n \rangle$ for integer $n$, and dimensions 1, 2, 3, 4, 5 are “hyperclosed”.

- Five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, dx \, dy.$$  \hspace{1cm} (20)

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0, 1]^5$ can be written in explicit form involving a $10^5$-character symbolic $J(3)$.

- We reduced the 5-dim box value to “only” $10^4$ characters.
A very recent result is that every box integral $\langle |\vec{r}|^n \rangle$ for integer $n$, and dimensions 1, 2, 3, 4, 5 are “hyperclosed”.

- **Five-dimensional box integrals** have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, dx \, dy. \quad (20)$$

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0,1]^5$ can be written in explicit form involving a $10^5$-character symbolic $J(3)$.

- We reduced the 5-dim box value to “only” $10^4$ characters.
What is that Dimension? Hyperclosure, 1.

A very recent result is that every box integral $\langle |\vec{r}|^n \rangle$ for integer $n$, and dimensions $1, 2, 3, 4, 5$ are "hyperclosed".

- **Five-dimensional box integrals** have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, dx \, dy.$$  

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0, 1]^5$ can be written in explicit form involving a $10^5$-character symbolic $J(3)$.

- We reduced the 5-dim box value to "only" $10^4$ characters.

=\[=\]
What is that Dimension? Hyperclosure, 1.

A very recent result is that every box integral $\langle |\vec{r}|^n \rangle$ for integer $n$, and dimensions 1, 2, 3, 4, 5 are "hyperclosed".

- **Five-dimensional box integrals** have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

\[
J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, dx \, dy. \tag{20}
\]

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0, 1]^5$ can be written in explicit form involving a $10^5$-character symbolic $J(3)$.

- We reduced the 5-dim box value to “only” $10^4$ characters.
A companion integral $J(2)$ also starts out with about $10^5$ characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8} \log 2 - \frac{7}{48} \zeta(3) + \frac{11}{24} \pi \text{Cl}_2 \left(\frac{\pi}{6}\right) - \frac{29}{24} \pi \text{Cl}_2 \left(\frac{5\pi}{6}\right),$$

where $\text{Cl}_2(\theta) := \sum_{n \geq 1} \frac{\sin(n\theta)}{n^2}$ is the simplest non-elementary Fourier series.

Thomas Clausen (1801-1885) learned to read at 12. He computed $\pi$ to 247 places in 1847 using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser at NYU, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper.
A companion integral $J(2)$ also starts out with about $10^5$ characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8} \log 2 - \frac{7}{48} \zeta(3) + \frac{11}{24} \pi \text{Cl}_2 \left( \frac{\pi}{6} \right) - \frac{29}{24} \pi \text{Cl}_2 \left( \frac{5\pi}{6} \right), \quad (21)$$

where $\text{Cl}_2(\theta) := \sum_{n \geq 1} \sin(n\theta)/n^2$ is the simplest non-elementary Fourier series.

Thomas Clausen (1801-1885) learned to read at 12. He computed $\pi$ to 247 places in 1847 using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser at NYU, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper.
A companion integral $J(2)$ also starts out with about $10^5$ characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8} \log 2 - \frac{7}{48} \zeta(3) + \frac{11}{24} \pi \text{Cl}_2 \left(\frac{\pi}{6}\right) - \frac{29}{24} \pi \text{Cl}_2 \left(\frac{5\pi}{6}\right),$$

(21)

where $\text{Cl}_2(\theta) := \sum_{n \geq 1} \sin(n\theta)/n^2$ is the simplest non-elementary Fourier series.

Thomas Clausen (1801-1885) learned to read at 12. He computed $\pi$ to 247 places in 1847 using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser at NYU, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper.
A companion integral $J(2)$ also starts out with about $10^5$ characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8} \log 2 - \frac{7}{48} \zeta(3) + \frac{11}{24} \pi \text{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24} \pi \text{Cl}_2\left(\frac{5\pi}{6}\right), \quad (21)$$

where $\text{Cl}_2(\theta) := \sum_{n \geq 1} \sin(n\theta)/n^2$ is the simplest non-elementary Fourier series).

Thomas Clausen (1801-1885) learned to read at 12. He computed $\pi$ to 247 places in 1847 using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser at NYU, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper.
11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk.  
  - especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals  
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W'_n(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that **Density**?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk.  
  - especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals  
  $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- **Radial densities** $p_n$ of a random planar walk.
  - especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W'_n(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk. 
  - especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W'_n(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk. Especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals: $\mu(1 + x_1 + \cdots + x_{n-1}) = W'_n(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities \( p_n \) of a random planar walk.
  - especially \( p_3, p_4, p_5 \) (as above with \( p_6 \)).
- Expectations and moments \( W_n(s) \).

This led Straub and JMB to make detailed study of:

- Mahler Measures \( \mu(P) \) and logsin integrals
  \[ \mu(1 + x_1 + \cdots + x_{n-1}) = W'_n(0) \text{ is known for } n = 3, 4, 5, 6. \]
- Multiple Mahler measures like \( \mu_n(1 + x + y) \) and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk. Especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals:
  $$\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that **Density**?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities \( p_n \) of a random planar walk. Especially \( p_3, p_4, p_5 \) (as above with \( p_6 \)).

- Expectations and moments \( W_n(s) \).

This led Straub and JMB to make detailed study of:

- Mahler Measures \( \mu(P) \) and logsin integrals
  - \( \mu(1 + x_1 + \cdots x_{n-1}) = W'_n(0) \) is known for \( n = 3, 4, 5, 6 \).

- Multiple Mahler measures like \( \mu_n(1 + x + y) \) and QFT.

- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk. especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals $\mu(1 + x_1 + \cdots x_{n-1}) = W'_n(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical uniform random walks in the plane:

- Radial densities $p_n$ of a random planar walk. especially $p_3, p_4, p_5$ (as above with $p_6$).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is known for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.
Part II (as time permits) and Conclusions

Hypergeometric evaluations of the densities of short random walks


Conclusions

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n+1,1)$ in terms of $L_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!
Part II (as time permits) and Conclusions

Hypergeometric evaluations of the densities of short random walks


Conclusions

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n+1,1)$ in terms of $L_s^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!
Part II (as time permits) and Conclusions

**Part II** *Hypergeometric evaluations of the densities of short random walks*


**Conclusions**

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n + 1, 1)$ in terms of $L^{(2n-3)}_{2n}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!
Part II (as time permits) and Conclusions

Part II *Hypergeometric evaluations of the densities of short random walks*


Conclusions

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n+1,1)$ in terms of $L_{s_{2n}}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!

J.M. Borwein  Meetings with Special Functions
Part II (as time permits) and Conclusions

Part II  Hypergeometric evaluations of the densities of short random walks

Conclusions

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n+1, 1)$ in terms of $L_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!
Conclusions

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n + 1, 1)$ in terms of $L_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!
Conclusions

1. We still lack a complete accounting of $\mu_n(1 + x + y)$ and are trying to resolve “the crisis of the 6th root in QFT.”
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(2n + 1, 1)$ in terms of $L_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA’s dynamic dictionary).
6. Thank you!