A Cyclic Douglas-Rachford Iteration Scheme

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March 11, 2013

Abstract

In this paper we present a Douglas-Rachford inspired iteration scheme which can be applied to $N$-set convex feasibility problems in Hilbert space. Our main result is weak convergence of the method to a point whose nearest point projections onto each of the $N$ sets coincide. In the case of affine subspaces, norm convergence is obtained.

1 Introduction

Throughout this paper,

$H$ is a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and induced norm $\| \cdot \|$. We use $w.$ to denote weak convergence.

Given a set $S \subseteq H$ and point $x \in H$, the best approximation to $x$ from $S$ is a point $p \in S$ such that

$$\|p - x\| = d(x, S) := \inf_{s \in S} \| x - s \|.$$ 

If there exists such a $p$, for any $x \in H$, then $S$ is said to be proximal. Additionally, if $p$ is unique then $S$ is said to be Chebyshev. In the latter case, the projection onto $S$ is the operator $P_S : H \to S$ which maps $x$ to its unique nearest point in $S$ and we write $P_S(x) = p$. The reflection about $S$ is the operator $R_S : H \to H$ defined by $R_S := 2P_S - I$ where $I$ denotes the identity operator which maps any $x \in H$ to itself.

Fact 1.1. Let $C \subseteq H$ be closed and convex.

(i) (Kolmogorov’s Criterion) Then $C$ is Chebyshev and

$$P_C(x) = p \iff \langle x - p, c - p \rangle \leq 0 \text{ for all } c \in C.$$
(ii) (Translation formula) For $y \in \mathcal{H}$, $P_{y+C}(x) = y + P_C(x - y)$.

(iii) (Dilation formula) For $0 \neq \lambda \in \mathbb{R}$, $P_{\lambda C}(x) = \lambda P_C(x/\lambda)$.

(iv) If $C$ is a subspace then $P_C$ is linear.

(v) If $C$ is an affine subspace then $P_C$ is affine.

Proof. See, for example, [7, Theorem 3.14, Proposition 3.17, Corollary 3.20], [24, Theorem 2.8, Exercise 5.2(i), Theorem 3.1, Exercise 5.10] or [32, Theorem 2.1.3, Theorem 2.1.6].

We consider the $N$-set convex feasibility problem:

\[
\text{Find } x \in \bigcap_{i=1}^{N} C_i \neq \emptyset \text{ where } C_i \subseteq \mathcal{H} \text{ are closed and convex.} \tag{1}
\]

Many optimization problems can be cast in this framework, either directly or as a suitable relaxation if a desired bound on the quality of the solution is known a priori. A common approach to solving convex feasibility problems is the use of projection algorithms. These iterative methods assume that the projections onto each of the individual sets are relatively simple to compute. Some well-known projection methods include von Neumann’s alternating projections method [33, 26, 16, 3, 6], the Douglas-Rachford method [20, 29, 10] and Dykstra’s method [22, 15, 4]. Of course, there are many variants. For a review we refer the reader to [2, 5, 19, 32, 24, 13].

On certain classes of problems, various projection methods coincide with each other, and with other known techniques. For example, if the sets are closed affine subspaces, alternating projections = Dykstra’s method [15]. If the sets are hyperplanes, alternating projections = Dykstra’s method = Kaczmarz’s method [19]. If the sets are half-spaces, alternating projections = the method Agmon, Motzkin and Schoenberg (MAMS), and Dykstra’s method = Hildreth’s method [24, Chapter 4]. Applied to the phase retrieval problem, alternating projections = error reduction, Dykstra’s method = Fienup’s BIO, and Douglas-Rachford method = Fienup’s HIO [8].

Given $A, B \subseteq \mathcal{H}$ define the 2-set Douglas-Rachford operator $T_{A,B} : \mathcal{H} \to \mathcal{H}$ by

\[
T_{A,B} := \frac{I + R_B R_A}{2}.
\]

Theorem 1.2 (Douglas-Rachford [20], Lions-Mercier [29]). Let $A, B \subseteq \mathcal{H}$ be closed and convex. For any $x_0 \in \mathcal{H}$, the sequence $T_{A,B}^n x_0$ converges weakly to a point $x$ such that $P_A x \in A \cap B$.

Continued interest in the Douglas-Rachford iteration is in part due to its excellent performance on various problems involving one or more non-convex sets. For example, in phase retrieval problems arising in the context of image reconstruction [8, 9]. The
method has also been successfully applied to NP-complete combinatorial problems including Boolean satisfiability [23, 25] and Sudoku [31, 23]. In contrast, von Neumann’s alternating projection method applied to such problems often fails to converge satisfactorily. For progress on the behaviour of non-convex alternating projections, we refer the reader to [28, 11, 27, 21].

Recently, Borwein and Sims [14] provided limited theoretical justification for non-convex Douglas-Rachford iterations, proving local convergence for a prototypical Euclidean case involving a sphere and an affine subspace. For the two-dimensional case of a circle and a line, Borwein and Aragón [1] were able to give an explicit region of convergence. Even more recently, a local version of firm nonexpansivity has been utilized by Hesse and Luke [27] to obtain local convergence of the Douglas-Rachford method in limited non-convex framework. However, their results do not directly overlap with the work of Aragón, Borwein and Sims (for details see [27, Example 43]).

Most projection algorithms can be extended to the $N$-set convex feasibility problem without significant modification. An exception is the Douglas-Rachford method, for which only application to 2-set feasibility problems has so far been successfully investigated. For applications involving $N > 2$ sets, an equivalent 2-set feasibility problem can, however, be posed in the product space $\mathcal{H}^N$, as we shall revisit later.

The aim of this paper is to introduce and study a cyclic Douglas-Rachford iteration scheme, which can applied directly to the $N$-set feasibility problem without recourse to a product space formulation.

The paper is organized as follows: Section 2 investigate properties of operators which are compositions of firmly nonexpansive operators. In Section 3 we introduce the cyclic Douglas-Rachford iteration scheme, proving weak convergence to a point whose projection onto each of the constraint sets coincide. In Section 4 the result is strengthened for closed affine constraints. In Section 5, the method is applied numerically to feasibility problems with ball/sphere constraints.

2 Properties of Nonexpansive Operators

In this section we study properties of sequences generated by nonexpansive and firmly nonexpansive mappings.

We first recall some definitions. Let $T : \mathcal{H} \rightarrow \mathcal{H}$. We denote the set of fixed points of $T$ by $\text{Fix} T = \{x : Tx = x\}$. We say $T$ is asymptotically regular if $T^n x - T^{n+1} x \rightarrow 0$, in norm, for all $x \in \mathcal{H}$.

Let $D \subseteq \mathcal{H}$ and $T : D \rightarrow \mathcal{H}$. We say $T$ is nonexpansive if

$$
\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in D
$$
Proof. We prove the
Lemma 2.2. Let
we require in Section 3.
nonexpansive operators. While our conclusions can be obtained as an easy consequence
firmly nonexpansive (see [17, Example 4.2.5]).
(i.e. 1-Lipschitz). We say
exists
k
such that
0
 Remark 4.2.4, Corollary 4.3.6].
Proof. See, for example, [7, Proposition 4.8, Corollary 4.10, Remark 4.24], or [32, Theorem
The class of nonexpansive mappings is closed under convex combinations, compositions, etc. However, the class of firmly nonexpansive mappings is not so well behaved. For example, just the composition of two projections onto subspaces need not be firmly nonexpansive (see [17, Example 4.2.5]).

We now consider asymptotic behaviour of operators that are the composition of firmly nonexpansive operators. While our conclusions can be obtained as an easy consequence of a recent more general result [12], for completeness we include a proof the specific cases we require in Section 3.

Lemma 2.2. Let \( T : D \to \mathcal{H} \) be firmly nonexpansive with \( \text{Fix} T \neq \emptyset \). Then \( T \) is asymptotically regular.

Proof. Let \( z \in \text{Fix}(T) \). Since \( T \) is nonexpansive,
\[
\|T^{n+1}x - z\| = \|T(T^nx) - Tz\| \leq \|T^nx - z\|
\]
hence \( \lim_{n \to \infty} \|T^nx - z\| \) exists. Since \( T \) is firmly nonexpansive,
\[
\|T^{n+1}x - z\|^2 + \|(I - T)(T^nx)\|^2 = \|T(T^nx) - Tz\|^2 + \|(I - T)(T^nx) - (I - T)z\|^2
\]
\[
\leq \|T^nx - z\|^2.
\]
Taking limits as \( n \to \infty \) shows \( \|(I - T)(T^nx)\|^2 \to 0 \).

Lemma 2.3. Let \( T_i : \mathcal{H} \to \mathcal{H} \), for each \( i \), and define \( T := T_r \ldots T_2 T_1 \). Suppose there exists \( k_i > 0 \) such that
\[
\|(x - y) - (T_ix - T_iy)\|^2 \leq k_i(\|x - y\|^2 - \|T_ix - T_iy\|^2),
\]
for each \( x, y \in \mathcal{H} \). Then there exists \( k > 0 \) such that
\[
\|(x - y) - (Tx - Ty)\|^2 \leq k(\|x - y\|^2 - \|Tx - Ty\|^2).
\]

Proof. We prove the \( r = 2 \) case, with the full result following by induction.
\[
\|(x - y) - (T_2T_1x - T_2T_1y)\|^2
\]
\[
\leq \|(x - y) - (T_1x - T_1y)\| + \|(T_1x - T_1y) - (T_2T_1x - T_2T_1y)\|^2
\]
\[
\leq (2 \max\{\|(x - y) - (T_1x - T_1y)\|, \|(T_1x - T_1y) - (T_2T_1x - T_2T_1y)\|\})^2
\]
\[
\leq 4\|(x - y) - (T_1x - T_1y)\|^2 + \|(T_1x - T_1y) - (T_2T_1x - T_2T_1y)\|^2
\]
\[
\leq 4 \max\{k_1, k_2\}(\|x - y\|^2 - \|T_1x - T_1y\|^2 + \|T_1x - T_2y\|^2 - \|T_2T_1x - T_2T_1y\|^2)
\]
\[
\leq 4 \max\{k_1, k_2\}(\|x - y\|^2 - \|T_2T_1x - T_2T_1y\|^2).
\]
As a consequence of Lemma 2.3 we have the following.

**Corollary 2.4.** Let $T_i : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive, for each $i$, and define $T := T_r \ldots T_2 T_1$. If $\text{Fix} T \neq \emptyset$ then $T$ is asymptotically regular.

**Proof.** By Lemma 2.3,

$$\sum_{n=0}^{N} \|(T^n x - T^n z) - (T^{n+1} x - T^{n+1} z)\|^2 \leq k \sum_{n=0}^{N} \left(\|T^n x - T^n z\|^2 - \|T^{n+1} x - T^{n+1} z\|^2\right)$$

$$\leq k \left(\|x - z\|^2 - \|T^{N+1} x - T^{N+1} z\|^2\right) \leq k \|x - z\|^2.$$

Thus $\|(T^n x - T^n z) - (T^{n+1} x - T^{n+1} z)\|^2 \to 0$. In particular choosing $z \in \text{Fix} T$ implies $T^n x - T^{n+1} x \to 0$.

**Remark 2.5.** Alternatively, Corollary 2.4 could be obtained by appealing to a recent result of Bauschke, Martín-Márquez, Moffat and Wang [12, Theorem 4.6]. They show that any composition of asymptotically regular firmly nonexpansive mappings is asymptotically regular, even when $\text{Fix} T = \emptyset$.

\[\square\]

We now show that sequences generated by asymptotically regular, nonexpansive mappings, converge. The result is well-known and attributed to Opial [30]. For completeness, we include a proof of the general result, and of the specialization we require in Section 3.

A Banach space is said to have the **Opial property** if, for any sequence $(x_n)$ such that $x_n \rightharpoonup x$, and any $y \neq x$, we have

$$\liminf_{n \to \infty} \|x_n - y\| > \liminf_{n \to \infty} \|x_n - x\|.$$

**Lemma 2.6.** Every Hilbert space has the Opial property.

**Proof.** Since weakly convergent sequences are bounded, both limits are finite. Now

$$\|x_n - y\|^2 = \|x_n - x\|^2 + \|x - y\|^2 + 2\langle x_n - x, x - y \rangle.$$

By weak convergence, $\langle x_n - x, x - y \rangle \to 0$. The result now follows by taking the lim inf on both sides. \[\square\]

Let $T : D \to \mathcal{H}$. We say $T$ is **demiclosed** if its graph in $D \times \mathcal{H}$ is sequentially closed in the Cartesian product topology induced by the weak topology on $D$ and the strong topology on $\mathcal{H}$. That is, if $x_n \rightharpoonup x$ and $Tx_n \to y$ then $Tx = y$.

**Lemma 2.7.** Let $T : D \to \mathcal{H}$ be nonexpansive. Then $I - T$ is demiclosed.

**Proof.** Let $(x_n) \subseteq C$ be a sequence weakly convergent to $x \in C$, and $x_n - Tx_n$ norm convergent to $y \in \mathcal{H}$. Suppose $x \neq y + Tx$. Then

$$\liminf_{n \to \infty} \|x_n - y - Tx\| = \liminf_{n \to \infty} \|Tx_n - Tx\| \leq \liminf_{n \to \infty} \|x_n - x\| \overset{\text{Lem. 2.6}}{\leq} \liminf_{n \to \infty} \|x_n - (y + Tx)\|.$$

This is a contradiction, hence, $x = y + Tx$ which implies $(I - T)x = y$. \[\square\]
We say a sequence \((x_n)\) is \textit{Fejér monotone} \cite{5} with respect to \(S \subseteq \mathcal{H}\), if, for each \(n\),
\[
\|x_{n+1} - y\| \leq \|x_n - y\| \text{ for all } y \in S.
\]

If \((x_n)\) is Fejér monotone with respect to \(S\), it is straightforward to see that \((x_n)\) is bounded and that \(\lim_{n \to \infty} \|x_n - y\|\) exists, for each \(y \in S\).

\textbf{Theorem 2.8 (Opial \cite{30}).} \textit{Let }\(T : \mathcal{H} \to \mathcal{H}\) \textit{be nonexpansive, asymptotically regular, and }\textit{Fix} \(T \neq \emptyset\). \textit{Then for any }\(x \in \mathcal{H}\), \(T^n x\) converges weakly to an element of \(\text{Fix} \ T\).

\textit{Proof.} Define \(x_n := T^n x_0\). Since \(T\) is nonexpansive, \((x_n)\) is Fejér monotone with respect to \(\text{Fix} \ T\). Therefore \((x_n)\) is bounded and \(\lim_{n \to \infty} \|x_n - y\|\) exists, for any \(y \in \text{Fix} \ T\).

Since \((x_n)\) is bounded, it contains a weakly convergent subsequence \((x_{n_k})\) such that \(x_{n_k} \overset{w}{\rightharpoonup} x\) for some \(x \in \mathcal{H}\). By asymptotic regularity, \((I - T)x_{n_k} \to 0\). By Lemma 2.7, \((I - T)x = 0 \implies x \in \text{Fix} \ T\).

Suppose \((x_n)\) is not weakly convergent to \(x\). Then there exists a subsequence \((x_{n_l})\) such that \(x_{n_l} \overset{w}{\rightharpoonup} y \neq x\). Consider the sequence \((a_n)\) defined by
\[
a_n := \|x_n - x\|^2 + \|x_n - y\|^2 = \|x\|^2 - \|y\|^2 + 2\langle x_n, y - x \rangle.
\]

Since \(x, y \in \text{Fix} \ T\), \(a := \lim_{n \to \infty} a_n\) exists. By consider the subsequences \((a_{n_k})\) and \((a_{n_l})\) we see that \(a = -\|x - y\|^2\) and \(a = \|x - y\|^2\). Thus \(a = 0\) and \(x = y\). \(\square\)

As a simple consequence of Theorem 2.8, we obtain the following Corollary for use in Section 3.

\textbf{Corollary 2.9.} \textit{Let }\(T_i : \mathcal{H} \to \mathcal{H}\) \textit{be firmly nonexpansive with }\textit{Fix} \(T_i \neq \emptyset\), \textit{for each }\(i\), \textit{and define }\(T := T_r \ldots T_2 T_1\). \textit{For any }\(x \in \mathcal{H}\), \(T^n x\) converges weakly to an element of \(\text{Fix} \ T\).

\textit{Proof.} Since \(T\) is the composition of nonexpansive operators, it too is nonexpansive. By Lemma 2.2 and Corollary 2.4, \(T\) is asymptotically regular. The result now follows from Theorem 2.8. \(\square\)

We now provide a characterization of \(\text{Fix} \ T\), when \(T\) is the composition of firmly nonexpansive operators having a common fixed point.

\textbf{Lemma 2.10.} \textit{Let }\(T : \mathcal{H} \to \mathcal{H}\) \textit{be firmly nonexpansive. Then for any }\(x, y \in \mathcal{H}\),
\[
\|Tx - Ty\| = \|x - y\| \text{ if and only if } Tx - Ty = x - y.
\]

\textit{Proof.} Suppose \(Tx - Ty \neq x - y\). Then \((I - T)x - (I - T)y \neq 0\) and hence,
\[
\|x - y\|^2 \geq \|(I - T)x - (I - T)y\|^2 + \|Tx - Ty\|^2 > \|Tx - Ty\|^2.
\]
The converse is trivial. \(\square\)

\textbf{Lemma 2.11.} \textit{Let }\(T_i : \mathcal{H} \to \mathcal{H}\) \textit{be firmly nonexpansive, for each }\(i\), \textit{and define }\(T := T_r \ldots T_2 T_1\). \textit{Then for any }\(x, y \in \mathcal{H}\),
\[
Tx - Ty = x - y
\]
if and only if
\[
(T_i \ldots T_1)x - (T_i \ldots T_1)y = x - y, \text{ for each } i \leq r.
\]
Proof. By nonexpansivity, for each \( i < r \),
\[
\|Tx - Ty\| \leq \|(T_{i+1} \ldots T_1)x - (T_{i+1} \ldots T_1)y\| \leq \|(T_i \ldots T_1)x - (T_i \ldots T_1)y\| \leq \|x - y\|.
\]
If \( Tx - Ty = x - y \) then Lemma 2.10 implies
\[
(T_{i+1} \ldots T_1)x - (T_{i+1} \ldots T_1)y = (T_i \ldots T_1)x - (T_i \ldots T_1)y = x - y.
\]
The converse is trivial. \( \Box \)

Lemma 2.12. Let \( T_i : \mathcal{H} \rightarrow \mathcal{H} \) be firmly nonexpansive, for each \( i \), and define \( T := T_r \ldots T_2 T_1 \). If \( \bigcap_{i=1}^r \text{Fix}(T_i \ldots T_1) \neq \emptyset \) then \( \text{Fix} T = \bigcap_{i=1}^r \text{Fix} T_i \).

Proof. Clearly, \( \text{Fix} T \supseteq \bigcap_{i=1}^r \text{Fix} T_i \). To prove the opposite inclusion, let \( x \in \text{Fix} T \) and let \( y \in \bigcap_{i=1}^r \text{Fix}(T_i \ldots T_1) \). Then \( Tx - Ty = x - y \). By Lemma 2.11, for each \( i \),
\[
(T_i \ldots T_1)x - (T_i \ldots T_1)y = x - y = (T_i \ldots T_1)x = x,
\]
since \( y \in \text{Fix}(T_i \ldots T_1) \). Hence, inductively, \( T_i x = x \) for each \( i \). \( \Box \)

### 3 Cyclic Douglas-Rachford Iterations

We are now ready to introduce a new projection algorithm, the cyclic Douglas-Rachford iteration scheme. Let \( C_1, C_2, \ldots, C_N \subseteq \mathcal{H} \) and define \( T_{[C_1 C_2 \ldots C_N]} : \mathcal{H} \rightarrow \mathcal{H} \) by
\[
T_{[C_1 C_2 \ldots C_N]} := T_{C_1, C_2} \ldots T_{C_2, C_3} T_{C_3, C_4} = \frac{1 + R_{C_1} R_{C_N}}{2} \left( \frac{1 + R_{C_N} R_{C_{N-1}}}{2} \right) \ldots \left( \frac{1 + R_{C_3} R_{C_2}}{2} \right) \left( \frac{1 + R_{C_2} R_{C_1}}{2} \right).
\]

Given \( x_0 \in \mathcal{H} \), the cyclic Douglas-Rachford method iterates by setting \( x_{n+1} = T_{[C_1 C_2 \ldots C_N]} x_n \).

Remark 3.1. In the two set case, the cyclic Douglas-Rachford operator becomes
\[
T_{[C_1 C_2]} = T_{C_2, C_1} T_{C_1, C_2} = \left( \frac{1 + R_{C_1} R_{C_2}}{2} \right) \left( \frac{1 + R_{C_2} R_{C_1}}{2} \right).
\]

That is, it does not coincide with the classic Douglas-Rachford scheme. It can, however, be considered a symmetrized version. \( \diamond \)

Where there is no ambiguity, we take indices modulo \( N \), and abbreviate \( T_{C_i, C_j} \) by \( T_{i,j} \), and \( T_{[C_1 C_2 \ldots C_N]} \) by \( T_{[1 \ldots N]} \).

Recall the following characterization of fixed points of the Douglas-Rachford operator.

Lemma 3.2. Let \( A, B \subseteq \mathcal{H} \) be closed and convex. Then \( P_A(\text{Fix} T_{A,B}) \subseteq A \cap B \).
Proof. If \( \text{Fix} T_{A,B} = \emptyset \) then the result is trivial. Else,
\[
x \in \text{Fix} T_{A,B} \iff \frac{x + R_B R_A x}{2} = x \iff P_A x = P_B R_A x.
\]

We are now ready to present our main result, regarding convergence of the cyclic Douglas-Rachford scheme.

**Theorem 3.3** (Cyclic Douglas-Rachford). Let \( C_1, C_2, \ldots, C_n \subseteq \mathcal{H} \) be closed and convex with nonempty intersection. For any \( x_0 \in \mathcal{H} \), the sequence \( T_{[1 \ldots N]}^n x_0 \) converges weakly to a point \( x \) such that \( P_{C_i} x = P_{C_j} x \) for all \( i,j \). Moreover, \( P_{C_j} x \in \cap_{i=1}^N C_i \), for each \( j \).

**Proof.** By Corollary 2.9, \( T_{[1 \ldots N]}^n x_0 \) converges weakly to a point \( x \in \text{Fix} T_{[1 \ldots N]} \). By Lemma 2.12,
\[
\text{Fix} T_{[1 \ldots N]} = \bigcap_{i=1}^N \text{Fix} T_{i,i+1}.
\]

By Lemma 3.2, \( P_{C_i} x \in C_{i+1} \) for each \( i \). Now we compute
\[
\frac{1}{2} \sum_{i=1}^N \| P_{C_i} x - P_{C_{i-1}} x \|^2 = \langle x, 0 \rangle + \frac{1}{2} \sum_{i=1}^N \left( \| P_{C_i} x \|^2 - 2 \langle P_{C_i} x, P_{C_{i-1}} x \rangle + \| P_{C_{i-1}} x \|^2 \right)
\]
\[
= \left( \langle x, \sum_{i=1}^N (P_{C_{i-1}} x - P_{C_i} x) \rangle - \sum_{i=1}^N \langle P_{C_i} x, P_{C_{i-1}} x \rangle + \sum_{i=1}^N \| P_{C_i} x \|^2 \right)
\]
\[
= \sum_{i=1}^N \langle x - P_{C_i} x, P_{C_{i-1}} x - P_{C_i} x \rangle \leq 0.
\]

Thus, \( P_{C_i} x = P_{C_{i-1}} x \) for each \( i \). \( \square \)

**Remark 3.4.** From the proof of Theorem 3.3, it is clear the same assertion holds replacing \( T_{[1 \ldots N]} \) with \( T \) such that the following three properties hold:

1. \( T = (T_M \ldots T_2 T_1) \), is nonexpansive and asymptotically regular,
2. \( \text{Fix} T = \bigcap_{j=1}^M \text{Fix} T_j \neq \emptyset \),
3. \( P_{C_j}(\text{Fix} T_j) \subseteq C_{j+1}, \) for each \( j \).

Some examples include:

- \( T_{\{A_1 A_2 \ldots A_M\}} \) where \( A_j \in \{C_1, C_2 \ldots C_N\} \), such that each \( C_i \) appear in the sequence \( A_1, A_2, \ldots, A_M \) at least once.

- \( T \) is any product of \( P_{C_1}, P_{C_2}, \ldots, P_{C_N} \), such that each projection appears in said product at least once. In particular, setting \( T = P_{C_N} \ldots P_{C_2} P_{C_1} \) we recover Bregman’s result [16].

- \( T_j = (I + P_j)/2 \) where \( P_j \) is a product of \( P_{C_1}, P_{C_2}, \ldots, P_{C_N} \), such that for each \( i \) there exists a \( j \) such that \( P_j = (P_{C_i} \ldots) \). A special case is,
\[
T = \left( \frac{I + P_{C_1} P_{C_N}}{2} \right) \ldots \left( \frac{I + P_{C_3} P_{C_2}}{2} \right) \left( \frac{I + P_{C_2} P_{C_1}}{2} \right).
\]
• If $T_1, T_2, \ldots, T_M$ are nonexpansive operators satisfying properties 1, 2, 3, then replacing $T_j$ with the relaxation $\alpha_j I + (1 - \alpha_j) T_j$ where $\alpha_j \in (0, 1/2]$, for each $j$ [7, Remark 4.27].

Of course, there are many other applicable variants.

We now investigate the cyclic Douglas-Rachford iteration in a special case.

**Corollary 3.5.** Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with nonempty intersection. If $y \in C_i$ then $T_i y = P_{C_i} y$. In particular, if $x_0 \in C_1$, the cyclic Douglas-Rachford trajectory coincides with that of von Neumann’s alternating projection method.

**Proof.** For any $y \in \mathcal{H}$, $T_i y = P_{C_i} y \iff R_{C_i} y = R_{C_{i+1}} R_{C_i} y$. If $y \in C_i$ then $R_{C_i} y = y$. In particular,

$$T_{[1 \ldots N]} y = T_{N,1} \ldots T_{3,2} T_{2,1} y = P_{C_1} P_{C_2} \ldots P_{C_N} y \in C_1.$$

**Remark 3.6.** If $x_0 \notin C_1$, then the cyclic Douglas-Rachford trajectory need not coincide with von Neumann’s alternating projection method. We give an example involving two closed subspaces with codimension 1 (see Figure 1). Define

$$C_1 := \{ x \in \mathcal{H} : \langle a_1, x \rangle = 0 \}, \quad C_2 := \{ x \in \mathcal{H} : \langle a_2, x \rangle = 0 \},$$

where $a_1, a_2 \in \mathcal{H}$ such that $\langle a_1, a_2 \rangle > 0$. By scaling if necessary, we may assume that $\|a_1\| = \|a_2\| = 1$. Then one has,

$$P_{C_1} x = x - \langle a_1, x \rangle a_1, \quad P_{C_2} x = x - \langle a_2, x \rangle a_2,$$

and

$$T_{1,2} x = x + 2P_{C_2} P_{C_1} x - (P_{C_1} x + P_{C_2} x), \quad T_{2,1} x = x + 2P_{C_1} P_{C_2} x - (P_{C_2} x + P_{C_1} x).$$

If $x$ is such that $\langle a_i, x \rangle < 0$, for each $i$, then $\langle a_i, T_{1,2} x \rangle < 0$, for each $i$, since

$$\langle a_2, T_{1,2} x \rangle = \langle a_2, x - P_{C_1} x \rangle = \langle a_2, \langle a_1, x \rangle a_1 \rangle = \langle a_2, a_1 \rangle \langle a_1, x \rangle < 0,$$

$$\langle a_1, T_{1,2} x \rangle = \langle a_1, x - P_{C_2} x + 2\langle a_1, P_{C_2} P_{C_1} x \rangle \rangle = \langle a_1, a_2 \rangle \langle a_2, x \rangle + 2\langle a_1, P_{C_1} x - \langle a_2, P_{C_1} x \rangle a_2 \rangle$$

$$= \langle a_1, a_2 \rangle \langle a_2, x \rangle - 2\langle a_1, a_2 \rangle \langle a_2, P_{C_1} x \rangle$$

$$= \langle a_1, a_2 \rangle \langle a_2, x \rangle - 2\langle a_1, a_2 \rangle \langle a_2, x - \langle a_1, x \rangle a_1 \rangle$$

$$= \langle a_1, a_2 \rangle \langle a_2, x \rangle + 2\langle a_1, a_2 \rangle^2 \langle a_1, x \rangle < 0.$$

Similarly, $\langle a_i, T_{2,1} x \rangle < 0$, for each $i$. Together, this implies, $\langle a_i, T_{1,2} x \rangle < 0$, for each $i$. In particular, if $x_0$ is chosen such $\langle a_i, x_0 \rangle < 0$, for each $i$, then none of the cyclic Douglas-Rachford iterates lie in $C_1$ or $C_2$.

A second example, involving a ball and an affine subspace is illustrated in Figure 2 below.
Figure 1: An interactive *Cinderella* applet showing a cyclic Douglas-Rachford trajectory which does not coincide with von Neumann’s alternating projection method. Each green dots represents a 2-set Douglas-Rachford iteration.

Figure 2: An interactive *Cinderella* applet showing a cyclic Douglas-Rachford trajectory which does not coincide with von Neumann’s alternating projection method. Each green dots represents a 2-set Douglas-Rachford iteration.
Remark 3.7 (A product version). We now consider a product formulation of (1). Define two subsets of $\mathcal{H}^N$:

$$C := C_1 \times C_2 \times \cdots \times C_N, \quad D := \{(x, x, \ldots, x) \in \mathcal{H}^N : x \in \mathcal{H}\},$$

which are both closed and convex (in fact $D$ is a subspace). Consider the 2-set convex feasibility problem

$$\text{Find } x \in C \cap D \subseteq \mathcal{H}^N. \quad (3)$$

Then (1) is equivalent to (3) in the sense that

$$x \in \bigcap_{i=1}^N C_i \iff (x, x, \ldots, x) \in C \cap D.$$

Further the projections, and hence reflections, are easily computed since

$$P_C x = \prod_{i=1}^N P_{C_i} x, \quad P_D x = \prod_{i=1}^N \left( \frac{1}{N} \sum_{j=1}^N x_j \right).$$

Let $x_0 \in D$ and define $x_n := T_{[D,C]} x_{n-1}$. Then Corollary 3.5 yields

$$T_{[D,C]} x_n = P_D P_C x_n = \left( \frac{1}{N} \sum_{i=1}^N P_{C_i}, \frac{1}{N} \sum_{i=1}^N P_{C_i}, \ldots, \frac{1}{N} \sum_{i=1}^N P_{C_i} \right).$$

That is, the cyclic Douglas-Rachford coincides with averaged projections.

In general, the iteration is

$$T_{[D,C]} x = x - P_D x + 2P_D P_C T_{D,C} x - P_C T_{D,C} x + P_C R_D x - P_D P_C R_D x. \quad (4)$$

If $x = (x_1, x_2, \ldots, x_N)$, then the $i$th coordinate of (4) can be expressed as

$$(T_{[D,C]} x)_i = x_i - \frac{1}{N} \sum_{j=1}^N x_j + 2 \frac{1}{N} \sum_{j=1}^N P_{C_j} (T_{D,C} x)_j - P_{C_i} (T_{D,C} x)_i$$

$$+ P_{C_i} \left( \frac{2}{N} \sum_{j=1}^N x_j - x_i \right) - \frac{1}{N} \sum_{j=1}^N P_{C_j} \left( \frac{2}{N} \sum_{k=1}^N x_k - x_j \right),$$

where

$$(T_{D,C} x)_j = x_j - \frac{1}{N} \sum_{k=1}^N x_k + P_{C_j} \left( \frac{2}{N} \sum_{k=1}^N x_k - x_j \right),$$

which is a considerably more complex formula.

Let $A, B \subseteq \mathcal{H}$. Recall that $(x, y) \in A \times B$ is a best approximation pair relative to $(A, B)$ if

$$\|x - y\| = \inf \{\|a - b\| : a \in A, b \in B\}.$$
Remark 3.8. Consider \( C_1 = B_H := \{ x \in H : \| x \| \leq 1 \} \) and \( C_2 = \{ y \} \) for some \( y \in H \). Then
\[
T_{[1/2]} x = x - P_{C_1} x + P_{C_1} (y - x + P_{C_1} x),
\]
where \( P_{C_1} z = z \) if \( z \in C_1 \), and \( z/\| z \| \) otherwise.

Then \( x \in \text{Fix} T_{[1/2]} \iff P_{C_1} x = P_{C_1} (y - x + P_{C_1} x) \). Hence,

- If \( x \in C_1 \) then \( x = P_{C_1} y \) and \( \| x \| = 1 \).
- If \( y - x + P_{C_1} x \in C_1 \) then \( x = y \) and \( \| y - x + P_{C_1} x \| = 1 \).
- Else, \( \| x \| > 1 \) and \( \| y - x + P_{\mathcal{A}} x \| > 1 \), which implies
  \[
  x = \lambda y \quad \text{where} \quad \lambda = \left( \frac{\| x \|}{\| y - x + P_{C_1} x \| + \| x \| - 1} \right) \in (0, 1).
  \]

Altogether, \( \text{Fix} T_{[1/2]} \subseteq \{ \lambda y + (1 - \lambda) P_{\mathcal{A}} y : \lambda \in [0, 1] \} \).

In particular, if \( C_1 \cap C_2 \neq \emptyset \), then \( P_{C_1} y = y \) and Theorem 2.8 implies weak convergence of the cyclic Douglas-Rachford scheme to \( y \), the unique element of \( C_1 \cap C_2 \).

On the other hand, if \( C_1 \cap C_2 = \emptyset \), Theorem 2.8 cannot be invoked to guarantee convergence. However, if \( x \in \text{Fix} T_{[1/2]} \), the above characterization implies \( P_{C_1} x = P_{C_1} y \), and thus \( (P_{C_1} x, P_{C_2} x) \) is a best approximation pair relative to \( (C_1, C_2) \).

If instead, \( C_1 = S_H := \{ x \in H : \| x \| = 1 \} \). Then a similar analysis can be performed. If \( x \in \text{Fix} T_{[1/2]} \) such that \( x, y - x + P_{C_1} x \neq 0 \), then

- If \( x \in C_1 \) then \( x = P_{C_1} y \).
- If \( y - x + P_{C_1} x \in C_1 \) then \( x = y \).
- Else, \( x = \lambda y \) where
  \[
  \lambda = \left( \frac{\| x \|}{\| y - x + P_{C_1} x \| + \| x \| - 1} \right) \geq \left( \frac{\| x \|}{\| y - x \| + \| P_{C_1} x \| + \| x \| - 1} \right) > 0.
  \]
In this case, \( (P_{C_1} x, P_{C_2} x) \) is a best approximation pair relative to \( (C_1, C_2) \).

Experiments with interactive Cinderella applets suggest similar behaviour of the cyclic Douglas-Rachford method applied to other problems with \( C_1 \cap C_2 = \emptyset \). For example, see Figure 4. This suggests the following:

Conjecture 3.9. Let \( C_1, C_2 \subseteq H \) be closed and convex with \( C_1 \cap C_2 = \emptyset \). Suppose that a best approximation pair relative to \( (C_1, C_2) \) exists. Then the cyclic Douglas-Rachford scheme converges weakly to a point \( x \) such that \( (P_{C_1} x, P_{C_2} x) \) is a best approximation pair relative to \( (C_1, C_2) \).
We note that, if there exists an integer $n$ such that either $T_{[1/2]}^{2n}x_0 \in C_1$ or $T_{[1/2]}^{2n+1}x_0 \in C_2$, Remark 3.6 implies that the cyclic Douglas-Rachford scheme coincides with von Neumann’s alternating projection method. In this case, Conjecture 3.9 holds by [18, Theorem 2]. In this connection, see [3, 4]. It is not hard to think of non-convex settings in which Conjecture 3.9 is false.

4 Affine Constraints

In this section we show that the conclusions of Theorem 3.3 can be strengthened when the constraints are closed affine subspaces. The same techniques were used by Halperin [26] to show that von Neumann’s alternating projection, applied to finitely closed subspace, converged in norm to the projection on the intersection of the subspaces.\footnote{The Math Review MR0141978 points out that Kakutani had earlier proven weak convergence.}
Lemma 4.1. Let \( T : \mathcal{H} \to \mathcal{H} \) be linear and nonexpansive. Then
\[
\mathcal{H} = \text{Fix } T \bigoplus \text{range} (I - T).
\]

Proof. Fix \( T = \ker (I - T) \) is a closed subspace and \((\ker (I - T))^\perp = \text{range} (I - T)\). Since every closed subspace of a Hilbert space is complemented, the result follows.

We first prove norm convergence for closed subspaces. The translation formula (Fact 1.1) then allows an easy extension to the affine case.

Theorem 4.2. Let \( C_1, C_2, \ldots, C_N \subseteq \mathcal{H} \) be closed subspaces. Then, in norm
\[
\lim_{n \to \infty} T_n^{C_1 \ldots C_N} x = P_{\text{Fix} T_{[c_1 \ldots c_N]}} x.
\]

Proof. For convenience, denote \( T_{[c_1 \ldots c_N]} \) by \( T \). For any \( x \in \mathcal{H} \), asymptotic regularity implies
\[
\|T^n x - T^{n+1} x\| \to 0 \implies T^n (I - T)x \to 0 \implies T^n z \to 0 \text{ for } z \in \text{range} (I - T).
\]
Since \( T \) is nonexpansive, the same is true for \( z \in \text{range} (I - T) \). By Lemma 4.1, \( x = y + z \) for some \( y \in \text{Fix } T = \ker (I - T) \) and some \( z \in \text{range} (I - T) \). Projections onto closed subspaces are linear (Fact 1.1), hence so is \( T \). Therefore
\[
T^n x = T^n y + T^n z = y + T^n z \to y.
\]

Corollary 4.3. Let \( C_1, C_2, \ldots, C_N \subseteq \mathcal{H} \) be closed affine subspaces with nonempty intersection. Then, in norm
\[
\lim_{n \to \infty} T_n^{C_1 \ldots C_N} x = P_{\text{Fix} T_{[c_1 \ldots c_N]}} x.
\]

Proof. Let \( c \in \bigcap_{i=1}^N C_i \). Since \( C_i \) are affine we may write \( C_i = c + C_i' \) where \( C_i' \) is a closed subspace. Since projections, and hence reflections, onto closed subspaces (resp. closed affine sets) are linear (resp. translates) (Fact 1.1),
\[
T_{[c_1 c_2]} x = \frac{x + R_{C_2} R_{C_1} x}{2} = \frac{x + c - R_{C_2'} c + R_{C_2'} (c + R_{C_1} (x - c))}{2} = \frac{x + c + R_{C_2'} R_{C_1} (x - c)}{2} = c + \frac{(x - c) + R_{C_2'} R_{C_1} (x - c)}{2} = c + T_{[c_1' c_2']} (x - c).
\]
A second iteration yields
\[
T_{[c_2 c_3]} T_{[c_1 c_2]} x = c + T_{[c_2' c_3']} (T_{[c_1 c_2]} x - c) = c + T_{[c_2' c_3']} (c + T_{[c_1' c_2']} (x - c) - c) = c + T_{[c_2' c_3']} T_{[c_1' c_2']} (x - c).
\]
Therefore
\[
T_n^{C_1 \ldots C_N} x = c + T_n^{C_1 \ldots C_N} (x - c) \to c + P_{\text{Fix} T_{[c_1' c_2' \ldots c_N']}} (x - c) = P_{\text{Fix} T_{[c_1 \ldots c_N]}} x.
\]
This completes the proof.
Remark 4.4. For the case of two closed affine subspaces, the iteration becomes
\[ T_{B,A}T_{A,B} = \frac{2I + R_BR_A + R_AR_B}{4} = \frac{T_{A,B} + T_{B,A}}{2}. \]

This can be considered as an averaged Douglas-Rachford iteration. Since \( T_{A,B} \) and \( T_{B,A} \) can be computed independently, the iteration can be parallelized.

The result cannot be generalized in the obvious way for \( N > 2 \) closed affine subspaces. For instance, when \( N = 3 \),
\[
T_{[123]} = T_{3,1}T_{2,3}T_{1,2} = I - (P_{C_1} + P_{C_2} + P_{C_3}) + (P_{C_1}P_{C_3} + P_{C_2}P_{C_1} + P_{C_3}P_{C_2} + P_{C_3}P_{C_1} + P_{C_1}P_{C_2})
- (P_{C_3}P_{C_2}P_{C_1} + P_{C_1}P_{C_3}P_{C_2} + P_{C_1}AP_{C_3}P_{C_1} + P_{C_1}P_{C_2}P_{C_1}) + 2P_{C_1}P_{C_3}P_{C_2}P_{C_1},
\]
which includes a term which is the composition of four projection operators.

5 Numerical Experiments

In this Section we present the results of some computational experiments. These are not intended to be a complete computational study, but simply a first demonstration of viability of the method.

Computations were performed using Python 2.7.3 on an Intel Core i7 Q740 1.73GHz (single threaded) with 6GB of memory, running 64-bit Ubuntu 12.10. The following conditions were used.

- Stopping criterion:
  \[ \|x_n - T_{[12...N]}x_n\| < \epsilon \text{ where } \epsilon = 10^{-6}. \]
- An iteration limit of 1000 was enforced.
- After termination, the quality of the solution was measured by
  \[ \text{error} = \sum_{i=1}^{N} \|P_{C_i}x - P_{C_i}x\|^2. \]

Two experiments were performed. Namely we considered the feasibility problems:

Find \( x \in \bigcap_{i=1}^{N} C_i \subseteq \mathbb{R}^n \) where \( C_i = x_i + r_iB_H := \{y : \|x_i - y\| \leq r_i\} \), \quad (P1)

Find \( x \in \bigcap_{i=1}^{N} C_i \subseteq \mathbb{R}^n \) where \( C_i = x_i + r_iS_H := \{y : \|x_i - y\| = r_i\} \). \quad (P2)

Here \( B_H \) (resp. \( S_H \)) denotes the closed unit ball (resp. unit sphere).

To ensure the tested instances were feasible, constraints were generated as followed.
• Ball constraints: Randomly choose \( x_i \in [-5, 5]^n \) and \( r_i \in [\|x_i\|, \|x_i\| + 0.1] \).

• Sphere constraints: Randomly choose \( x_i \in [-5, 5]^n \) and set \( r_i = \|x_i\| \).

In both cases, the intersection contains the origin.

Note, if \( C_i \) is a sphere constraint then \( P_{C_i}(x_i) = C_i \), i.e. the nearest point is not unique and \( P_{C_i} \) is a set-valued mapping. In this situation, a random nearest point was choose from \( C_i \). In every other case, \( P_{C_i} \) is single valued.

Results are tabulated in Table 5. Illustrations of low dimensional examples are shown in Figures 7 and 8.

![Figure 5](image.png)

Figure 5: An interactive Cinderella applet using the cyclic Douglas-Rachford method to solve a feasibility problem with two sphere constraints. Each green dot represents a 2-set Douglas-Rachford iteration.

The cyclic Douglas-Rachford method easily solves both problems. Solution for 1000 dimensional instances, with varying numbers of constraints, could be obtained in under half-a-second, with an error of less than 0.0005. Instances involving fewer constraints required a greater number of iterations before termination. This can be explained by noting that each application of \( T_{[12...N]} \) applies a 2-set Douglas-Rachford operator \( N \) times, and hence iterations for instances with a greater number of constraints are more computationally expensive.

The method performed slightly better on (P1) compared to (P2). This might well be predicted. For in (P1), all constraint sets are convex, hence convergence is guaranteed by Theorem 3.3. However, in (P2), the constraints are non-convex, thus Theorem 3.3 cannot be evoked. The good performance of the standard Douglas-Rachford scheme on constraints of this kind, provide motivation for investigating the performance of cyclic Douglas-Rachford as a heuristic.
Figure 6: An interactive *Cinderella* applet using the cyclic Douglas-Rachford method to solve a feasibility problem with a near-tangent line and a sphere/ball constraint. Each green dot represents a 2-set Douglas-Rachford iteration.

Figure 7: Cyclic Douglas-Rachford algorithm applied to a 3-set feasibility problem in $\mathbb{R}^2$. The constraint sets are colored in blue, red and yellow. Each arrow represents a 2-set Douglas-Rachford iteration.

HTML versions of the interactive *Cinderella* applets are available at:

Table 1: Results for $N$ Balls in $\mathbb{R}^n$. The mean (max) from 10 trials are reported.

<table>
<thead>
<tr>
<th>Dimension ($n$)</th>
<th>Constraints ($N$)</th>
<th>Iterations</th>
<th>Time (s)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.6 (5)</td>
<td>0.004 (0.004)</td>
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Table 2: Results for $N$ Spheres in $\mathbb{R}^n$. The mean (max) from 10 trials are reported.

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<th>Constraints ($N$)</th>
<th>Iterations</th>
<th>Time (s)</th>
<th>Error</th>
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<td>100</td>
<td>8.0 (8)</td>
<td>0.097 (0.098)</td>
<td>0.000 (0.000)</td>
</tr>
<tr>
<td>500</td>
<td>200</td>
<td>5.0 (5)</td>
<td>0.120 (0.121)</td>
<td>0.000 (0.000)</td>
</tr>
<tr>
<td>500</td>
<td>500</td>
<td>3.0 (3)</td>
<td>0.180 (0.181)</td>
<td>0.000 (0.000)</td>
</tr>
<tr>
<td>1000</td>
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<td>81.3 (82)</td>
<td>0.121 (0.122)</td>
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<tr>
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<td>20</td>
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<td>0.000 (0.000)</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
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<td>0.000 (0.000)</td>
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<tr>
<td>1000</td>
<td>1000</td>
<td>2.0 (2)</td>
<td>0.329 (0.413)</td>
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</table>
Figure 8: Cyclic Douglas-Rachford algorithm applied to a 3-set feasibility problem in $\mathbb{R}^3$. The constraint sets are colored in blue, red and yellow. Each arrow represents a 2-set Douglas-Rachford iteration.
References


