L_p Norms and the Sinc Function

D. Borwein, J. M. Borwein and I. E. Leonard

March 6, 2009

1 Introduction

The *sinc function* is a real valued function defined on the real line $\mathbb{R}$ by the following expression:

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

This function is important in many areas of computing science, approximation theory, and numerical analysis. For example, it is used in interpolation and approximation of functions, approximate evaluation of transforms (e.g. Hilbert, Fourier, Laplace, Hankel, and Mellon transforms as well as the fast Fourier transform). It is used in finding approximate solutions of differential and integral equations, in image processing (it is the Fourier transform of the box filter and central to the understanding of the Gibbs phenomenon [12]), in signal processing and information theory. Much of this is nicely described in [7].

The first explicit appearance of the sinc function in approximation theory was probably in the use of the *Whittaker cardinal functions* $C(f, h)$ to approximate functions analytic on an interval or on a contour. Given a function f which is defined on the real line $\mathbb{R}$, the function $C(f, h)$ is defined by

$$C(f, h) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)$$

whenever the series converges, where the stepsize $h > 0$ and where

$$S(k, h)(x) = \frac{\sin[(\frac{\pi}{h}(x - kh)]}{\frac{\pi}{h}(x - kh)},$$

that is, $S(k, h)(x) = \operatorname{sinc}\big(\frac{\pi}{h}(x - kh)\big)$. See, for example, [11].

The object of this note is to study the behavior and properties of the following function

$$I(p) = \sqrt{p} \int_0^{\infty} \left| \frac{\sin x}{x} \right|^p dx$$

for $1 < p < \infty$. Note that this function is only defined for $p > 1$, since $\int_0^{\infty} \frac{\sin x}{x} dx$ is conditionally convergent. Indeed

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2} \qquad \text{while} \qquad \int_0^{\infty} \left| \frac{\sin x}{x} \right| dx = +\infty,$$

see [12].

This integral arises, for example, in the L_p approximation of real valued functions by Whittaker cardinal functions, and is important in estimating the error made in the approximation. It also arises in many other computational problems, and it is surprising that so little is known about it.

Various properties of the function $I(p)$ are known. For example, the behavior of $I(p)$ for large p is known:

$$\lim_{p\to\infty} I(p) = \lim_{p\to\infty} \sqrt{p} \int_0^{\infty} \left| \frac{\sin x}{x} \right|^p dx = \sqrt{\frac{3\pi}{2}}$$

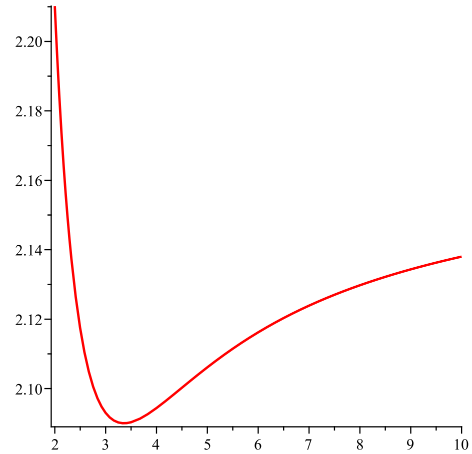

Figure 1: The function I on $[2, 10]$

This result, obtained independently by A. Meir and I. E. Leonard, is in principle not new (see equation 3). We provide a self-contained proof below as part of our more general result in Theorem 1.

Also, for integer p, the integral

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^p dx$$

can be calculated explicitly. In fact, for $n \geq 1$ we have

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx = \frac{1}{(n-1)!} \cdot \frac{\pi}{2^n} \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} (n-2k)^{n-1}$$

This result is most definitely not new, it can be found in Bromwich [4, Exercise 22, p. 518], where it is attributed to Wolstenholme, and in many other places—including two relatively recent articles on integrals of more general products of sinc functions [2, 3].

Thus, if p is an even integer, then we have a closed form expression for $I(p)$, and in this case the values of $I(p)$ can be calculated exactly:

$$I(p) = \sqrt{p} \int_0^\infty \left(\frac{\sin x}{x}\right)^p dx = \sqrt{p} \cdot \frac{1}{(p-1)!} \cdot \frac{\pi}{2^p} \cdot \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^k \binom{p}{k} (p-2k)^{p-1}. \tag{1}$$

In particular $I(2) = \pi/\sqrt{2}$, $I(4) = 2\pi/3$ and $I(6) = 11\sqrt{6}\pi/40$. That said, this sum is very difficult to use numerically for large p. Not only are the rational factors growing rapidly but it contains extremely large terms of alternating sign and consequently dramatic cancelations. For example

$$I(36) = \frac{73150940186053320492582118865887 1713}{10630810665006321944101493145600000 00}\,\pi,$$

and $I(10) = Q_{100}\,\pi$ where Q_{100} is a rational number whose numerator and denominator both have roughly 150 digits. Similarly $I(12) = Q_{144}\,\pi$ where Q_{144} is comprised of 240 digit integers. We also note that numerical integration of $I(p)$ even to single precision is not easy and so (1) provides a very good confirmation of numerical integration results. We challenge the reader to numerically confirm the limit at infinity to 8 places.

The behavior of $I(p)$ for intermediate values of p is not fully established. It had been conjectured that $I(p)$ had a global minimum at $p = 4$, however, (very) recent computations using both *Maple* and *Mathematica* suggest that the global minimum, and unique critical point, is at approximately $p = 3.36...$ as illustrated in Figure 1.

Although it is known that $\lim_{p \to 1^+} I(p) = +\infty$, and that $\lim_{p \to \infty} I(p) = \sqrt{\frac{3\pi}{2}}$, it is not known precisely how the asymptote $y = \sqrt{\frac{3\pi}{2}}$ is approached, although both numerical and graphical evidence strongly suggest the following conjecture:

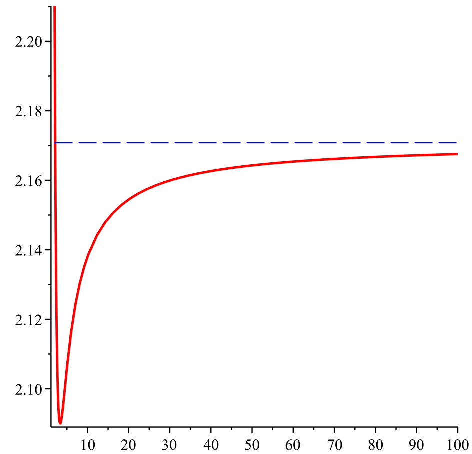

Figure 2: The function I and its limiting value on $[2, 100]$

Conjecture *I is increasing for p above the conjectured global minimum near 3.36 and concave for p above an inflection point near 4.469.*

This is shown in Figure 2 in which the dashed line has height $\sqrt{\frac{3\pi}{2}}$. Moreover, in Theorem 2 we shall prove

$$I(p) > \sqrt{\frac{3\pi}{2}}\frac{2p}{2p+1} > \sqrt{\frac{3\pi}{2}}\left(1 - \frac{1}{2p}\right), \tag{2}$$

for all $p > 1$.

We conclude this introduction by observing that one can derive the existence of an asymptotic expansion for $I(p)$ from a general result of Olver [10] on asymptotics of integrals using critical point theory and contour integration. Specialized to our case, [10, Theorem 7.1, p. 127] (with $q = 1$ and $p = \log(\sin(x)/x)$ on $[-\pi, \pi]$) establishes the existence of real constants c_s such that

$$\begin{aligned} I(p) &\sim \frac{1}{2}\sqrt{p}\int_{-\pi}^{\pi}\left|\frac{\sin(x)}{x}\right|^p dx \\ &\sim \sqrt{\frac{3\pi}{2}} - \frac{3}{20}\sqrt{\frac{3\pi}{2}}\frac{1}{p} + \sum_{s=2}^{\infty} c_s \frac{1}{p^s} + \cdots \end{aligned} \tag{3}$$

as $p \to \infty$. From this one may deduce that $I(p)$ is concave and increasing for sufficiently large values of p—consistent with our stronger conjecture—as (3) may be differentiated termwise.

2 Our Main Results

In order to study the properties of the function $I(p)$, we consider first the functions

$$\varphi_n(p) = \int_0^\infty \left(\log\left|\frac{\sin x}{x}\right|\right)^n \cdot \left|\frac{\sin x}{x}\right|^p dx$$

for $p > 1$ and n a nonnegative integer. We write

$$\varphi(p) = \varphi_0(p) = \int_0^\infty \left|\frac{\sin x}{x}\right|^p dx.$$

In Lemma 1 below we confirm that $\varphi(p)$ is analytic in a region containing $(1, \infty)$ and that its n-th derivative for $p > 1$ is given by $\varphi^{(n)}(p) = \varphi_n(p)$.

Then in Theorem 1 we shall use induction to prove the following result for n a nonnegative integer:

$$\lim_{p\to\infty} p^{n+\frac{1}{2}}\,\varphi^{(n)}(p) = (-1)^n\sqrt{\frac{3}{2}}\,\Gamma\left(n + \frac{1}{2}\right).$$

The base case, $n = 0$, for our induction is established in Lemma 2 below. It uses Laplace's method for determining asymptotic behavior of an integral for large values of a parameter p, see, e.g., [6, p. 60].

Lemma 1 *For $p - 1 > z > 1 - p$,*

$$\varphi(p - z) = \sum_{n=0}^{\infty} (-1)^n \varphi_n(p) \frac{z^n}{n!}.$$

In particular, $\varphi(p)$ is analytic in a region containing $(1, \infty)$ and its n-th derivative for $p > 1$ is given by

$$\varphi^{(n)}(p) = \varphi_n(p).$$

Proof. We have

$$\begin{aligned}
\varphi(p - z) &= \int_0^{\infty} \left|\frac{\sin x}{x}\right|^{p-z} dx = \int_0^{\infty} dx \sum_{n=0}^{\infty} \left(-\log\left|\frac{\sin x}{x}\right|\right)^n \cdot \left|\frac{\sin x}{x}\right|^p \frac{z^n}{n!} & (4)\\
&= \sum_{n=0}^{\infty} \int_0^{\infty} \left(-\log\left|\frac{\sin x}{x}\right|\right)^n \cdot \left|\frac{\sin x}{x}\right|^p \frac{z^n}{n!}\, dx = \sum_{n=0}^{\infty} (-1)^n \varphi_n(p) \frac{z^n}{n!}, & (5)
\end{aligned}$$

the inversion of sum and integral in (4) being justified as follows:
Case i. $p - 1 > z \geq 0$. All the terms involved are nonnegative.
Case ii. $0 > z > 1 - p$. By *Case i*

$$\varphi(p - |z|) = \int_0^{\infty} dx \sum_{n=0}^{\infty} \left(-\log\left|\frac{\sin x}{x}\right|\right)^n \cdot \left|\frac{\sin x}{x}\right|^p \frac{|z|^n}{n!} < \infty.$$

Thus (5) yields the Taylor series for $\varphi(p - z)$ at $z = 0$, and the final conclusion follows. □

Lemma 2

$$\lim_{p\to\infty} I(p) = \lim_{p\to\infty} \sqrt{p}\,\varphi(p) = \sqrt{\frac{3\pi}{2}}. \tag{6}$$

Proof. Let $a > 0$, then for $p > 1$ we have

$$I(p) = \sqrt{p} \int_0^{\infty} \left|\frac{\sin x}{x}\right|^p dx = \sqrt{p} \int_0^{a} \left|\frac{\sin x}{x}\right|^p dx + \sqrt{p} \int_a^{\infty} \left|\frac{\sin x}{x}\right|^p dx.$$

We show first that

$$\lim_{p\to\infty} \sqrt{p} \int_a^{\infty} \left|\frac{\sin x}{x}\right|^p dx = 0. \tag{7}$$

It suffices to consider the case $0 < a < 1$; since for $a \geq 1$, we have

$$\sqrt{p} \int_a^{\infty} \left|\frac{\sin x}{x}\right|^p dx \leq \lim_{b\to\infty} \sqrt{p} \int_a^{b} \frac{1}{x^p}\, dx = \frac{\sqrt{p}}{p-1} \cdot \frac{1}{a^{p-1}} \longrightarrow 0$$

as $p \to \infty$.

Now, for $a < x < 1$, we have

$$0 < \frac{\sin x}{x} < \frac{\sin a}{a} < 1,$$

and it follows that

$$\begin{aligned}
0 < \sqrt{p} \int_a^{\infty} \left|\frac{\sin x}{x}\right|^p dx &\leq \sqrt{p} \int_a^{1} \left|\frac{\sin x}{x}\right|^p dx + \sqrt{p} \int_1^{\infty} \left|\frac{\sin x}{x}\right|^p dx \\
&\leq (1 - a)\sqrt{p} \left|\frac{\sin a}{a}\right|^p + \frac{\sqrt{p}}{p-1} \longrightarrow 0
\end{aligned}$$

as $p \to \infty$. This establishes (7).

We next use the following easily proved results [9, 8]:

$$1 - \frac{x^2}{6} \le \frac{\sin x}{x} \le 1 - \frac{x^2}{6} + \frac{x^4}{120} \quad \text{for all real} \quad x, \tag{8}$$

and

$$\int_0^1 (1-u^2)^p \, du = \frac{\sqrt{\pi}}{2} \frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}. \tag{9}$$

where the equality is a special case of a beta-function evaluation (see also [12, Theorem 7.69]). It follows from (8) and (9) that

$$\int_0^{\sqrt{6}} \left|\frac{\sin x}{x}\right|^p dx \ge \sqrt{6} \int_0^1 (1-u^2)^p \, du = \sqrt{\frac{3\pi}{2}} \frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}, \tag{10}$$

and hence that

$$\liminf_{p\to\infty} I(p) \ge \lim_{p\to\infty} \sqrt{\frac{3\pi}{2}} \frac{\sqrt{p}\,\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}. \tag{11}$$

Now, in order to get an appropriate inequality for the limsup, we note that for any $w > 1$ such that

$$W = 2\sqrt{5}\sqrt{\left(1 - \frac{1}{w}\right)} \le \sqrt{6},$$

we have

$$\frac{\sin x}{x} \le 1 - \frac{x^2}{6w} \quad \text{for} \quad 0 < x < W, \tag{12}$$

whence

$$\int_0^W \left|\frac{\sin x}{x}\right|^p dx \le \sqrt{6w} \int_0^{\frac{W}{\sqrt{6w}}} (1-u^2)^p \, du \le \sqrt{6w} \int_0^1 (1-u^2)^p \, du = \sqrt{\frac{3\pi w}{2}} \frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}. \tag{13}$$

It follows from (7) and (13) that

$$\limsup_{p\to\infty} I(p) \le \lim_{p\to\infty} \sqrt{\frac{3\pi w}{2}} \frac{\sqrt{p}\,\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}, \tag{14}$$

and therefore from (11) and (14), for $w > 1$ we have

$$\lim_{p\to\infty} \sqrt{\frac{3\pi}{2}} \frac{\sqrt{p}\,\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)} \le \liminf_{p\to\infty} I(p) \le \limsup_{p\to\infty} I(p) \le \lim_{p\to\infty} \sqrt{\frac{3\pi w}{2}} \frac{\sqrt{p}\,\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}. \tag{15}$$

Letting $p \to \infty$ in (15), since for any $a > 0$, we have

$$\lim_{a\to\infty} \frac{\sqrt{a}\,\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a+1)} = 1,$$

from [8, Problem 2, p. 45] or (23), we obtain

$$\sqrt{\frac{3\pi}{2}} \le \liminf_{p\to\infty} I(p) \le \limsup_{p\to\infty} I(p) \le \sqrt{\frac{3\pi w}{2}}, \tag{16}$$

for all $w > 1$. Finally, letting $w \to 1^+$, we get the desired equation (6). $\square$

We are now ready for our more general result.

Theorem 1 *For all natural numbers n we have*

$$\begin{aligned}\lim_{p\to\infty} p^{n+\frac{1}{2}}\,\varphi^{(n)}(p) &= \lim_{p\to\infty} p^{n+\frac{1}{2}} \int_0^\infty \left(\log\left|\frac{\sin x}{x}\right|\right)^n \cdot \left|\frac{\sin x}{x}\right|^p dx \\ &= (-1)^n\sqrt{\frac{3}{2}}\,\Gamma\left(n+\frac{1}{2}\right). \end{aligned} \tag{17}$$

Proof. The first equality was noted above. We proceed to establish equation (17) by induction. The proof of the base case was given in Lemma 1.

For the inductive step of the proof, we assume that for a given nonnegative integer n, we have

$$\lim_{p\to\infty} p^{n+\frac{1}{2}}\,\varphi^{(n)}(p) = (-1)^n\sqrt{\frac{3}{2}}\,\Gamma\left(n+\frac{1}{2}\right).$$

It is easily verified that $x < -\log(1-x) < \dfrac{x}{1-x}$ for $0<x<1,$ and setting $x = 1-\left|\dfrac{\sin t}{t}\right|^p$, that

$$1-\left|\frac{\sin t}{t}\right|^p < -\log\left|\frac{\sin t}{t}\right|^p < \frac{1-\left|\frac{\sin t}{t}\right|^p}{\left|\frac{\sin t}{t}\right|^p}$$

for all but countably many values of t.

For $q>p+1$, multiplying these inequalities by the nonnegative term

$$(-1)^n\left(\log\left|\frac{\sin t}{t}\right|\right)^n \cdot \left|\frac{\sin t}{t}\right|^q,$$

we have

$$\begin{aligned} 0 \le (-1)^n\left(\log\left|\frac{\sin t}{t}\right|\right)^n\left(\left|\frac{\sin t}{t}\right|^q - \left|\frac{\sin t}{t}\right|^{p+q}\right) &< -(-1)^n p\left(\log\left|\frac{\sin t}{t}\right|\right)^{n+1} \cdot \left|\frac{\sin t}{t}\right|^q \\ &< (-1)^n\left(\log\left|\frac{\sin t}{t}\right|\right)^n\left(\left|\frac{\sin t}{t}\right|^{q-p} - \left|\frac{\sin t}{t}\right|^q\right) \end{aligned}$$

for the same values of t, and integrating over $(0,\infty)$ yields

$$(-1)^n\left(\varphi^{(n)}(q)-\varphi^{(n)}(p+q)\right) \le -(-1)^n p\,\varphi^{(n+1)}(q) \le (-1)^n\left(\varphi^{(n)}(q-p)-\varphi^{(n)}(q)\right),$$

and hence

$$\begin{aligned} (-1)^n\left(\frac{q^{n+\frac{1}{2}}\varphi^{(n)}(q)}{p\,q^{n+\frac{1}{2}}} - \frac{(p+q)^{n+\frac{1}{2}}\varphi^{(n)}(p+q)}{p\,(p+q)^{n+\frac{1}{2}}}\right) &\le -(-1)^n\frac{q^{n+1+\frac{1}{2}}\varphi^{(n+1)}(q)}{q^{n+1+\frac{1}{2}}} \\ \le (-1)^n\left(\frac{(q-p)^{n+\frac{1}{2}}\varphi^{(n)}(q-p)}{p\,(q-p)^{n+\frac{1}{2}}} - \frac{q^{n+\frac{1}{2}}\varphi^{(n)}(q)}{p\,q^{n+\frac{1}{2}}}\right). \end{aligned} \tag{18}$$

Now let $q=kp$, where $k>2$ is fixed, then (18) becomes

$$\begin{aligned} (-1)^n\left(k\,q^{n+\frac{1}{2}}\varphi^{(n)}(q) - \frac{k\,(p+q)^{n+\frac{1}{2}}\varphi^{(n)}(p+q)}{(1+\frac{1}{k})^{n+\frac{1}{2}}}\right) &\le -(-1)^n q^{n+1+\frac{1}{2}}\varphi^{(n+1)}(q) \\ &\le (-1)^n\left(k\frac{(q-p)^{n+\frac{1}{2}}\varphi^{(n)}(q-p)}{(1-\frac{1}{k})^{n+\frac{1}{2}}} - k\,q^{n+\frac{1}{2}}\varphi^{(n)}(q)\right). \end{aligned}$$

Next let $q \to \infty$, keeping $k > 2$ fixed, so that $p \to \infty$ and $q - p = (k-1)p \to \infty$. It follows from the inductive hypothesis that

$$\lim_{q\to\infty} q^{n+\frac{1}{2}}\varphi^{(n)}(q) = \lim_{q\to\infty}(p+q)^{n+\frac{1}{2}}\varphi^{(n)}(p+q) = \lim_{q\to\infty}(q-p)^{n+\frac{1}{2}}\varphi^{(n)}(q-p) = (-1)^n\sqrt{\frac{3}{2}}\,\Gamma\left(n+\frac{1}{2}\right),$$

and therefore

$$\begin{aligned}\left(k - \frac{k}{(1+\frac{1}{k})^{n+\frac{1}{2}}}\right)\sqrt{\frac{3}{2}}\,\Gamma\left(n+\frac{1}{2}\right) &\le \liminf_{q\to\infty}(-1)^{n+1}q^{n+1+\frac{1}{2}}\varphi^{(n+1)}(q) \le \limsup_{q\to\infty}(-1)^{n+1}q^{n+1+\frac{1}{2}}\varphi^{(n+1)}(q) \\ &\le \left(\frac{k}{(1-\frac{1}{k})^{n+\frac{1}{2}}} - k\right)\sqrt{\frac{3}{2}}\,\Gamma\left(n+\frac{1}{2}\right). \qquad (19)\end{aligned}$$

Since

$$\lim_{k\to\infty}\left(k - \frac{k}{(1+\frac{1}{k})^{n+\frac{1}{2}}}\right) = \lim_{k\to\infty}\left(\frac{k}{(1-\frac{1}{k})^{n+\frac{1}{2}}} - k\right) = \lim_{t\to 0}\frac{1-(1+t)^{-n-\frac{1}{2}}}{t} = n + \frac{1}{2},$$

it follows from (19) that

$$\lim_{q\to\infty}(-1)^{n+1}q^{n+1+\frac{1}{2}}\varphi^{(n+1)}(q) = \sqrt{\frac{3}{2}}\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\frac{3}{2}}\,\Gamma\left(n+1+\frac{1}{2}\right),$$

and this completes the proof of the inductive step. □

3 Final Remarks

Our proof of Theorem 1 shows both that

$$\lim_{p\to\infty} p^{n+\frac{1}{2}}\varphi^{(n)}(p) = a_n \qquad (20)$$

exists and determines the value of a_n. If we know in advance that the limit exists for every nonnegative integer n, then we can use Lemmas 1 and 2 to write

$$\lim_{p\to\infty}\sqrt{p}\,\varphi(p(1+x)) = \lim_{p\to\infty}\sum_{n=0}^{\infty} p^{n+\frac{1}{2}}\varphi^{(n)}(p)\frac{x^n}{n!} = \frac{\sqrt{3\pi/2}}{\sqrt{1+x}}$$

for $1 - \frac{1}{p} > x > \frac{1}{p} - 1$, and then justify the exchange of limit and sum, and expand the final term to obtain

$$\sum_{n=0}^{\infty} a_n\frac{x^n}{n!} = \sum_{n=0}^{\infty}\sqrt{\frac{3}{2}}(-1)^n\Gamma\left(n+\tfrac{1}{2}\right)\frac{x^n}{n!}.$$

Comparing coefficients of the above two exponential generating functions yields the desired valuation

$$a_n = \sqrt{\frac{3}{2}}(-1)^n\Gamma\left(n+\tfrac{1}{2}\right). \qquad (21)$$

In fact, to justify the exchange by means of the series version of Lebesgue's theorem on dominated convergence one needs to establish something like

$$\left|\frac{p^{n+\frac{1}{2}}\varphi^{(n)}(p)}{n!}\right| \le M$$

with M a positive constant independent of n and p, and this requires an inequality such as the right-hand side of (19) (with q replaced by p and n by $n-1$) used in the given proof of Theorem 1.

Another way of determining the value of a_n in (20) if we know it exists for every n, is to proceed via L'Hospital's rule as follows:

$$a_{n-1} = \lim_{p\to\infty} \frac{\varphi^{(n-1)}(p)}{p^{-n+\frac{1}{2}}} = \lim_{p\to\infty} \frac{\varphi^{(n)}(p)}{-(n-\frac{1}{2})p^{-n-\frac{1}{2}}} = -\frac{a_n}{n-\frac{1}{2}},$$

whence, by Lemma 2,

$$a_n = (-1)^n a_0 \prod_{k=1}^{n} \left(k - \frac{1}{2}\right) = \sqrt{\frac{3}{2}}(-1)^n \Gamma\left(n+\tfrac{1}{2}\right)$$

which is (20) again.

One advantage of our explicit proof of Lemma 2 over Olver's asymptotic result in (3) is that it is easily exploited to establish (2).

Theorem 2 *For all $p > 1$ we have*

$$I(p) > \sqrt{\frac{3\pi}{2}}\frac{2p}{2p+1} > \sqrt{\frac{3\pi}{2}}\left(1 - \frac{1}{2p}\right). \tag{22}$$

Proof. For $x > 0$ and $0 < s < 1$, Abromowitz and Stegun [1] records (as (5.6.4) in the new web version) that

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}. \tag{23}$$

Hence, from (10) and (23) we obtain for $p > 1$ that

$$\begin{aligned} I(p) &> \sqrt{p}\sqrt{\frac{3\pi}{2}}\frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)} = \sqrt{\frac{3\pi}{2}}\frac{\sqrt{p}}{2p+1}\frac{\Gamma(p+1)}{\Gamma\left(p+\frac{1}{2}\right)} \\ &> \sqrt{\frac{3\pi}{2}}\frac{2p}{2p+1} > \sqrt{\frac{3\pi}{2}}\left(1-\frac{1}{2p}\right). \end{aligned}$$

Here, for the penultimate inequality, we have used the left-hand inequality in (23) with $x = p, s = 1/2$. □

Note that (22) implies that

$$\|\mathrm{sinc}\|_p > \left(\frac{2\sqrt{6p\,\pi}}{2p+1}\right)^{1/p}$$

when sinc is viewed as a function in $L_p([-\infty,\infty])$. We finish by observing that the lower bound is asymptotically of the correct order, and leave as an open question whether similar explicit techniques to those in Theorem 1 can be used to establish the second-order term in the asymptotic expansion (3) or the concavity properties conjectured in the introduction.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, NBS (now NIST), 1965. See also `http://dlmf.nist.gov/`.

[2] D. Borwein and J. M. Borwein, "Some remarkable properties of sinc and related integrals," *Ramanujan J.,* **5** (2001), 73–90.

[3] D. Borwein, J. M. Borwein, and B. Mares, "Multi-variable sinc integrals and volumes of polyhedra," *Ramanujan J.,* **6** (2002), 189–208.

[4] T. J. Bromwich, *Theory of Infinite Series,* First Edition 1908, Second Edition 1926, Blackie & Sons, Glasgow.

[5] H. S. Carslaw, *An Introduction to the Theory of Fourier's Series and Integrals*, Third Revised Edition, Dover Publications Inc., New Jersey, 1950.

[6] N. G. de Bruijn, *Asymptotic Methods in Analysis*, Second Edition, North-Holland Publishing Co., Amsterdam, 1961.

[7] W. B. Gearhart and H. S. Schultz, "The function $\frac{\sin(x)}{x}$," *The College Mathematics Journal,* **2** (1990), 90–99.

[8] P. Henrici, *Applied and Computational Complex Analysis Volume 2,* John Wiley & Sons, Inc., New York, 1977.

[9] I. E. Leonard and James Duemmel, "More–and Moore–Power Series without Taylor's Theorem," *The American Mathematical Monthly,* **92** (1985), 588–589.

[10] F. W. J. Olver, *Asymptotics and Special Functions* (AKP Classics), Second Edition, AK Peters, Nattick, Mass, 1997.

[11] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer Series in Computational Mathematics **20**, Springer–Verlag, New York, 1993.

[12] K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

Acknowledgements We wish to thank Amram Meir and David Bailey for very useful discussions during the preparation of this note.

David Borwein obtained two B.Sc. degrees from Witwatersrand University, one in engineering in 1945 and the other in mathematics in 1948. From University College London (UK) he received a Ph.D. in 1950 and a D.Sc. in 1960. He has been at the University of Western Ontario since 1963 with an emeritus title since 1989. His main area of research has been classical analysis, particularly summability theory.
Department of Mathematics, The University of Western Ontario, London, ONT, N6A 5B7, Canada. Email: dborwein@uwo.ca

Jonathan M. Borwein received his B.A. from the University of Western Ontario in 1971 and a D.Phil. from Oxford in 1974, both in mathematics. He currently holds a Laureate Professorship at University of Newcastle and a Canada Research Chair in the Faculty of Computer Science at Dalhousie University. His primary current research interests are in nonlinear functional analysis, optimization, and experimental (computationally-assisted) mathematics.
School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia and Faculty of Computer Science and Department of Mathematics, Dalhousie University, Halifax NS, B3H 2W5, Canada. Email: jonathan.borwein@newcastle.edu.au

Isaac E. Leonard received his B.A. and M.A. from the University of Pennsylvania in 1961 and 1963, both in physics. He received his M.Sc. and Ph.D. from Carnegie-Mellon University in 1969 and 1973, both in mathematics; and a B.Sc. in Computer Science from the University of Alberta in 1994. He currently holds a position as a Sessional Lecturer with the Department of Mathematical and Statistical Sciences at the University of Alberta, and has taught courses in the Computing Science and Electrical and Computer Engineering Departments at the University of Alberta. His current research interests are in analysis of algorithms and numerical analysis.
Department of Mathematical and Statistical Sciences, The University of Alberta, Edmonton, AB, T6G 2G1, Canada Email: isaac@math.ualberta.ca