Invariant Differential Operators on Siegel-Jacobi Space and Maass-Jacobi Forms

Jae-Hyun Yang

Inha University
jhyang@inha.ac.kr

NOAH'S On the Beach
Newcastle, Australia
March 15 (Thursday), 2012

Introduction

Let

$$\mathbf{H}_n = \left\{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0 \right\}$$

be the Siegel upper half plane and let

$$\mathbf{H}_{n,m} = \mathbf{H}_n \times \mathbb{C}^{(m,n)}$$

be the Siegel-Jacobi space.

Notations : Here $F^{(m,n)}$ denotes the set of all $m \times n$ matrices with entries in a commutative ring F and ${}^tA$ denotes the transpose of a matrix A. For an $n \times m$ matrix B and an $n \times n$ matrix A, we write $A[B] = {}^tBAB$.

Let

$$Sp(n,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \right\}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $Sp(n,\mathbb{R})$ acts on $\mathbf{H}_n$ transitively by

$$M \circ \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \qquad (1)$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbf{H}_n$. Therefore

$$Sp(n,\mathbb{R})/U(n) \cong \mathbf{H}_n$$

is a (Hermitian) symmetric space.

Let

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda, \mu, \kappa) \,\Big|\, \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

be the Heisenberg group. Let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

be the **Jacobi group** with the multiplication law

$$(M_0, (\lambda_0, \mu_0, \kappa_0)) \cdot (M, (\lambda, \mu, \kappa))$$
$$= \Big(M_0 M, \big(\tilde{\lambda}_0 + \lambda, \tilde{\mu}_0 + \mu, \kappa_0 + \kappa + \tilde{\lambda}_0 {}^t\mu - \tilde{\mu}_0 {}^t\lambda \big) \Big),$$

where $(\tilde{\lambda}_0, \tilde{\mu}_0) = (\lambda_0, \mu_0)M$. Then G^J acts on the **Siegel-Jacobi space** $\mathbf{H}_{n,m}$ transitively by

$$\Big(M, (\lambda, \mu, \kappa)\Big) \cdot (\Omega, Z) \qquad (2)$$
$$= \Big(M \circ \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\Big),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),\ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbf{H}_{n,m}$. Thus

$$G^J / K^J \cong \mathbf{H}_{n,m}$$

is a **non-reductive** complex manifold, where

$$K^J = U(n) \times \mathrm{Sym}(n, \mathbb{R}).$$

Let Γ_* be an arithmetic subgroup of $Sp(n, \mathbb{R})$ and $\Gamma_*^J = \Gamma_* \ltimes H_{\mathbb{Z}}^{(n,m)}$. For instance, $\Gamma_* = Sp(n, \mathbb{Z})$. Here

$$H_{\mathbb{Z}}^{(n,m)} = \Big\{ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)} \,\Big|\, \lambda, \mu, \kappa \text{ integral } \Big\}.$$

We have the following **natural problems** :

Problem I : Find the spectral decomposition of

$$L^2(\Gamma_*^J \backslash \mathbf{H}_{n,m})$$

for the Laplacian $\Delta_{n,m}$ on $\mathbf{H}_{n,m}$ or a commuting set $\mathbb{D}_*$ of G^J-invariant differential operators on $\mathbf{H}_{n,m}$.

Problem II : Decompose the regular representation $R_{\Gamma_*^J}$ of G^J on $L^2(\Gamma_*^J \backslash G^J)$ into irreducibles.

The above problems are very important arithmetically and geometrically. However the above problems are very **difficult** to solve at this moment. One of the reason is that it is difficult to deal with Γ_*. Unfortunately the unitary dual of $Sp(n,\mathbb{R})$ is not known yet for $n \geq 3$.

For a coordinate $(\Omega, Z) \in \mathbf{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_n$ and $Z = (z_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
\Omega =& X + iY, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \text{ real}, \\
Z =& U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\
d\Omega =& (d\omega_{\mu\nu}), \quad d\overline{\Omega} = (d\overline{\omega}_{\mu\nu}), \\
dZ =& (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}),
\end{aligned}$$

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{\omega}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial X} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial x_{\mu\nu}} \right), \quad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial U} = \begin{pmatrix} \frac{\partial}{\partial u_{11}} & \cdots & \frac{\partial}{\partial u_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial u_{1n}} & \cdots & \frac{\partial}{\partial u_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial V} = \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \cdots & \frac{\partial}{\partial v_{m1}} \\ \vdots & \ddots & \cdots \\ \frac{\partial}{\partial v_{1n}} & \cdots & \frac{\partial}{\partial v_{mn}} \end{pmatrix}.$$

1. Invariant metrics on $\mathbf{H}_{n,m}$

We recall that for a positive real number A, the metric

$$\boxed{ds^2_{n;A} = A \cdot \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big)}$$

is a $Sp(n,\mathbb{R})$-invariant Kähler metric on $\mathbf{H}_n$ introduced by C. L. Siegel (cf. [8], 1943).

Theorem 1 (J.-H. Yang [16], 2005). For any two positive real numbers A and B, the following metric

$$\begin{aligned} & ds^2_{n,m;A,B} \\ = \; & A \cdot \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\ & \quad + B \cdot \Big\{ \mathrm{tr}\Big(Y^{-1}\, {}^tV\, V\, Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\ & \qquad + \mathrm{tr}\Big(Y^{-1}\, {}^t(dZ)\, d\overline{Z}\Big) \\ & \qquad - \mathrm{tr}\Big(V\, Y^{-1} d\Omega\, Y^{-1}\, {}^t(d\overline{Z})\Big) \\ & \qquad - \mathrm{tr}\Big(V\, Y^{-1} d\overline{\Omega}\, Y^{-1}\, {}^t(dZ)\Big) \Big\} \end{aligned}$$

is a Riemannian metric on $\mathbf{H}_{n,m}$ which is invariant under the action (2) of G^J.

For the case $n = m = A = B = 1$, we get

$$\begin{aligned} & ds^2_{1,1;1,1} \\ = \ & \frac{y + v^2}{y^3}\left(dx^2 + dy^2 \right) + \frac{1}{y}\left(du^2 + dv^2 \right) \\ & -\frac{2v}{y^2}\left(dx\,du + dy\,dv \right). \end{aligned}$$

Lemma A. The following differential form

$$dv_{n,m} = \frac{[dX] \wedge [dY] \wedge [dU] \wedge [dV]}{(\det Y)^{n+m+1}}$$

is a G^J-invariant volume element on $\mathbf{H}_{n,m}$, where

$$[dX] = \wedge_{\mu\le\nu}\, dx_{\mu\nu}, \quad [dY] = \wedge_{\mu\le\nu}\, dy_{\mu\nu},$$
$$[dU] = \wedge_{k,l}\, du_{kl}, \quad [dV] = \wedge_{k,l}\, dv_{kl}.$$

Proof. The proof follows from the fact that

$$(\det Y)^{-(n+1)}[dX] \wedge [dY]$$

is a $Sp(n,\mathbb{R})$-invariant volume element on $\mathbf{H}_n$. (cf. [9]) $\square$

2. Laplacians on $\mathbb{H}_{n,m}$

Hans Maass(cf. [3], 1953) proved that for a positive real number A, the differential operator

$$\Delta_n = \frac{4}{A} \cdot \mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega} \right)$$

is the Laplacian of $\mathbb{H}_n$ for the metric $ds^2_{n;A}$.

[3] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **26** (1953), 44–68.

Theorem 2 (J.-H. Yang [16], 2005). For any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $ds^2_{n,m;A,B}$ is given by

$$\begin{aligned} & \Delta_{n,m;A,B} \\ = \ & \frac{4}{A}\left\{ \mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega} \right) \right. \\ & \qquad + \mathrm{tr}\left(V Y^{-1}\, {}^tV\, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}}\right) \frac{\partial}{\partial Z} \right) \\ & \qquad + \mathrm{tr}\left(V\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial Z} \right) \\ & \qquad \left. + \mathrm{tr}\left({}^tV\, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}}\right) \frac{\partial}{\partial \Omega} \right) \right\} \\ & + \frac{4}{B}\, \mathrm{tr}\left(Y \frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) \right). \end{aligned}$$

For the case $n = m = A = B = 1$, we get

$$\begin{aligned}\Delta_{1,1;1,1} &= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ &\quad + (\, y + v^2\,) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ &\quad + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right).\end{aligned}$$

Remark : $ds^2_{n,m;A,B}$ and $\Delta_{n,m;A,B}$ are expressed in terms of the **trace form**. !!!

3. Invariant differential operators on $\mathbb{H}_{n,m}$

Let $T_{n,m} = T_n \times \mathbb{C}^{(m,n)}$. Then the adjoint action of K^J on the Lie algebra $\mathfrak{g}^J$ of G^J induces the natural action of K on $T_{n,m}$ defined by

$$\boxed{h \cdot (\omega, z) = \left(h\,\omega\,{}^t h,\, z\,{}^t h\right),} \qquad (3)$$

where $h \in K,\ \omega \in T_n,\ z \in \mathbb{C}^{(m,n)}$. Then this action induces naturally the action ρ of K on the polynomial algebra

$$\mathrm{Pol}_{m,n} = \mathrm{Pol}\,(T_{n,m}).$$

We denote by $\mathrm{Pol}_{m,n}^K$ the subalgebra of $\mathrm{Pol}_{m,n}$ consisting of all K-invariants of the action ρ of K.

We also denote by

$$\mathbb{D}(\mathbf{H}_{n,m})$$

the algebra of all differential operators on $\mathbf{H}_{n,m}$ which is invariant under the action (2) of the Jacobi group G^J. Then we can show that there exists a natural linear bijection

$$\Theta_{n,m} : \mathrm{Pol}_{m,n}^K \longrightarrow \mathbb{D}(\mathbf{H}_{n,m}) \qquad (4)$$

of $\mathrm{Pol}_{m,n}^K$ onto $\mathbb{D}(\mathbf{H}_{n,m})$.

The map $\mathfrak{S}_{n,m}$ is described explicitly as follows.

We put $N_\star = n(n+1) + 2mn$. Let $\left\{\eta_\alpha \,|\, 1 \leq \alpha \leq N_\star\right\}$ be a basis of $T_{n,m}$. If $P \in \mathrm{Pol}_{m,n}^K$, then

$$\left(\Theta_{n,m}(P)f\right)(gK)$$
$$= \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha\right) K\right)\right]_{(t_\alpha)=0},$$

We propose the following natural problems.

Problem 1. Find the generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 2. Find all the relations among a given set of generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 3. Find an efficient way to express the images of the above invariant polynomials under the Helgason map $\Theta_{n,m}$ explicitly.

Problem 4. Decompose $\mathrm{Pol}_{n,m}$ into K-irreducibles.

Problem 5. Construct explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$. Or construct explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$.

Problem 6. Find all the relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 7. Is $\mathrm{Pol}_{n,m}^{U(n)}$ finitely generated ? Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated ?

For a coordinate (ω, z) in $T_{1,1}$, we write $\omega = x + i\,y,\ z = u + i\,v,\ x, y, u, v$ real.

Theorem 3 (J.-H. Yang, 2003). The algebra $\text{Pol}_{1,1}^{U(1)}$ is generated by

$$\begin{aligned} f_1(\omega, z) &= \frac{1}{4}\omega\,\overline{\omega} = \frac{1}{4}\Big(x^2 + y^2\Big), \\ f_2(\omega, z) &= z\,\overline{z} = u^2 + v^2, \\ f_3(\omega, z) &= \frac{1}{2}\,\text{Re}\left(z^2\overline{\omega}\right) = \frac{1}{2}\left(u^2 - v^2\right)x + uvy, \\ f_4(\omega, z) &= \frac{1}{2}\,\text{Im}\,(z^2\overline{\omega}) = \frac{1}{2}\left(v^2 - u^2\right)y + uvx. \end{aligned}$$

Using Formula (4) the author calculated explicitly the images

$$D_1 = \Theta_{1,1}(f_1), \quad D_2 = \Theta_{1,1}(f_2),$$

$$D_3 = \Theta_{1,1}(f_3), \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi)$$

of f_1, f_2, f_3 and f_4 under the Helgason map $\Theta_{1,1}$.

Theorem 4 (J.-H. Yang, 2003). The algebra $\mathbb{D}(\mathbf{H}_1 \times \mathbb{C})$ is generated by the following differential operators

$$D_1 = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),$$

$$D_2 = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$D_3 = 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} - y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \left(v \frac{\partial}{\partial v} + 1 \right) D_2$$

and

$$D_4 = y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} D_2,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$D_1 D_2 - D_2 D_1 = 2\,y^2 \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) - 4\,y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2\left(v \frac{\partial}{\partial v} D_2 + D_2\right) = 2D_3.$$

In particular, the algebra $\mathbb{D}(\mathbf{H}_1 \times \mathbb{C})$ is not commutative.

Hiroyuki Ochiai (2011) proved the following results.

Proposition 5 (H. Ochiai, 2011). We have the following relation

$$f_3^2 + f_4^2 = f_1 f_2^2 \tag{5}$$

This relation exhausts all the relations among the generators f_1, f_2, f_3 and f_4 of $\mathbb{D}(\mathbb{H}_{1,1})$. I generalized this fact to the case $n = 1$ and $m \geq 1$.

Proposition 6 (H. Ochiai, 2011). The action of $U(n)$ on $\mathrm{Pol}_{1,1}$ is *not* multiplicity-free.

Theorem 7 (H. Ochiai, 2011). We have the following relations

$$(a)\quad [D_1, D_2] = 2D_3$$

$$(b)\quad [D_1, D_3] = 2D_1D_2 - 2D_3$$

$$(c)\quad [D_2, D_3] = -D_2^2$$

$$(d)\quad [D_4, D_i] = 0, \quad i = 1, 2, 3$$

$$(e)\quad D_3^2 + D_4^2 = D_2D_1D_2$$

These five relations exhaust all the relations among D_1, D_2, D_3 and D_4.

We present the following *basic* K-invariant polynomials in $\mathrm{Pol}_{n,m}^{U(n)}$.

$$
\begin{aligned}
q_j(\omega,z) &= \mathrm{tr}((\omega\,\overline{\omega})^{j+1}), \quad 0 \le j \le n-1,\\
\alpha_{kp}^{(j)}(\omega,z) &= (z\,(\overline{\omega}\omega)^{j}\,{}^t\overline{z})_{kp}, \quad 0 \le j \le n-1,\\
&\qquad\qquad 1 \le k \le p \le m,\\
\beta_{lq}^{(j)}(\omega,z) &= (z\,(\overline{\omega}\omega)^{j}\,{}^t\overline{z})_{lq}, \quad 0 \le j \le n-1,\\
&\qquad\qquad 1 \le l < q \le m,\\
f_{kp}^{(j)}(\omega,z) &= \mathrm{Re}\,(z\,(\overline{\omega}\omega)^{j}\,\overline{\omega}\,{}^t z)_{kp}, \quad 0 \le j \le n-1,\\
&\qquad\qquad 1 \le k \le p \le m,\\
g_{kp}^{(j)}(\omega,z) &= \mathrm{Im}\,(z\,(\overline{\omega}\omega)^{j}\,\overline{\omega}\,{}^t z)_{kp}, \quad 0 \le j \le n-1,\\
&\qquad\qquad 1 \le k \le p \le m,
\end{aligned}
$$

where $\omega \in T_n$ and $z \in \mathbb{C}^{(m,n)}$.

Minoru Itoh (2012) solved Problem 1 and Problem 2.

Theorem 8 (M. Itoh, 2012). $\mathrm{Pol}_{n,m}^{U(n)}$ is generated by the above polynomials.

4. Examples of Explicit Invariant Differential Operators

$$\mathbb{D} = \mathrm{tr}\left(Y \, \frac{\partial}{\partial Z} \, {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) \right)$$

$$\begin{aligned} \mathbb{M} \;=\; & \mathrm{tr}\left(Y \, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega} \right) \\ & + \mathrm{tr}\left(V Y^{-1} \, {}^tV \, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}}\right) \, \frac{\partial}{\partial Z} \right) \\ & + \mathrm{tr}\left(V \, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial Z} \right) \\ & + \mathrm{tr}\left({}^tV \, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}}\right) \frac{\partial}{\partial \Omega} \right) \end{aligned}$$

$$\mathbb{K} = \det(Y) \, \det\left(\frac{\partial}{\partial Z} \, {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) \right)$$

$$\mathbb{T} = {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) Y \frac{\partial}{\partial Z}$$

$$\mathbb{T}_{kl} = \sum_{i,j=1}^{n} y_{ij} \frac{\partial^2}{\partial \overline{z}_{ki} \partial z_{lj}}, \quad 1 \leq k, l \leq m$$

In the case $n = 1$ and $m = 1$, I gave explicit invariant differential operators of degree 3.

5. Maass-Jacobi Forms

Definition Let

$$\Gamma_{n,m} := Sp(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral } \right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) $\quad f$ is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$.

(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X+iY,Z)| \leq C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

It is natural to propose the following problems.

Problem A : Construct Maass-Jacobi forms. Study the behaviour of the eigenvalues of Maass-Jacobi forms.

Problem B : Find all the eigenfunctions of $\Delta_{n,m;A,B}$ and their eigenvalues.

6. Partial Cayley transform

Let

$$\mathbf{D}_n = \left\{ W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - W\overline{W} > 0 \right\}$$

be the generalized unit disk of degree n. We let

$$\mathbf{D}_{n,m} = \mathbf{D}_n \times \mathbb{C}^{(m,n)}$$

be the Siegel-Jacobi disk.

We define the **partial Cayley transform**

$$\Phi_* : \mathbf{D}_{n,m} \longrightarrow \mathbf{H}_{n,m}$$

by

$$\Phi_*(W, \eta) = \tag{6}$$

$$\boxed{\left(i(I_n + W)(I_n - W)^{-1},\, 2\, i\, \eta\, (I_n - W)^{-1} \right),}$$

where $W \in \mathbf{D}_n$ and $\eta \in \mathbb{C}^{(m,n)}$. It is easy to see that Φ_* is a biholomorphic mapping.

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.$$

We now consider the group G_*^J defined by

$$G_*^J = T_*^{-1} G^J T_*.$$

Then G_*^J acts on $\mathbf{D}_{n,m}$ transitively by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu, \kappa) \right) \cdot (W, \eta) = \qquad (7)$$

$$\Big((PW+Q)(\overline{Q}W+\overline{P})^{-1}, (\eta+\lambda W+\mu)(\overline{Q}W+\overline{P})^{-1} \Big).$$

Theorem 9 (J.-H. Yang [17], 2005). The action (2) of G^J on $\mathbf{H}_{n,m}$ is compatible with the action (12) of G_*^J on $\mathbf{D}_{n,m}$ through the partial Cayley transform Φ_*. More precisely, if $g_0 \in G^J$ and $(W, \eta) \in \mathbf{D}_{n,m}$,

$$g_0 \cdot \Phi_*(W, \eta) = \Phi_*\Big(g_* \cdot (W, \eta) \Big),$$

where $g_* = T_*^{-1} g_0 T_*$.

7. Invariant Differential Operators on $\mathbb{D}_{n,m}$

For a coordinate $(W,\eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned} dW &= (dw_{\mu\nu}), \qquad d\overline{W} = (d\overline{w}_{\mu\nu}), \\ d\eta &= (d\eta_{kl}), \qquad d\overline{\eta} = (d\overline{\eta}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial w_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{w}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial \overline{\eta}} = \left(\frac{\partial}{\partial \overline{\eta}_{kl}} \right).$$

Theorem 10 (J.-H. Yang [18], 2005). The following metric $d\tilde{s}^2_{n,m;A,B}$ defined by

$$
\begin{aligned}
&\frac{1}{4}\, d\tilde{s}^2_{n,m;A,B} = \\
&A\,\mathrm{tr}\Big((I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W}\Big) \\
+\ & B \Big\{ \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t(d\eta)\, d\overline{\eta}\,\Big) \\
+\ & \mathrm{tr}\Big((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1} dW \\
&\quad (I_n - \overline{W}W)^{-1}\, {}^t(d\overline{\eta})\Big) \\
+\ & \mathrm{tr}\Big((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1} d\overline{W} \\
&\quad (I_n - W\overline{W})^{-1}\, {}^t(d\eta)\,\Big) \\
-\ & \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \eta\, (I_n - \overline{W}W)^{-1} \\
&\quad \overline{W} dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \\
-\ & \mathrm{tr}\Big(W (I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, \overline{\eta}\, (I_n - W\overline{W})^{-1} \\
&\quad dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \\
+\ & \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \overline{\eta}\, (I_n - W\overline{W})^{-1} \\
&\quad dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \\
+\ & \mathrm{tr}\Big((I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\, \eta\, \overline{W}\, (I_n - W\overline{W})^{-1} \\
&\quad dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \\
+\ & \mathrm{tr}\Big((I_n - \overline{W})^{-1} (I_n - W)(I_n - \overline{W}W)^{-1} \\
&\quad {}^t\overline{\eta}\, \eta\, (I_n - \overline{W}W)^{-1}\, (I_n - \overline{W})(I_n - W)^{-1} \\
&\quad dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \Big\}
\end{aligned}
$$

$$- \; B\,\mathrm{tr}\Big((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}$$
$$\,{}^t\overline{\eta}\,\eta\,(I_n - W)^{-1}\,dW(I_n - \overline{W}W)^{-1}d\overline{W}\,\Big)$$

is a Riemannian metric on $\mathbf{D}_{n,m}$ which is invariant under the action (12) of G_*^J.

If $n = m = A = B = 1$, then $d\tilde{s}^2 = d\tilde{s}^2_{1,1;1,1}$ is given by

$$\begin{aligned}
\frac{1}{4}\,d\tilde{s}^2 \;&=\; \frac{dW\,d\overline{W}}{(1-|W|^2)^2} + \frac{1}{(1-|W|^2)}\,d\eta\,d\overline{\eta} \\
&+\; \frac{(1+|W|^2)|\eta|^2 - \overline{W}\eta^2 - W\overline{\eta}^2}{(1-|W|^2)^3}\,dW\,d\overline{W} \\
&+\; \frac{\eta\overline{W} - \overline{\eta}}{(1-|W|^2)^2}\,dW\,d\overline{\eta} \\
&+\; \frac{\overline{\eta}W - \eta}{(1-|W|^2)^2}\,d\overline{W}\,d\eta.
\end{aligned}$$

Theorem 11 (J.-H. Yang [18], 2005). The Laplacian $\tilde{\Delta} = \tilde{\Delta}_{n,m;A,B}$ of the above metric $d\tilde{s}^2_{n,m;A,B}$ is given by

$$
\begin{aligned}
\tilde{\Delta} = A \Bigg\{ & \mathrm{tr}\left[(I_n - W\overline{W})^t \Big((I_n - W\overline{W}) \frac{\partial}{\partial \overline{W}} \Big) \frac{\partial}{\partial W} \right] \\
+ \ & \mathrm{tr}\left[{}^t(\eta - \overline{\eta}\, W)\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W} W) \frac{\partial}{\partial W} \right] \\
+ \ & \mathrm{tr}\left[(\overline{\eta} - \eta\, \overline{W})^t \Big((I_n - W\overline{W}) \frac{\partial}{\partial \overline{W}} \Big) \frac{\partial}{\partial \eta} \right] \\
- \ & \mathrm{tr}\left[\eta \overline{W} (I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right] \\
- \ & \mathrm{tr}\left[\overline{\eta} W (I_n - \overline{W} W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right] \\
+ \ & \mathrm{tr}\left[\overline{\eta} (I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right] \\
+ \ & \mathrm{tr}\left[\eta \overline{W} W (I_n - \overline{W} W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right] \Bigg\} \\
+ \ & B \cdot \mathrm{tr}\left[(I_n - \overline{W} W) \frac{\partial}{\partial \eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) \right].
\end{aligned}
$$

If $n = m = A = B = 1$, we get

$$\begin{aligned}
\tilde{\Delta}_{1,1;1,1} \;=\;\; & (1-|W|^2)^2 \frac{\partial^2}{\partial W \partial \overline{W}} \\
& + (1-|W|^2)\, \frac{\partial^2}{\partial \eta \partial \overline{\eta}} \\
& + (1-|W|^2)(\eta - \overline{\eta}\, W)\, \frac{\partial^2}{\partial W \partial \overline{\eta}} \\
& + (1-|W|^2)(\overline{\eta} - \eta\, \overline{W})\, \frac{\partial^2}{\partial \overline{W} \partial \eta} \\
& - (\overline{W}\, \eta^2 + W \overline{\eta}^2)\, \frac{\partial^2}{\partial \eta \partial \overline{\eta}} \\
& + (1+|W|^2)|\eta|^2\, \frac{\partial^2}{\partial \eta \partial \overline{\eta}}.
\end{aligned}$$

The main ingredients for the proof of Theorem 10 and Theorem 11 are the partial Cayley transform (Theorem 9), Theorem 1 and Theorem 2.

Let $\mathbb{D}(\mathbf{D}_{n,m})$ be the algebra of all differential operators $\mathbf{D}_{n,m}$ invariant under the action (12) of G^J_*. By Theorem 9, we have the algebra isomorphism

$$\mathbb{D}(\mathbf{D}_{n,m}) \cong \mathbb{D}(\mathbf{H}_{n,m}).$$

We give some examples of explicit invariant differential operators on $\mathbb{D}_{n,m}$.

$$\mathbb{S} = \mathrm{tr}\left((I_n - \overline{W}W)\frac{\partial}{\partial\eta}{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

$$\mathbb{K}_{\mathbb{D}_{n,m}} = \det(I_n - W\overline{W})\,\det\left(\frac{\partial}{\partial\eta}{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

$$\mathbb{T}^{\mathbb{D}} := {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}$$

$$\mathbb{T}^{\mathbb{D}}_{kl} = \sum_{i,j=1}^{n}\left(\delta_{ij} - \sum_{r=1}^{n}\overline{w}_{ir}\,w_{jr}\right)\frac{\partial^2}{\partial\overline{\eta}_{ki}\partial\eta_{lj}}$$

$$(1 \le k, l \le m)$$

V. References

[1] **R. Langlands**, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Math. 544, Springer-Verlag, Berlin and New York (1976).

[2] **H. Maass**, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funtionalgleichungen*, Math. Ann. **121** (1949), 141-183.

[3] **H. Maass**, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **26** (1953), 44–68.

[4] **H. Maass**, *Siegel modular functions and Dirichlet series*, Lecture Notes in Math. 216, Springer-Verlag, Berlin and New York (1971).

[5] **H. Minkowski**, *Gesammelte Abhandlungen,* Chelsea, New York (1967).

[6] **D. Mumford**, *Tata Lectures on Theta I,* Progress in Math. **28**, Boston-Basel-Stuttgart (1983).

[7] **A. Selberg**, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. B. **20** (1956), 47-87.

[8] **C. L. Siegel**, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. **41, vol. II**, Springer-Verlag (1966), 274-359.

[9] **C. L. Siegel**, *Topics in Complex Function Theory*, Wiley-Interscience, New York, vol. III (1973).

[10] **A. Weil**, *Sur certains groupes d'operateurs unitaires (French)*, Acta Math. **111** (1964), 143–211.

[11] **J.-H. Yang**, *The Siegel-Jacobi operator,* Abh. Math. Sem. Univ. Hamburg **59** (1993), 135–146.

[12] **J.-H. Yang**, *Singular Jacobi Forms,* Trans. Amer. Math. Soc. **347 (6)** (1995), 2041–2049.

[13] **J.-H. Yang**, *Construction of vector valued modular forms from Jacobi forms,* Canadian J. of Math. **47 (6)** (1995), 1329–1339.

[14] **J.-H. Yang**, *A Geometrical Theory of Jacobi Forms of Higher Degree,* Proceedings of Symposium on Hodge Theory and Algebraic Geometry (Editor: Tadao Oda), Sendai, Japan

(1996), 125–147 or Kyungpook Math. J. **40** (2000), 209–237 or arXiv:math.NT/0602267.

[15] **J.-H. Yang**, *The Method of Orbits for Real Lie Groups,* Kyungpook Math. J. **42 (2)** (2002), 199–272 or arXiv:math.RT/0602056.

[16] **J.-H. Yang**, *Invariant metrics and Laplacians on Siegel-Jacobi spaces,* Journal of Number Theory, **127** (2007), 83-102 or arXiv:math. NT/0507215 v2.

[17] **J.-H. Yang**, *The partial Cayley transform of the Siegel-Jacobi disk* , J. Korean Math. Soc. **45 (3)** (2008), 781-794 or arXiv:math.NT /0507216 v2.

[18] **J.-H. Yang**, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* arXiv:math. NT /0507217 v2 or Chinese Annals of Mathematics, Series B, Vol. 31, No. 1 (2010), 85-100.

[19] **J.-H. Yang**, *A note on a fundamental domain for Siegel-Jacobi space*, Houston Journal of Mathematics, Vol. **32**, No. 3 (2006), 701–712 or arXiv:math.NT/0507218.

[20] **J.-H. Yang**, *Invariant differential operators on Siegel-Jacobi spaces,* arXiv:1107.0509 v1 [math.NT] 4 Jul 2011.

Thank You Very Much !!!

★ ★ ★ ★ ★ ★ ★ ★

★ ★ ★ ★ ★ ★ ★ ★

★ ★ ★ ★ ★ ★ ★ ★

★ ★ ★ ★ ★ ★ ★ ★