The Life of \( \pi \): History and Computation
A Talk for Pi Day or Other Days

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University of Newcastle

3.14 pm, March 14, 2016
Revised 28.01.16 for Western 08.04.16
The Life of Pi: From this extended online presentation we shall sample

- Pi in popular culture: Pi Day — 3.14 (.15 in 2015)
- Why Pi? From utility to ... normality.
- Recent computations and digit extraction methods.
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Outline. We will cover Some of:

1. 25. Pi’s Childhood
   Links and References
   Babylon, Egypt and Israel
   Archimedes Method circa 250 BCE
   Precalculus Calculation Records
   The Fairly Dark Ages

2. 44. Pi’s Adolescence
   Infinite Expressions
   Mathematical Interlude, I
   Geometry and Arithmetic

3. 49. Adulthood of Pi
   Machin Formulas
   Newton and Pi
   Calculus Calculation Records
   Mathematical Interlude, II
   Why Pi? Utility and Normality

4. 80. Pi in the Digital Age
   Ramanujan-type Series
   The ENIACalculator
   Reduced Complexity Algorithms
   Modern Calculation Records
   A Few Trillion Digits of Pi

5. 114. Computing Individual Digits of \( \pi \)
   BBP Digit Algorithms
   Mathematical Interlude, III
   Hexadecimal Digits
   BBP Formulas Explained
   BBP for Pi squared — in base 2 and base 3
Introduction: \textbf{Pi is ubiquitous}

- The desire to understand $\pi$, the challenge, and originally the need, to calculate ever more accurate values of $\pi$, the ratio of the circumference of a circle to its diameter, has captured mathematicians — \textit{great and less great} — for eons.

- And, especially recently, $\pi$ has provided \textit{compelling examples} of computational mathematics.

\begin{center}
\begin{quote}
Pi, uniquely in mathematics, is pervasive in popular culture and the popular imagination.

In this talk I shall intersperse a largely chronological account of $\pi$’s mathematical and numerical status with examples of its ubiquity.
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In this talk I shall intersperse a **largely chronological account** of $\pi$’s mathematical and numerical status with examples of its ubiquity.
The Life of Pi Teaches a Great Deal:

We shall learn that scientists are humans and see a lot:

- of important mathematics;
- of its history and philosophy;
- about the evolution of computers and computation;
- of general history, philosophy and science;
- proof and truth (certainty and likelihood);
- of just plain interesting — sometimes weird — stuff.
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Mnemonics for Pi Abound: Piems — Word lengths give digits

Now I, even I, would celebrate
(3 1 4 1 5 9)
In rhymes inapt, the great
(2 6 5 3 5)
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before
Left men for guidance
How to circles mensurate.

“When you’re young, it comes naturally, but when you get a little older, you have to rely on mnemonics.”

– punctuation is always ignored
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*Life of Pi* (2001):

Yann Martel’s 2002 Booker Prize novel starts

‘‘My name is Pi scine Molitor Patel known to all as Pi Patel. For good measure I added \( \pi = 3.14 \) and I then drew a large circle which I sliced in two with a diameter, to evoke that basic lesson of geometry.’’

- 1706. Notation of \( \pi \) introduced by William Jones.
- 1737. Leonhard Euler (1707-83) popularized \( \pi \).
  - One of the three or four greatest mathematicians of all times:
  - He introduced much of our modern notation: \( \int, \Sigma, \phi, e, \Gamma, \ldots \).
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Wife of Pi (2013)
Pi: the **Source Book** (1997)

- **Berggren, Borwein and Borwein**, 3rd Ed, Springer, **2004**. (3,650 years of copyright releases. *E-rights* for Ed. 4 are in process.)
  - [MacTutor](http://www-gap.dcs.st-and.ac.uk/~history) at [www-gap.dcs.st-and.ac.uk/~history](http://www-gap.dcs.st-and.ac.uk/~history) is a good informal mathematical history source.
  - See also [www.cecm.sfu.ca/~jborwein/pi_cover.html](http://www.cecm.sfu.ca/~jborwein/pi_cover.html).
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Keanu Reeves, *Neo*, only has **314** seconds to enter "The Source."

(Do we need Parts 4 and 5?)

▶ From [http://www.freakingnews.com/Pi-Day-Pictures--1860.asp](http://www.freakingnews.com/Pi-Day-Pictures--1860.asp)
Roger Ebert gave the film 3.5 stars out of 4: “Pi is a thriller. I am not very thrilled these days by whether the bad guys will get shot or the chase scene will end one way instead of another. You have to make a movie like that pretty skillfully before I care.

“But I am thrilled when a man risks his mind in the pursuit of a dangerous obsession.”
Pi the Movie (1998): a Sundance screenplay winner

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Pi the URL

Pi to one MILLION decimal places

3.141592653589793238462643383279502884197169399375105820974944592.com/
This 2005 URL seems to have disappeared.
2014: $\pi$ Day turns 26: Our book *Pi and the AGM* is 27

- From www.google.com/trends?q=Pi+
  - H, E, D, C: “Pi Day March 14 (3.14, get it?)”
  - G,F: A ‘PI’, and the Seattle PI dies
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- 1988. *Pi Day* was Larry Shaw’s gag at the Exploratorium (SF).

- 2003. Schools running our award-winning applet nearly crashed SFU. It recites Pi fast in many languages
  - http://oldweb.cecm.sfu.ca/pi/yapPing.html
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Google Search for “Pi Day 2013”

1. Pi Day
   www.timeanddate.com › Calendar › Holidays


2. News for “Pi day 2013”

3. Celebrate Pi Day 2013 – with Pie
   Patch.com - 8 hours ago

   A perfect day for math geeks, Einstein lovers, and admirers of pie.

4. Celebrate Pi Day 2013 with Fredericksburg Pizza
   Patch.com - 22 hours ago

5. Pi Day 2013: A Celebration of the Mathematical Constant 3.1415926535...
   Patch.com - 1 day ago

   millburn.patch.com/.../celebrate-pi-day-2013-wit... - United States

   9 hours ago – A perfect day for math geeks, Einstein lovers, and admirers of pie.

7. Pi Day 2013: A Celebration of the Mathematical Constant ...
   manassas.patch.com/.../pi-day-2013-a-celebration... - United States

   2 days ago – March 14, or 3-14, is Pi Day – a day to celebrate the mathematical constant 3.14. What Pi Day activities do you have planned?

8. “Pi” Day 2013 - FunCheapSF.com
   sf.funcheapsf.com › City Guide

   2 days ago – Pi Day 2013 Any day can be a holiday, so why not look to math for some inspiration. Pi day (March 14th...3/14...3.14) seeks to celebrate π ...

9. Pi Day 2013 | Facebook
   www.facebook.com/events/181240568664057/

   Thu, 14 Mar - Everywhere,,

   Celebrate mathematics by celebrating Pi Day! Pi is the ratio of the circumference of a circle to its diameter (3.14159265...) For more info see: http://www.piday.org ...

10. Pi Day 2013: Events, Activities, & History | Exploratorium
    www.exploratorium.edu/learning_studio/pi/

    Welcome to our twenty-fifth annual Pi Day! Help us celebrate this never-ending number (3.14159 ... ) and Einstein’s birthday as well. On the afternoon of March ...
To solve the puzzle, first note that the clue for 28 DOWN is \textit{March 14, to Mathematicians}, to which the answer is \textit{PIDAY}. Moreover, roughly a dozen other characters in the puzzle are $\pi=\text{PI}$.

For example, the clue for 5 down was \textit{More pleased with the six character answer HAP$\pi$ER}.

(... -- -- -- -- -- -- -- -- -- -- -- -- -- -- -- ...)

(MSNBC Thanksgiving 1997)
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\[ \ldots - - - - - - - - - - - - \ldots \]

(Borweins and Plouffe | Pi Art | A Fine Book | Puzzle)

(MSNBC Thanksgiving 1997)
The Puzzle (By Permission)

The New York Times Crossword

Edited by Will Shortz
No. 0314

Across
1. Enlighten
2. A couple CBS specials
3. 1972 Broadway musical
4. Meld
5. Evad
6. Area
7. Surface again as a road
8. Pirate of France, briefly
9. Camera feature
10. B McClatchy, perhaps
11. River to the Lepanto Sea
12. Leg of a right triangle
13. Medical procedure, in brief
14. "Wiseass of Fortune" option
15. It gets bigger at right
16. "Hold your horses"
17. Shade
18. Europe/Asia border town
19. Suffocates
20. Learning
21. Broadside, e.g.
22. Rick with the 1976 "Hit
23. "Dread Cruze"
24. Year's targets
25. Enrages
26. Mustard nap
27. "Metals" or "enings"
28. Meat share
29. Pen
30. More pleased
31. Troubled with
32. "Star Trek" star
33. Enterprise
34. Fry
35. Mrs. Lincoln's earring
36. "Help! Help!
37. "I've eaten
38. "Horse"
39. "Is he"
40. "Big"
41. "There"
42. "Hello"
43. "Eat"
ANSWER TO PREVIOUS PUZZLE

J.M. Borwein
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The Simpsons (Permission refused by Fox)

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- See also “Springfield Theory,” (Science News, June 10, 2006) at www.aarms.math.ca/ACMN/links,
  Mouthful of Pi, http://tvtropes.org/pmwiki/pmwiki.php/Main/MouthfulOfPi and
  http://www.recordholders.org/en/list/memory.html#pi. The record is now over 80,000.
The Simpsons (Permission refused by Fox)

Apu: I can recite pi to 40,000 places. The last digit is 1. Homer: Mmm... pie. ("Marge in Chains." May 6, 1993)

National Pi Day 3.12.2009: The first successful Pi Law

H.RES.224

Latest Title: Supporting the designation of Pi Day, and for other purposes.
Cosponsors (15)
Latest Major Action: 3/12/2009 Passed/agreed to in House. Status: On motion to suspend the rules and agree to the resolution Agreed to by the Yeas and Nays: (2/3 required): 391 - 10 (Roll no. 124).

2007-2011. Chairman of House Committee on Science and Technology.

1897. Indiana Bill 246 was fortunately shelved.
Attempt to legislate value(s) of Pi and charge royalties started in the ‘Committee on Swamps’.

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Captions: To celebrate Pi Day 2008, the San Francisco Exploratorium made a Pi thing with more than 4,000 colored beads on it, each color representing a digit from 0 to 9. (Credit: Daniel Fareld/CNET)

Washington politicians took time from bailouts and earmark-laden spending packages on Wednesday for what might seem like an unproductive act officially designating a National Pi Day.

That’s Pi as in ratio-of-a-circle’s-circumference-to-diameter, better known as the mathematical constant beginning with 3.14159.
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Life of Pi (CARMA)
On Pi Day, one number 'reeks of mystery'

By Elizabeth Landau, CNN
March 12, 2010 12:36 p.m. EST

Sunday is Pi Day, on which math geeks celebrate the number representing the ratio of circumference to diameter of a circle.

(CNN) -- The sound of meditation for some people is full of deep breaths or gentle humming. For Marc Umile, it's "3.14159265358979..."

Whether in the shower, driving to work, or walking down the street, he'll mentally rattle off digits of pi to pass the time. Holding 10th place in the world for pi memorization -- he typed out 15,314 digits from memory in 2007 -- Umile meditates through one of the most beloved and mysterious numbers in all of mathematics.

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On Pi Day, one number 'reeks of mystery'

By Elizabeth Landau, CNN
March 12, 2010 12:36 p.m. EST

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Google's homage to 3.14.10
Judge rules “Pi is a non-copyrightable fact” on 3.14.2012

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Google (29-1-13) and US Gov’t (14-8-12) still both love $\pi$

U.S. Population Reaches 314,159,265, Or Pi Times 100 Million: Census

The Huffington Post | By Boreis Karraszu
Posted: 08/14/2012 4:03 am Updated: 08/14/2012 5:55 am

The U.S. population has reached a nifty and delightful milestone. Shortly after 2:29 p.m. on Tuesday, August 14, 2012, the U.S. population was exactly 314,159,265, or pi ($\pi$) times 100 million, the U.S. Census Bureau reports.

Pi ($\pi$) is a unique number in multiple ways. For one, it is the ratio of a circle’s circumference to its diameter. It is also an irrational number, meaning it goes on forever without ever repeating itself. Are you remembering how much you loved geometry class? You can check out Pi to one million places [here](http://example.com/pi-one-million).

Contestants will be offered $110,000 for a successful exploit delivered by a web page that achieves a browser or system level compromise in guest mode or as a tagged-in user. A $150,000 prize will be offered for a compromise with device persistence – guest to guest with intern swapout, delivered via a web page.

Hackers will need to demonstrate their attacks against a Wi-Fi-only model of Samsung’s Nexus S 5G.
Each year brings more $\pi$-trivia and serious stuff

1. September 2014. *Pencil, Paper and Pi* or where Shanks computation went wrong
   


3. 22.10.14. A mile of Pi on one piece of paper

   http://www.youtube.com/watch?v=0r3cEKZiLmg&feature=youtu.be
π Records *Always* Make The News

- By now you get the idea: \( \pi \) is everywhere ... also volumes, areas, lengths, probabilities, **everywhere**.
26. Links and References

1. The Pi Digit site: http://carma.newcastle.edu.au/bbp
2. Dave Bailey's Pi Resources: http://crd.lbl.gov/~dhbailey/pi/

---

The Infancy of Pi: Babylon, Egypt and Israel

2000 BCE. Babylonians used the approximation $\frac{22}{7} = 3.125$.

1650 BCE. Rhind papyrus: a circle of diameter nine has the area of a square of side eight:
$$\pi = \frac{256}{81} = 3.1604\ldots$$

- Pi is the only topic from the earliest strata of mathematics being actively researched today.

Some argue ancient Hebrews used $\pi = 3$:
Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (I Kings 7:23; 2 Chron. 4:2)

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Archimedes of Syracuse (c.287 – 212 BCE) was first to show that the “two Pi’s” are one in Measurement of the Circle (c.250 BCE):

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3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825
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Archimedes Method circa 250 BCE

The first rigorous mathematical calculation of $\pi$ was also due to Archimedes, who used a brilliant scheme based on doubling inscribed and circumscribed polygons

$$6 \mapsto 12 \mapsto 24 \mapsto 48 \mapsto 96$$

to obtain the bounds $3 \frac{10}{71} < \pi < 3 \frac{1}{7}$.

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Where Greece Was: **Magna Graecia**

1. Syracuse
2. Troy
3. Byzantium
   - Constantinople
4. Rhodes
   - (Helios)
5. Halicarnassus
   - (Mausolus)
6. Ephesus
   - (Artemis)
7. Athens
   - (Zeus)

The others of the Seven Wonders: Lighthouse of Alexandria, Pyramids of Giza, Gardens of Babylon
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  − After 1929. Painted over with gold icons and left in a wet bucket in a garden.
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“Archimedes used knowledge of levers and centres of gravity to envision ways of balancing geometric figures against one another which allowed him to compare their areas or volumes. He then used rigorous geometric argument to prove Method discoveries.”

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Archimedes from *The Method*

“... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.”
Let’s be Clear: $\pi$ Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* ‘knows’ this since

$$0 < \int_{0}^{1} \frac{(1 - x)^4 x^4}{1 + x^2} \, dx = \frac{22}{7} - \pi,$$

(1)

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

**Assume we trust it.** Then the integrand is strictly positive on $(0, 1)$, and the answer in (1) is an area and so strictly positive, despite millennia of claims that $\pi$ is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to $\pi$. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$
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Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

**Algorithm (Archimedes)**

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

\[
\begin{align*}
    a_{n+1} &= \frac{2a_nb_n}{a_n + b_n} \quad (H) \\
    b_{n+1} &= \sqrt{a_{n+1}b_n} \quad (G)
\end{align*}
\]

These tend to $\pi$, error decreasing by a factor of four at each step.

- The greatest mathematician (scientist) to live before the Enlightenment. To compute $\pi$ Archimedes had to invent many subjects — including numerical and interval analysis.
Archimedes Method circa 1800 CE

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Proving $\pi$ is not $\frac{22}{7}$

In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4 (1 - x)^4}{1 + x^2} dx = \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4 t - 4 \arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus proves (1).

QED

One can take this idea a bit further. Note that

$$\int_0^1 x^4 (1 - x)^4 dx = \frac{1}{630}.$$ (2)
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$$\int_0^1 x^4 (1 - x)^4 \, dx = \frac{1}{630}. \quad (2)$$
... Going Further

Hence

\[
\frac{1}{2} \int_0^1 x^4 (1 - x)^4 \, dx < \int_0^1 \frac{(1 - x)^4 x^4}{1 + x^2} \, dx < \int_0^1 x^4 (1 - x)^4 \, dx.
\]

Combine this with (1) and (2) to derive:

\[
\frac{223}{71} < \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260} < \frac{22}{7}
\]

and so re-obtain Archimedes’ famous

\[
3 \frac{10}{71} < \pi < 3 \frac{10}{70}.
\]
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2\pi = 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},
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good to 16 decimal places (using $3 \cdot 2^{28}$-gons).
Kuhnian ‘Paradigm Shifts’ and Normal Science

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## Precalculus π Calculations

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Digits</th>
</tr>
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<tbody>
<tr>
<td>Babylonians</td>
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<td>1</td>
</tr>
<tr>
<td>Egyptians</td>
<td>2000? BCE</td>
<td>1</td>
</tr>
<tr>
<td>Hebrews (1 Kings 7:23)</td>
<td>550? BCE</td>
<td>1</td>
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<td><strong>Archimedes</strong></td>
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<td>Ptolemy</td>
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<tr>
<td>Liu Hui</td>
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<td>5</td>
</tr>
<tr>
<td>Tsu Ch’ung Chi</td>
<td>480?</td>
<td>7</td>
</tr>
<tr>
<td>Al-Kashi</td>
<td>1429</td>
<td>14</td>
</tr>
<tr>
<td>Romanus</td>
<td>1593</td>
<td>15</td>
</tr>
<tr>
<td>Van Ceulen (Ludolph’s number*)</td>
<td>1615</td>
<td>35</td>
</tr>
</tbody>
</table>

* Used $2^{62}$-gons for 39 places/35 correct — published posthumously.
Ludolph’s Rebuilt Tombstone in Leiden

Ludolph van Ceulen (1540-1610)

- Destroyed several centuries ago; the plans remained.
Ludolph’s tombstone was reconsecrated July 5, 2000.

- Attended by Dutch royal family and 750 others.
- My brother lectured on Pi from halfway up to the pulpit.
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> If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division — assuming that he has sufficient gifts — then you will have to send him to Italy.
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• Why did Google want precisely this many pieces of the Pie?
Google Buys \((\text{Pi-3}) \times 100,000,000\) Shares

August 19, 2005

14,159,265 New Slices of Rich Technology

By JOHN MARKOFF

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45. Pi’s (troubled) Adolescence

1579. Modern mathematics dawns in Viète’s product

\[
\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots = \frac{2}{\pi}
\]  

— considered to be the first truly infinite formula — and in the first continued fraction given by Lord Brouncker (1620-1684):

\[
\frac{2}{\pi} = \frac{1}{1 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \cdots}}}}
\]
Eqn. (4) was based on John Wallis’ (1613-1706) ‘interpolated’ product:

\[
\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdots = \prod_{k=1}^{\infty} \frac{4k^2 - 1}{4k^2} = \frac{2}{\pi}
\]

which led to discovery of the *Gamma function* and much more.

- Christiaan Huygens (1629-1695) did not believe (5) before he checked it numerically.

> It’s a clue.
A never repeating or ending chain, the total timeless catalogue, elusive sequences, sum of the universe.
This riddle of nature begs:
Can the totality see no pattern, revealing order as reality’s disguise?


J.M. Borwein  
Life of Pi (CARMA)
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Mathematical Interlude I: the \textbf{Zeta Function}

Formula (5) follows from \textit{Euler’s product formula} for $\pi$,

\[
\frac{\sin(\pi x)}{x} = c \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)
\]

\text{(6)}

with $x = 1/2$, or by integrating $\int_0^{\pi/2} \sin^{2n}(t) \, dt$ by parts.

One may divine (6) — as Euler did — by \textit{considering $\sin(\pi x)$ as an ‘infinite’ polynomial} and obtaining a product in terms of the roots $0, \{1/n^2\}$. Euler argued that, like a polynomial, $c = \pi$ is the value at 0.

The coefficient of $x^2$ in the Taylor series is the sum of the roots:

$\zeta(2) := \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$.

Hence, $\zeta(2n) = \text{rational} \times \pi^{2n}$: so

$\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$

(using Bernoulli numbers)

\textbf{1976. Apéry showed $\zeta(3)$ irrational; and Zudilin (CARMA) has shown at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.}
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CATEGORY: By the numbers. CLUE: The phrase “How I want a drink, alcoholic of course” is often used to help memorize this.

ANSWER: What is Pi? FINAL SCORES:

Ray: $7,200 + $7,000 = $14,200 (What is Pi)
    (New champion: $14,200)

Stacey: $11,400 - $3,001 = $8,399 (What is no clue!?)
    (2nd place: $2,000)

Victoria: $12,900 - $9,901 = $2,999 (What is quadratic for)
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J.M. Borwein Life of Pi (CARMA)
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- **Ray:** $7,200 + $7,000 = $14,200 (What is Pi)
  (New champion: $14,200)
- **Stacey:** $11,400 - $3,001 = $8,399 (What is no clue!?)
  (2nd place: $2,000)
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- 17C Newton and Leibnitz discovered calculus ... and fought over priority (Machin adjudicated).
- It was instantly exploited to find formulas for \( \pi \).

One early use comes from the \( \arctan \) integral and series:\(^3\)

\[
\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots) \, dt
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\[
= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots
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\(^3\)Known to Madhava of Sangamagrama (c. 1350 – c. 1425) near Kerala. He probably computed 13 digits of Pi.
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Formally $x := 1$ gives the Gregory–Leibniz formula $\left(1671–74\right)$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

- Naively, this is useless — hundreds of terms produce two digits.
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By contrast, Euler’s (1738) trigonometric identity

$$\tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \quad (7)$$

produces the geometrically convergent:

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots$$

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John **Machin** (1680-1751) and Brook **Taylor** (1685-1731)

An even faster formula, found earlier by John Machin — Brook Taylor’s teacher — lies in the identity

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\frac{\pi}{4} = 4 \tan^{-1} \left( \frac{1}{5} \right) - \tan^{-1} \left( \frac{1}{239} \right).
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(9)

- Used in numerous computations of \( \pi \) (starting in 1706) culminating with Shanks’ computation of \( \pi \) to 707 decimals in 1873.

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Integrating term-by-term and combining the above:

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Newton’s (1643-1727) Annus Mirabilis

Newton used his formula to find **15 digits** of \( \pi \).

- As noted, he ‘apologized’ for “**having no other business at the time.**” A standard 1951 MAA chronology said, condescendingly, “*Newton never tried to compute \( \pi \).*"

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## Calculus $\pi$ Calculations: and an IBM 7090

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Digits</th>
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<tbody>
<tr>
<td>Sharp (and Halley)</td>
<td>1699</td>
<td>71</td>
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<tr>
<td><strong>Machin</strong></td>
<td>1706</td>
<td>100</td>
</tr>
<tr>
<td>Strassnitzky and Dase</td>
<td>1844</td>
<td>200</td>
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<td>Rutherford</td>
<td>1853</td>
<td>440</td>
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<tr>
<td><strong>W. Shanks</strong></td>
<td>1874</td>
<td>(707) 527</td>
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<td>Ferguson (Calculator)</td>
<td>1947</td>
<td>808</td>
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<tr>
<td>Reitwiesner et al. (ENIAC)</td>
<td>1949</td>
<td>2,037</td>
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<tr>
<td>Genuys</td>
<td>1958</td>
<td>10,000</td>
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<td><strong>D. Shanks</strong> and Wrench (IBM)</td>
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<tr>
<td>Guilloud and Bouyer</td>
<td>1973</td>
<td>1,001,250</td>
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An early vegetarian (who misused needles) next to the inventor of **Monte Carlo** methods.

1. Draw a unit square and inscribe a circle within: the area of the circle is $\frac{\pi}{4}$.

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Life of Pi (CARMA)
Monte Carlo Methods

- This is a Monte Carlo estimate (MC) for $\pi$.
- MC simulation: slow ($\sqrt{n}$) convergence — but great in parallel on Beowulf clusters.
- Used in Manhattan project ... the atomic-bomb predates digital computers!
Gauss (1777-1855), Johan Dase and William Shanks

In his teens, Viennese computer and 'kopfrechner' Dase (1824-1861) publicly demonstrated his skill by multiplying

\[ 79532853 \times 93758479 = 7456879327810587 \]

- in 54 seconds; 20-digits in 6 min; 40-digits in 40 min; 100-digit numbers in \(8\frac{3}{4}\) hours etc.
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In 1849-50 Dase made a seven-digit Tafel der natürlichen Logarithmen der Zahlen, asking the Hamburg Academy to fund factorization of integers between 7 and 10 million (evidence for the Prime Number Theorem).

- Now Gauss was impressed and recommended Dase be funded.

- 1861. When Dase died he had only reached 8,000,000.

One motivation for computations of π was very much in the spirit of modern experimental mathematics: to see if

- the decimal expansion of π repeats, meaning π was the ratio of two integers (a rational number),

- if π was the root of an integer polynomial (an algebraic number).

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Lambert (1728-77)  Legendre (1752-1833)  Lindemann (1852-1939)

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The Three **Construction Problems** of Antiquity

The other two are
doing the cube and
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- This settled *once and for all*, the ancient Greek question of whether the circle could be squared with ruler and compass.

- It cannot, because lengths of lines that can be constructed using ruler and compasses (*constructible numbers*) are necessarily algebraic, and squaring the circle is equivalent to constructing the value of $\pi$.

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The Irrationality of $\pi$, II

Ivan Niven's 1947 proof that $\pi$ is irrational. Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n!f(x)$ has integral coefficients and terms in $x$ of degree not less than $n$, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$; also for $x = \pi = a/b$, since $f(x) = f(a/b - x)$. By elementary calculus we have

$$\frac{d}{dx} \{ F'(x) \sin x - F(x) \cos x \}$$

$$= F''''(x) \sin x + F'(x) \sin x = f(x) \sin x$$
The Irrationality of $\pi$, II

and

$$\int_0^\pi f(x) \sin x \, dx = [F'(x) \sin x - F(x) \cos x]_0^\pi$$

$$= F(\pi) + F(0).$$

(10)

Now $F(\pi) + F(0)$ is an integer, since $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers. But for $0 < x < \pi$,

$$0 < f(x) \sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (10) is positive but arbitrarily small for $n$ sufficiently large. Thus (10) is false, and so is our assumption that $\pi$ is rational.

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This, exact transcription of Niven's proof, is an excellent intimation of more sophisticated irrationality and transcendence proofs.
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At the end of his story, Piscine (Pi) Molitor writes:

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Summation. Why Pi?  “Pi is Mount Everest.”

What motivates modern computations of $\pi$ — given that irrationality and transcendence of $\pi$ were settled a century ago?

- One motivation is the raw challenge of harnessing the stupendous power of modern computer systems.

  Programming is quite hard — especially on large, distributed memory computer systems: load balancing, communication needs, etc.

Substantial practical spin-offs accrue:

- Accelerating computations of $\pi$ sped up the fast Fourier transform (FFT) — heavily used in science and engineering.
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Pi Seems Normal: Things we sort of know about Pi

A walk on a billion hex digits of Pi with \textit{box dimension} 1.85343...

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Pi Seems Normal: Some million bit comparisons

Euler’s constant and a pseudo-random number

Fractal dimension of 10,000 random walks of 1 million steps

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Pi Seems Normal: Comparisons to Stoneham’s number \( \sum_{k \geq 1} 1/(3^k 2^k) \), 1

In base 2 Stoneham’s number is provably normal. It may be normal base 3.
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The X Chromosome (34K) and Chromosome One (10K).
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Erdös-Copeland number (base 2) and Champernowne number (base 10).

All pictures are thanks to Fran Aragon and Jake Fountain
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- The simple continued fraction for Pi is unbounded.
  - Euler found the 292.
- There are infinitely many sevens in the decimal expansion of Pi.
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- Or pretty much anything I have not told you.

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\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \frac{1}{1 + \ldots}}}}}}
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Decimal Digit Frequency: and "Johnny" von Neumann

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JvN (1903-57) at the Institute for Advanced Study
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1st von Neumann architecture machine

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**Hexadecimal Digit Frequency:** and Richard Crandall (Apple HPC)

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Changing Cognitive Tastes

Why in antiquity \( \pi \) was not measured to greater accuracy than \( \frac{22}{7} \) (with rope)?

It reflects not an inability, but rather a very different mindset to a modern (Baconian) experimental one — see Francis Bacon’s *De augmentis scientiarum* (1623).

- Gauss and Ramanujan did not exploit their identities for \( \pi \).
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where \( r(n) := \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} \).

- I can “discover” it using 30-digit arithmetic and check it to 1,000 digits in 0.75 sec, 10,000 digits in 4.01 min with two naive command-line instructions in Maple.

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  - It may be true for no good reason — it might just have no proof and be a very concrete Gödel-like statement.
Pi in High Culture (1993)

The admirable number pi:
three point one four one.
All the following digits are also initial,
five nine two because it never ends.
It can’t be comprehended six five three five at a glance,
eight nine by calculation,
seven nine or imagination,
not even three two three eight by wit, that is, by comparison
four six to anything else
two six four three in the world.
The longest snake on earth calls it quits at about forty feet.
Likewise, snakes of myth and legend, though they may hold out a bit longer.
The pageant of digits comprising the number pi doesn’t stop at the page’s edge.
It goes on across the table, through the air, over a wall, a leaf, a bird’s nest, clouds, straight into the sky, through all the bottomless, bloated heavens.

Oh how brief - a mouse tail, a pigtail - is the tail of a comet!
How feeble the star’s ray, bent by bumping up against space!
While here we have two three fifteen three hundred nineteen
my phone number your shirt size the year nineteen hundred and seventy-three the sixth floor
the number of inhabitants sixty-five cents
hip measurement two fingers a charade, a code, in which we find hail to thee, blithe spirit, bird thou never wert
alongside ladies and gentlemen, no cause for alarm, as well as heaven and earth shall pass away,
but not the number pi, oh no, nothing doing, it keeps right on with its rather remarkable five, its uncommonly fine eight, its far from final seven, nudging, always nudging a sluggish eternity to continue.

1996 Nobel Wislawa Szymborska (2-7-1923 1-2-2012)
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1950s. Commercial computers — and discovery of advanced algorithms for arithmetic — unleashed π.

1965. The new fast Fourier transform (FFT) performed high-precision multiplications much faster than conventional methods — viewing numbers as polynomials in $\frac{1}{10}$.

- Newton methods helped reduce time for computing $\pi$ to ultra-precision from millennia to weeks or days.

$$x \leftarrow x + x(1 - bx)$$  \hspace{1cm}  $$x \leftarrow x + x(1 - ax^2)/2$$

converts $1/b$ to $4 \times$  \hspace{1cm}  converts $1/\sqrt{a}$ to $6 \times$ (7 for $\sqrt{a}$)

▼ But until the 1980s all computer evaluations of $\pi$ employed classical formulas, usually of Machin-type.

Happily, MRI and FFT were discovered at the same time.
Computers Cease Being Human

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Newton’s method is self-correcting and quadratically convergent.

So we start close (to the left); and

We keep only the first half of each answer.
Newton's method is **self-correcting and quadratically convergent**.

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Ramanujan’s Seventy-Fifth Birthday Stamp.

- Truly new infinite series formulas were discovered by the self-taught Indian genius Srinivasa Ramanujan around **1910**.
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Ramanujan Series for $\frac{1}{\pi}$

One of these series is the remarkable:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)! (1103 + 26390k)}{(k!)^4 396^{4k}}$$

(12)

- Each term adds an additional eight correct digits.

◇ 1985. ‘Hacker’ Bill Gosper used (12) to compute 17 million digits of (the continued fraction for) $\pi$; and so the first proof of (12)!

1987. David and Gregory Chudnovsky found a variant:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}$$

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Some Series Can Save Significant Work

- Relatedly, the Ramanujan-type series:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left( \frac{(2n)!}{n!16^n} \right)^3 \frac{42n + 5}{16}. \quad (14)$$

allows one to compute the billionth binary digit of $1/\pi$, or the like, without computing the first half of the series.

Conjecture (Moore’s Law in Electronics Magazine 19 April, 1965)

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ENIAC: Electronic Numerical Integrator and Calculator, I

**SIZE/WEIGHT:** ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.

The ENIAC in the Smithsonian

- This Smithsonian 20Mb picture would require 100,000 ENIACs to store. [Moore’s Law!]
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ENIAC: Integrator and Calculator, II

SPEED/MEMORY: A 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000 — 1,000 times faster than any earlier machine. The first stored-memory computer, ENIAC could store 200 digits.

1949 ‘skunk-works’ computation of $\pi$ — suggested by von Neumann — to 2,037 places in 70 hrs.

Origin of the term ‘bug’?

ARCHITECTURE: Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line. The accumulators were connected to each other manually.
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Programming ENIAC in 1946
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Another formula of Euler for \( \text{arccot} \) is:

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x \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! \left(x^2 + 1\right)^{n+1}} = \arctan \left( \frac{1}{x} \right).
\]

As \( 10 \left( 18^2 + 1 \right) = 57^2 + 1 = 3250 \) we may rewrite the formula

\[
\frac{\pi}{4} = \arctan \left( \frac{1}{18} \right) + 8 \arctan \left( \frac{1}{57} \right) - 5 \arctan \left( \frac{1}{239} \right)
\]

used by Shanks and Wrench in 1961 for 100,000 digits, and by Guilloud and Boyer in 1973 for a million digits of Pi in the efficient form

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\pi = 864 \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! \, 325^{n+1}} + 1824 \sum_{n=0}^{\infty} \frac{(n!)^2 4^n}{(2n+1)! \, 3250^{n+1}} - 20 \arctan \left( \frac{1}{239} \right)
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where terms of the second series are just decimal shifts of the first.
Ballantine’s (1939) Series for \( \pi \)

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where terms of the second series are just \textit{decimal shifts} of the first.
5. A Million Decimals? Can \( \pi \) be computed to 1,000,000 decimals with the computers of today? From the remarks in the first section we see that the program which we have described would require times of the order of months. But since the memory of a 7090 is too small, by a factor of ten, a modified program, which writes and reads partial results, would take longer still. One would really want a computer 100 times as fast, 100 times as reliable, and with a memory 10 times as large. No such machine now exists.

There are, of course, many other formulas similar to (1), (2), and (5), and other programming devices are also possible, but it seems unlikely that any such modification can lead to more than a rather small improvement.

Are there entirely different procedures? This is, of course, possible. We cite the following: compute \( 1/\pi \) and then take its reciprocal. This sounds fantastic, but, in fact, it can be faster than the use of equation (2). One can compute \( 1/\pi \) by Ramanujan’s formula [8]:

\[
\frac{1}{\pi} = \frac{1}{4} \left( \frac{1133}{882} - \frac{22583}{882^2} + \frac{44043}{882^3} - \cdots \right)
\]

The first factors here are given by \((-1)^n (1133 + 214604k)\). A binary value of \( 1/\pi \) equivalent to 100,000D, can be computed on a 7090 using equation (6) in 6 hours instead of the 8 hours required for the application of equation (2). To reciprocate this value of \( 1/\pi \) would take about 1 hour. Thus, we can reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and, in any case, this small gain is quite inadequate for the present question.

One could hope for a theoretical approach to this question of optimization—a theory of the “depth” of numbers—but no such theory now exists. One can guess that \( \varepsilon \) is not as “deep” as \( \pi \), but try to prove it!

Such a theory would, of course, take years to develop. In the meantime—say, in 5 to 7 years—such a computer as we suggested above (100 times as fast, 100 times as reliable, and with 10 times the memory) will, no doubt, become a reality. At that time a computation of \( \pi \) to 1,000,000D will not be difficult.

Shanks (the 2nd) and Wrench: “A Million Decimals?” (1961)
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The first factors here are given by $(-1)^k (1123 + 21460k)$. A binary value of $1/\pi$ equivalent to 100,000D can be computed on a 7090 using equation (6) in 6 hours instead of the 8 hours required for the application of equation (2). To reciprocate this value of $1/\pi$ would take about 1 hour. Thus, we can reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and, in any case, this small gain is quite inadequate for the present question.

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* We have computed $1/\pi$ by (6) to over 5000D in less than a minute.
† We have computed $\pi$ on a 7090 to 100,263D by the obvious program. This takes 2.5 hours instead of the 8-hour run for $\pi$ by (2).
A *random walk* on $\pi$ (courtesy David and Gregory Chudnovsky)

- See Richard Preston’s: “The Mountains of Pi”, *New Yorker*, March 2, 1992 (AAAS-Westinghouse Award for Science Journalism);
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Reduced Complexity Methods

These series are much faster than classical ones, but the number of terms needed still increases linearly with the number of digits.

Twice as many digits correct requires twice as many terms of the series.

1976. Richard Brent of ANU-CARMA and Eugene Salamin independently found a reduced complexity algorithm for $\pi$.
- It takes $O(\log N)$ operations for $N$ digits.
- Uses arithmetic-geometric mean iteration (AGM) and other elliptic integral ideas due to Gauss and Legendre circa 1800.
- Gauss — and others — missed connection to computing $\pi$. 
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- Uses arithmetic-geometric mean iteration (AGM) and other elliptic integral ideas due to Gauss and Legendre circa 1800.
  - Gauss — and others — missed connection to computing \( \pi \).
Reduced Complexity Methods

These series are much faster than classical ones, but the number of terms needed still increases linearly with the number of digits.

Twice as many digits correct requires twice as many terms of the series.

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A Reduced Complexity Algorithm

Algorithm (Brent-Salamin AGM iteration)

Set \( a_0 = 1, b_0 = 1/\sqrt{2} \) and \( s_0 = 1/2 \). Calculate

\[
\begin{align*}
  a_k &= \frac{a_{k-1} + b_{k-1}}{2} \quad & (A) \\
  b_k &= \sqrt{a_{k-1}b_{k-1}} \quad & (G) \\
  c_k &= a_k^2 - b_k^2, \quad & s_k &= s_{k-1} - 2^k c_k \\
  \text{and compute} \quad p_k &= \frac{2a_k^2}{s_k}. \quad (15)
\end{align*}
\]

Then \( p_k \) converges quadratically to \( \pi \).

- Each step doubles the correct digits — successive steps produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 digits of \( \pi \).
  - 25 steps compute \( \pi \) to 45 million digits. But, steps must be performed to the desired precision.
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Four Famous Pi Guys: Salamin, Kanada, Bailey and Gosper in 1987

- To appear in Donald Knuth’s book of mathematics pictures.
And Some Others: Niven, Shanks (1917-96), Brent, Zudilin (込)
1985. Peter and I discovered algebraic algorithms of all orders:

**Algorithm (Cubic Algorithm)**

Set \( a_0 = 1/3 \) and \( s_0 = (\sqrt{3} - 1)/2 \). Iterate

\[
\begin{align*}
    r_{k+1} &= \frac{3}{1 + 2(1 - s_k^3)^{1/3}}, \\
    s_{k+1} &= \frac{r_{k+1} - 1}{2}, \\
    a_{k+1} &= r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1).
\end{align*}
\]

Then \( 1/a_k \) converges cubically to \( \pi \).

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The Borwein Brothers

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The Borwein Brothers
A Fourth Order Algorithm

Algorithm (Quartic Algorithm)

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

and

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3}y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then $1/a_k$ converges quartically to $\pi$.

- Using $4 \times \mathrm{‘plus’} \ 1 \div \mathrm{‘plus’} \ 2 \ 1/\sqrt{\cdot} = 19$ full precision $\times$ per step. So $20$ steps costs out at around $400$ full precision multiplications.

(This assumes intermediate storage. Additions are cheap)
A Fourth Order Algorithm

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ap_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2).
\end{align*}
\]

Then \( 1/a_k \) converges quartically to \( \pi \)

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y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \quad \text{and} \quad a_{k+1} = a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2).
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## Modern Calculation Records:

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Correct Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miyoshi and Kanada</td>
<td>1981</td>
<td>2,000,036</td>
</tr>
<tr>
<td>Kanada-Yoshino-Tamura</td>
<td>1982</td>
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</tr>
<tr>
<td>Gosper</td>
<td>1985</td>
<td>17,526,200</td>
</tr>
<tr>
<td>Bailey</td>
<td>Jan. 1986</td>
<td>29,360,111</td>
</tr>
<tr>
<td>Kanada and Tamura</td>
<td>Sep. 1986</td>
<td>33,554,414</td>
</tr>
<tr>
<td>Kanada et. al</td>
<td>Jan. 1987</td>
<td>134,217,700</td>
</tr>
<tr>
<td>Kanada and Tamura</td>
<td>Jan. 1988</td>
<td>201,326,551</td>
</tr>
<tr>
<td>Chudnovskys</td>
<td>May 1989</td>
<td>480,000,000</td>
</tr>
<tr>
<td>Kanada and Tamura</td>
<td>Jul. 1989</td>
<td>536,870,898</td>
</tr>
<tr>
<td>Kanada and Tamura</td>
<td>Nov. 1989</td>
<td>1,073,741,799</td>
</tr>
<tr>
<td>Chudnovskys</td>
<td>Aug. 1991</td>
<td>2,260,000,000</td>
</tr>
<tr>
<td>Chudnovskys</td>
<td>May 1994</td>
<td>4,044,000,000</td>
</tr>
<tr>
<td>Kanada and Takahashi</td>
<td>Oct. 1995</td>
<td>6,442,450,938</td>
</tr>
<tr>
<td>Kanada and Takahashi</td>
<td>Jul. 1997</td>
<td>51,539,600,000</td>
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<tr>
<td>Kanada and Takahashi</td>
<td>Sep. 1999</td>
<td>206,158,430,000</td>
</tr>
<tr>
<td>Kanada-Ushiro-Kuroda</td>
<td>Dec. 2002</td>
<td>1,241,100,000,000</td>
</tr>
<tr>
<td>Takahashi</td>
<td>Jan. 2009</td>
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<tr>
<td>Takahashi</td>
<td>Apr. 2009</td>
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</tr>
<tr>
<td>Bellard</td>
<td>Dec. 2009</td>
<td>2,699,999,990,000</td>
</tr>
<tr>
<td>Kondo and Yee</td>
<td>Aug. 2010</td>
<td>5,000,000,000,000</td>
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<tr>
<td>Kondo and Yee</td>
<td>Oct. 2011</td>
<td>10,000,000,000,000</td>
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<tr>
<td>Kondo and Yee</td>
<td>Dec. 2013</td>
<td>12,200,000,000,000</td>
</tr>
</tbody>
</table>
Moore’s Law Marches On

Computation of $\pi$ since 1975 plotted vs. Moore’s law predicted increase
An Amazing Algebraic Approximation to $\pi$

The transcendental number $\pi$ and the algebraic number $1/a_{20}$ actually agree for more than 1.5 trillion decimal places.

- $\pi$ and $1/a_{21}$ agree for more than six trillion decimal places.

1984. I found these on a 16K upgrade of an 8K double-precision TRS80-100 Radio Shack portable.


- Took 6 months to convince Seymour Cray; then ran on every CRAY before it left the factory.
- This iteration still gives me goose bumps. Especially when written out in full...
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\[ a_0 = 6 - 4 \sqrt{2}, \quad y_0 = \sqrt{2} - 1 \]

\[
y_1 = \frac{1 - \frac{4}{\sqrt{1-y_0^4}}}{1 + \frac{4}{\sqrt{1-y_0^4}}}, \quad a_1 = a_0 \left(1 + y_1\right)^4 - 2^3 y_1 \left(1 + y_1 + y_1^2\right)
\]

\[
y_2 = \frac{1 - \frac{4}{\sqrt{1-y_1^4}}}{1 + \frac{4}{\sqrt{1-y_1^4}}}, \quad a_2 = a_1 \left(1 + y_2\right)^4 - 2^5 y_2 \left(1 + y_2 + y_2^2\right)
\]

\[
y_3 = \frac{1 - \frac{4}{\sqrt{1-y_2^4}}}{1 + \frac{4}{\sqrt{1-y_2^4}}}, \quad a_3 = a_2 \left(1 + y_3\right)^4 - 2^7 y_3 \left(1 + y_3 + y_3^2\right)
\]

\[
y_4 = \frac{1 - \frac{4}{\sqrt{1-y_3^4}}}{1 + \frac{4}{\sqrt{1-y_3^4}}}, \quad a_4 = a_3 \left(1 + y_4\right)^4 - 2^9 y_4 \left(1 + y_4 + y_4^2\right)
\]

\[
y_5 = \frac{1 - \frac{4}{\sqrt{1-y_4^4}}}{1 + \frac{4}{\sqrt{1-y_4^4}}}, \quad a_5 = a_4 \left(1 + y_5\right)^4 - 2^{11} y_5 \left(1 + y_5 + y_5^2\right)
\]

\[
y_6 = \frac{1 - \frac{4}{\sqrt{1-y_5^4}}}{1 + \frac{4}{\sqrt{1-y_5^4}}}, \quad a_6 = a_5 \left(1 + y_6\right)^4 - 2^{13} y_6 \left(1 + y_6 + y_6^2\right)
\]

\[
y_7 = \frac{1 - \frac{4}{\sqrt{1-y_6^4}}}{1 + \frac{4}{\sqrt{1-y_6^4}}}, \quad a_7 = a_6 \left(1 + y_7\right)^4 - 2^{15} y_7 \left(1 + y_7 + y_7^2\right)
\]

\[
y_8 = \frac{1 - \frac{4}{\sqrt{1-y_7^4}}}{1 + \frac{4}{\sqrt{1-y_7^4}}}, \quad a_8 = a_7 \left(1 + y_8\right)^4 - 2^{17} y_8 \left(1 + y_8 + y_8^2\right)
\]

\[
y_9 = \frac{1 - \frac{4}{\sqrt{1-y_8^4}}}{1 + \frac{4}{\sqrt{1-y_8^4}}}, \quad a_9 = a_8 \left(1 + y_9\right)^4 - 2^{19} y_9 \left(1 + y_9 + y_9^2\right)
\]

\[
y_{10} = \frac{1 - \frac{4}{\sqrt{1-y_9^4}}}{1 + \frac{4}{\sqrt{1-y_9^4}}}, \quad a_{10} = a_9 \left(1 + y_{10}\right)^4 - 2^{21} y_{10} \left(1 + y_{10} + y_{10}^2\right)
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J.M. Borwein  Life of Pi (CARMA)
\[
y_1 = \frac{1 - 4\sqrt{1 - y_0^4}}{1 + 4\sqrt{1 - y_0^4}}, \quad a_1 = a_0 (1 + y_1)^4 - 2^3 y_1 \left(1 + y_1 + y_1^2\right)
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y_2 = \frac{1 - 4\sqrt{1 - y_1^4}}{1 + 4\sqrt{1 - y_1^4}}, \quad a_2 = a_1 (1 + y_2)^4 - 2^5 y_2 \left(1 + y_2 + y_2^2\right)
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\[
y_3 = \frac{1 - 4\sqrt{1 - y_2^4}}{1 + 4\sqrt{1 - y_2^4}}, \quad a_3 = a_2 (1 + y_3)^4 - 2^7 y_3 \left(1 + y_3 + y_3^2\right)
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y_4 = \frac{1 - 4\sqrt{1 - y_3^4}}{1 + 4\sqrt{1 - y_3^4}}, \quad a_4 = a_3 (1 + y_4)^4 - 2^9 y_4 \left(1 + y_4 + y_4^2\right)
\]

\[
y_5 = \frac{1 - 4\sqrt{1 - y_4^4}}{1 + 4\sqrt{1 - y_4^4}}, \quad a_5 = a_4 (1 + y_5)^4 - 2^{11} y_5 \left(1 + y_5 + y_5^2\right)
\]

\[
y_6 = \frac{1 - 4\sqrt{1 - y_5^4}}{1 + 4\sqrt{1 - y_5^4}}, \quad a_6 = a_5 (1 + y_6)^4 - 2^{13} y_6 \left(1 + y_6 + y_6^2\right)
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y_7 = \frac{1 - 4\sqrt{1 - y_6^4}}{1 + 4\sqrt{1 - y_6^4}}, \quad a_7 = a_6 (1 + y_7)^4 - 2^{15} y_7 \left(1 + y_7 + y_7^2\right)
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y_8 = \frac{1 - 4\sqrt{1 - y_7^4}}{1 + 4\sqrt{1 - y_7^4}}, \quad a_8 = a_7 (1 + y_8)^4 - 2^{17} y_8 \left(1 + y_8 + y_8^2\right)
\]

\[
y_9 = \frac{1 - 4\sqrt{1 - y_8^4}}{1 + 4\sqrt{1 - y_8^4}}, \quad a_9 = a_8 (1 + y_9)^4 - 2^{19} y_9 \left(1 + y_9 + y_9^2\right)
\]

\[
y_{10} = \frac{1 - 4\sqrt{1 - y_9^4}}{1 + 4\sqrt{1 - y_9^4}}, \quad a_{10} = a_9 (1 + y_{10})^4 - 2^{21} y_{10} \left(1 + y_{10} + y_{10}^2\right)
\]
\[
y_{11} = \frac{1 - 4\sqrt{1 - y_{10}^4}}{1 + 4\sqrt{1 - y_{10}^4}}, \quad a_{11} = a_{10} (1 + y_{11})^4 - 2^{23} y_{11} (1 + y_{11} + y_{11}^2)
\]
\[
y_{12} = \frac{1 - 4\sqrt{1 - y_{11}^4}}{1 + 4\sqrt{1 - y_{11}^4}}, \quad a_{12} = a_{11} (1 + y_{12})^4 - 2^{25} y_{12} (1 + y_{12} + y_{12}^2)
\]
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\]
\[
y_{14} = \frac{1 - 4\sqrt{1 - y_{13}^4}}{1 + 4\sqrt{1 - y_{13}^4}}, \quad a_{14} = a_{13} (1 + y_{14})^4 - 2^{29} y_{14} (1 + y_{14} + y_{14}^2)
\]
\[
y_{15} = \frac{1 - 4\sqrt{1 - y_{14}^4}}{1 + 4\sqrt{1 - y_{14}^4}}, \quad a_{15} = a_{14} (1 + y_{15})^4 - 2^{31} y_{15} (1 + y_{15} + y_{15}^2)
\]
\[
y_{16} = \frac{1 - 4\sqrt{1 - y_{15}^4}}{1 + 4\sqrt{1 - y_{15}^4}}, \quad a_{16} = a_{15} (1 + y_{16})^4 - 2^{33} y_{16} (1 + y_{16} + y_{16}^2)
\]
\[
y_{17} = \frac{1 - 4\sqrt{1 - y_{16}^4}}{1 + 4\sqrt{1 - y_{16}^4}}, \quad a_{17} = a_{16} (1 + y_{17})^4 - 2^{35} y_{17} (1 + y_{17} + y_{17}^2)
\]
\[
y_{18} = \frac{1 - 4\sqrt{1 - y_{17}^4}}{1 + 4\sqrt{1 - y_{17}^4}}, \quad a_{18} = a_{17} (1 + y_{18})^4 - 2^{37} y_{18} (1 + y_{18} + y_{18}^2)
\]
\[
y_{19} = \frac{1 - 4\sqrt{1 - y_{18}^4}}{1 + 4\sqrt{1 - y_{18}^4}}, \quad a_{19} = a_{18} (1 + y_{19})^4 - 2^{39} y_{19} (1 + y_{19} + y_{19}^2)
\]
\[
y_{20} = \frac{1 - 4\sqrt{1 - y_{19}^4}}{1 + 4\sqrt{1 - y_{19}^4}}, \quad a_{20} = a_{19} (1 + y_{20})^4 - 2^{41} y_{20} (1 + y_{20} + y_{20}^2).
\]
\begin{align*}
y_{11} &= \frac{1 - \sqrt[4]{1 - y_{10}^4}}{1 + \sqrt[4]{1 - y_{10}^4}}, \quad a_{11} = a_{10} (1 + y_{11})^4 - 2^{23} y_{11} (1 + y_{11} + y_{11}^2) \\
y_{12} &= \frac{1 - \sqrt[4]{1 - y_{11}^4}}{1 + \sqrt[4]{1 - y_{11}^4}}, \quad a_{12} = a_{11} (1 + y_{12})^4 - 2^{25} y_{12} (1 + y_{12} + y_{12}^2) \\
y_{13} &= \frac{1 - \sqrt[4]{1 - y_{12}^4}}{1 + \sqrt[4]{1 - y_{12}^4}}, \quad a_{13} = a_{12} (1 + y_{13})^4 - 2^{27} y_{13} (1 + y_{13} + y_{13}^2) \\
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y_{15} &= \frac{1 - \sqrt[4]{1 - y_{14}^4}}{1 + \sqrt[4]{1 - y_{14}^4}}, \quad a_{15} = a_{14} (1 + y_{15})^4 - 2^{31} y_{15} (1 + y_{15} + y_{15}^2) \\
y_{16} &= \frac{1 - \sqrt[4]{1 - y_{15}^4}}{1 + \sqrt[4]{1 - y_{15}^4}}, \quad a_{16} = a_{15} (1 + y_{16})^4 - 2^{33} y_{16} (1 + y_{16} + y_{16}^2) \\
y_{17} &= \frac{1 - \sqrt[4]{1 - y_{16}^4}}{1 + \sqrt[4]{1 - y_{16}^4}}, \quad a_{17} = a_{16} (1 + y_{17})^4 - 2^{35} y_{17} (1 + y_{17} + y_{17}^2) \\
y_{18} &= \frac{1 - \sqrt[4]{1 - y_{17}^4}}{1 + \sqrt[4]{1 - y_{17}^4}}, \quad a_{18} = a_{17} (1 + y_{18})^4 - 2^{37} y_{18} (1 + y_{18} + y_{18}^2) \\
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\end{align*}
“A Billion Digits is Impossible”

- Since 1988 used, with Salamin-Brent, by Kanada’s Tokyo team. Including: $\pi$ to 200 billion decimal digits in 1999 ... and records in 2009.

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Billions and Billions

I spent my entire fortune to buy this supercomputer.

What does it do?

It can calculate the value of pi to about a jillion decimal places...

A lot of people talk about the areas of circles, but I’m doing something about it.
Kirk asks:

"Aren't there some mathematical problems that simply can't be solved?"

And Spock 'fries the brains' of a rogue computer by telling it:

"Compute to the last digit the value of ... Pi."
Star Trek

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**Pi the Song:** from the album Aerial

**2005** Influential Singer-songwriter *Kate Bush* sings “Pi” on Aerial.

Sweet and gentle and sensitive man  
With an obsessive nature and deep fascination  
for numbers  
And a complete infatuation  
with the calculation of Pi  

**Chorus:** Oh he love, he love, he love  
He does love his numbers  
And they run, they run, they run him  
In a great big circle  
In a circle of infinity

“a sentimental ode to a mathematician, audacious in both subject matter and treatment. The chorus is the number sung to many, many decimal places.” [150 – wrong after 50] — Observer Review
2002. Kanada computed $\pi$ to over **1.24 trillion decimal digits**. His team first computed $\pi$ in **hex** (base 16) to **1,030,700,000,000** places, using good old Machin type relations:

$$
\pi = 48 \tan^{-1} \frac{1}{49} + 128 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239} \\
+ 48 \tan^{-1} \frac{1}{110443} \\
\quad \text{ (Takano, pop-song writer 1982)}
$$

$$
\pi = 176 \tan^{-1} \frac{1}{57} + 28 \tan^{-1} \frac{1}{239} - 48 \tan^{-1} \frac{1}{682} \\
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Back to the Future

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• The decimal expansion was checked by converting it back to hex.
  – Base conversion require pretty massive computation.

• Six times as many digits as before: hex and decimal ran 600 hrs on same 64-node Hitachi — at roughly 1 Tflop/sec (2002).

• 2002 hex-pi computation record broken 3 times in 2009 — quite spectacularly. We will see that:

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- He tried a complete verification computation, but it failed;
- He had used hexadecimal and so the first could be ‘partially’ checked using his BBP series (17) below.

This took **131 days** but he only used a single 4-core workstation with a lot of storage and even more human intelligence!

- For full details of this feat and of Takahashi’s most recent computation one can look at Wikipedia /wiki/Chronology_of_computation_of_pi
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- August 2010. On a home built $18,000 machine, Kondo (hardware engineer, below) and Yee (undergrad software) nearly doubled this to 5,000,000,000,000 places. The last 30 are

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Two New Pi Guys: Alex Yee and his Elephant

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Mario Livio (JPL) in 01-31-2013 *HuffPost*

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There is probably no number in mathematics (with the possible exception of e) that is more celebrated than the one equal to the ratio of a circle’s circumference to its diameter. This number is denoted by the Greek letter π (pi). Pi is approximately equal to 3.14159, but its decimal representation neither ends nor settles into a repeating pattern. In fact, on Oct. 16, 2011, Alexander J. Yee and Shigeru Kondo completed the task of using a custom-built computer (shown in Fig. 1) for 371 days, to calculate π to 10 trillion digits. To appreciate this accuracy, let me note that if we wanted to express the radius of the observable universe in terms of the radius of the hydrogen atom, about 40 digits would have sufficed.
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Figure 1. The computer used by Alexander Yee and Shigeru Kondo to calculate \( \pi \) to 10 trillion digits (reproduced by permission from Alexander Yee).
1971. One might think everything of interest about computing $\pi$ has been discovered. This was Beckmann’s view in *A History of $\pi$*. Yet, the Salamin-Brent quadratic iteration was found only five years later. Higher-order algorithms followed in the 1980s.

1990. Rabinowitz and Wagon found a ‘spigot’ algorithm for $\pi$: It ‘drips’ individual digits (of $\pi$ in any desired base) using all previous digits.

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What BBP Does?

Prior to 1996, most folks thought to compute the \(d\)-th digit of \(\pi\), you had to generate the (order of) the entire first \(d\) digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of \(\pi\). In 1996, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of \(\pi\). It produces:

  - a modest-length string hex or binary digits of \(\pi\), beginning at any position, *using no prior bits*;
    1. is implementable on any modern computer;
    2. requires no multiple precision software;
    3. requires very little memory; and has
    4. a computational cost growing only slightly faster than the digit position.
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    - 4 a computational cost *growing only slightly faster than the digit position*. 
What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$
\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right) \tag{16}
$$

- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (16) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

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\pi = 4 {} _2 F _1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5
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where $2 {} _2 F _1 (1, 1/4; 5/4, -1/4) = 0.955933837 \ldots$ is a Gauss hypergeometric function.
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What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right)$$

$$\text{(16)}$$

- The millionth hex digit (four millionth binary digit) of $\pi$ can be found in under 30 secs on a fairly new computer in Maple (not C++) and the billionth in 10 hrs.

Equation (16) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$\pi = 4 \, _2F_1 \left( 1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{2} \right) - \log 5$$

where $$_2F_1(1, 1/4; 5/4, -1/4) = 0.955933837 \ldots$$ is a Gauss hypergeometric function.
Edge of Computation Prize Finalist

BBP was the only mathematical finalist (of about 40) for the first *Edge of Computation Science Prize*
- Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...

Won by David Deutsch — discoverer of Quantum Computing.
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1. It includes most known BBP formulas.
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Below are the results obtained using the interactive calculator.
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Below are the results obtained using the interactive calculator.

```
Please enter a digit to calculate: [00000]

Digits are [68AC8FCFB80]
Calculated in 1.033 seconds.
```
Matthew Tam has built an interactive website.
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Mathematical Interlude: III. (Maple, Mathematica and Human)

Proof of (16). For $0 < k < 8$,

$$
\int_{0}^{1/\sqrt{2}} \frac{x^{k-1}}{1 - x^8} \, dx = \int_{0}^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i} \, dx = \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i (8i + k)}.
$$

Thus, one can write

$$
\sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right)
$$

$$
= \int_{0}^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8} \, dx,
$$

which on substituting $y := \sqrt{2}x$ becomes

$$
\int_{0}^{1} \frac{16y - 16}{y^4 - 2y^3 + 4y - 4} \, dy = \int_{0}^{1} \frac{4y}{y^2 - 2} \, dy - \int_{0}^{1} \frac{4y - 8}{y^2 - 2y + 2} \, dy = \pi.
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QED
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Tuning BBP Computation

- **1997.** Fabrice Bellard of INRIA computed 152 bits of $\pi$ starting at the trillionth position; in 12 days on 20 workstations working in parallel over the Internet.

Bellard used the following variant of (16):

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k + 1)} - \frac{1}{64} \sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k} \left( \frac{32}{4k + 1} + \frac{8}{4k + 2} + \frac{1}{4k + 3} \right)$$

This frequently-used formula is a little faster than (16).

Colin Percival (L) and Fabrice Bellard (R)
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Abstract

We present a new record on computing specific bits of \( \pi \), the mathematical constant, and discuss performing such computations on Apache Hadoop clusters. The new record represented in hexadecimal is

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August 27, 2012 Ed Karrel found 25 hex digits of $\pi$ starting after the $10^{15}$ position

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See [www.karrels.org/pi/](http://www.karrels.org/pi/),
Everything Doubles Eventually

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BBP Formulas Explained

Base-$b$ BBP numbers are constants of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)b^k}, \quad (18)$$

where $p(k)$ and $q(k)$ are integer polynomials and $b = 2, 3, \ldots$.

- I illustrate why this works in binary for $\log 2$. We start with:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}, \quad (19)$$

as discovered by Euler.

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- **The key**: the numerator in (20), $2^{d-k} \mod k$, can be found rapidly by binary exponentiation, performed modulo $k$. So,

$$3^{17} = (((3^2)^2)^2)^2 \cdot 3$$

uses only 5 multiplications, not the usual 16. Moreover, $3^{17}$ mod 10 is done as $3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3$. 

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**J.M. Borwein**  
Life of Pi (CARMA)
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Catalan’s Constant $G$: and BBP for $G$ in Binary

The simplest number not proven irrational is

$$G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots,$$

$$\frac{\pi^2}{12} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

2009. $G$ is calculated to 31.026 billion digits. Records often use:

$$G = \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{(2n)_n(2n+1)^2} + \frac{\pi}{8} \log(2 + \sqrt{3}) \text{ (Ramanujan)} \quad (21)$$

- holds since $G = -T(\frac{\pi}{4}) = -\frac{3}{2} T(\frac{\pi}{12})$ where $T(\theta) := \int_0^\theta \log \tan \sigma d\sigma$.

An 18 term binary BBP formula for $G = 0.9159655941772190 \ldots$ is:

Eugene Catalan (1818-94)—a revolutionary
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Eugene Catalan (1818-94) – a revolutionary
A Better Formula for $G$

A 16 term formula in concise BBP notation is:

$$\begin{align*}
G &= P(2, 4096, 24, \vec{v}) \\
\vec{v} &:= (6144, -6144, -6144, 0, -1536, -3072, -768, 0, -768, -384, 192, 0, -96, 96, 96, 0, 24, 48, 12, 0, 12, 6, -3, 0)
\end{align*}$$

It takes almost exactly $\frac{8}{9}$th the time of 18 term formula for $G$.

- This makes for a very cool calculation
- Since we can not prove $G$ is irrational, Who can say what might turn up?
What About **Base Ten**?

- The first integer logarithm with no known binary BBP formula is $\log 23$ (since $23 \times 89 = 2^{10} - 1$).

Searches conducted by numerous researchers for base-ten formulas have been unfruitful. Indeed:

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Searches conducted by numerous researchers for base-ten formulas have been unfruitful. Indeed:

**2004.** D. Borwein (my father), W. Gallway and I showed there are no BBP formulas of the *Machin-type* of (16) for \( \pi \) if base is not a power of **two**.

- Bailey and Crandall have shown connections between the existence of a \( b \)-ary BBP formula for \( \alpha \) and its base \( b \) normality (via a dynamical system conjecture).
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Pi Photo-shopped: a 2010 PiDay Contest

“Noli Credere Pictis”
Thanks to Dave Broadhurst, a ternary BBP formula exists for $\pi^2$ (unlike $\pi$):

$$\pi^2 = \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \times \left\{ \frac{243}{(12k + 1)^2} - \frac{405}{(12k + 2)^2} - \frac{81}{(12k + 4)^2} \right. \\
- \frac{27}{(12k + 5)^2} - \frac{72}{(12k + 6)^2} - \frac{9}{(12k + 7)^2} \left. \right\}$$
Thanks to Dave Broadhurst, a ternary BBP formula exists for $\pi^2$ (unlike $\pi$):

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A Partner Binary BBP Formula for $\pi^2$

$$\pi^2 = \frac{9}{8} \sum_{k=0}^{\infty} \frac{1}{26^k} \left\{ \frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right\}$$

- We do not fully understand why $\pi^2$ allows BBP formulas in two distinct bases.

- $4\pi^2$ is the area of a sphere in three-space (L).
- $\frac{1}{2}\pi^2$ is the volume inside a sphere in four-space (R).
  - So in binary we are computing these fundamental physical constants.
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IBM’s New **Record** Results

Algorithm (What We Did)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I, have obtained and *(nearly) confirmed:*

1. **106** digits of $\pi^2$ base 2 at the **ten trillionth** place base 64
2. **94** digits of $\pi^2$ base 3 at the **ten trillionth** place base 729
3. **150** digits of $G$ base 2 at the **ten trillionth** place base 4096

on a **4-rack BlueGene/P system** at IBM’s Benchmarking Centre in Rochester, Minn, USA.
The 3 Records Use Over 1380 CPU Years (135 rack days)

An enormous amount of delicate computation: **1380 years** is a long time. Suppose a spanking new IBM single-core PC went back **1381 years**.

- It would find itself in **632 CE**.
- The year that Mohammed died, and the Caliphate was established. If it then calculated π nonstop:
  - Through the Crusades, black plague, Moguls, Renaissance, discovery of America, Gutenberg, Reformation, invention of steam, Napoleon, electricity, WW2, the transistor, fiber optics,...
- With no breaks or break-downs:
- It would have finished in **2012**.

August 2013, *Notices of the AMS*
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August 2013, Notices of the AMS
IBM’s New Results: $\pi^2$ base 2

Algorithm (10 trillionth digits of $\pi^2$ in base 64 — in 230 years)

1. The calculation took, on average, 253529 seconds per thread. It was broken into 7 “partitions” of 2048 threads each. For a total of $7 \cdot 2048 \cdot 253529 = 3.6 \cdot 10^9$ CPU seconds.

2. On a single Blue Gene/P CPU it would take 115 years!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{7 \cdot 2048 \cdot 253529}{4096 \cdot 60 \cdot 60 \cdot 24} = 10.3$ “rack days”.

3. The verification run took the same time (within a few minutes): 106 base 2 digits are in agreement.

base-8 digits = 75|60114505303236475724500005743262754530363052416350634|573227604
60114505303236475724500005743262754530363052416350634|22021056612
IBM’s New Results: $\pi^2$ base 3

Algorithm (10 trillionth digits of $\pi^2$ in base 729 — in 414 years)

1. The calculation took, on average, 795773 seconds per thread. It was broken into 4 “partitions” of 2048 threads each. For a total of $4 \cdot 2048 \cdot 795773 = 6.5 \cdot 10^9$ CPU seconds.

2. On a single Blue Gene/P CPU it would take 207 years!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{4 \cdot 2048 \cdot 795773}{4096 \cdot 60 \cdot 60 \cdot 24} = 18.4$ “rack days”.

3. The verification run took the same time (within a few minutes): 94 base 3 digits are in agreement.

base-9 digits = 001|12264485064548583177111135210162856048323453468|10565567|635862
12264485064548583177111135210162856048323453468|04744867|134524345
IBM’s New Results: $G$ base 2

Algorithm (10 trillionth digits of $G$ in base 4096 — in 735 years)

1. The calculation took, on average, 707857 seconds per thread. It was broken into 8 “partitions” of 2048 threads each. For a total of $8 \cdot 2048 \cdot 707857 = 1.2 \cdot 10^{10}$ CPU seconds.

2. On a single Blue Gene/P CPU it would take 368 years!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{8 \cdot 2048 \cdot 707857}{4096 \cdot 60 \cdot 60 \cdot 24} = 32.8$ “rack days”.

3. The verification run will take the same time (within a few minutes): xxx base 2 digits will be in agreement.

base-8 digits = 0176|347050537747770511226133716201252573272173245226000177545727
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx|xxx
Thank You, One and All, and Happy Birthday, Albert


J.M. Borwein  
Life of Pi (CARMA)
139. Links and References

1. The Pi Digit site: http://carma.newcastle.edu.au/bbp
2. Dave Bailey's Pi Resources: http://crd.lbl.gov/~dhbailey/pi/

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