Quantitative Version of the Distribution of Eigenvalues of the Hecke Operators

Wang Yingnan

Joint work with Lau Yuk-Kam

March 14th, 2012

Let S_k be the set of all normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2, \mathbb{Z})$.

In this talk, f will always denote some element of S_k i.e. f is a normalized primitive holomorphic cusp forms. Let $\lambda_f(n)$ be the eigenvalue of f under the nth normalized Hecke operator T_n, $n = 1, 2, 3, \ldots$.

The Generalized Ramanujan Conjecture

The generalized Ramanujan conjecture indicates that

$$|\lambda_f(p)| \leq 2$$

for all primes p which was proved by Deligne in 1974. This inequality is also called Deligne's bound.

The Sato-Tate Conjecture

The asymptotic distribution of Hecke eigenvalues $\lambda_f(p)$ as the primes p vary is an interesting problem.

Inspired by the Sato-Tate conjecture, Serre conjectured that the Hecke eigenvalues $\lambda_f(p)$, $p \leq x$, are equidistributed with respect to the Sato-Tate measure

$$d\mu = \frac{1}{2\pi}\sqrt{4-t^2}dt$$

as $x \to \infty$. More precisely, for any interval $[\alpha, \beta] \subset [-2, 2]$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \{p \leq x : \lambda_f(p) \in [\alpha, \beta]\} = \int_\alpha^\beta d\mu,$$

where $\pi(x)$ denotes the number of primes not bigger than x.

In 2006, Nagoshi proved that the Sato-Tate conjecture holds on average of primitive holomorphic cusp forms.

Theorem (Nagoshi)

Suppose that $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Then for any interval $[\alpha, \beta] \subset [-2, 2]$, we have

$$\lim_{x\to\infty} \frac{1}{|S_k|\pi(x)} \# \{\lambda_f(p) \in [\alpha, \beta] : f \in H_k \text{ and } p \leq x\} = \int_{\alpha}^{\beta} d\mu,$$

where $|S_k|$ denotes the cardinality of S_k.

In 2011, the conjecture was proved by Barnet-Lamb, Geraghty, Harris and Taylor.

A "Vertical" Version of the Sato-Tate Conjecture

Naturally, we may ask whether for a fixed prime p, the Hecke eigenvalues $\lambda_f(p)$, $f \in S_k$ follow some similar distribution law as $k \to \infty$.

A "Vertical" Version of the Sato-Tate Conjecture

Naturally, we may ask whether for a fixed prime p, the Hecke eigenvalues $\lambda_f(p)$, $f \in S_k$ follow some similar distribution law as $k \to \infty$.

In 1997, Serre and Conrey/Duke/Farmer independently found that they are equidistributed with respect to the p-adic measure

$$d\mu_p = \frac{p+1}{2\pi} \frac{\sqrt{4-x^2}}{(p^{1/2}+p^{-1/2})^2 - x^2}\, dx.$$

Theorem (Serre&Conrey/Duke/Farmer)

For any interval $[\alpha, \beta] \subset [-2, 2]$,

$$\lim_{k \to \infty} \frac{1}{|S_k|} \# \{f \in S_k : \lambda_f(p) \in [\alpha, \beta]\} = \int_{\alpha}^{\beta} d\mu_p.$$

Theorem (Serre&Conrey/Duke/Farmer)

For any interval $[\alpha, \beta] \subset [-2, 2]$,

$$\lim_{k \to \infty} \frac{1}{|S_k|} \# \{ f \in S_k : \lambda_f(p) \in [\alpha, \beta] \} = \int_\alpha^\beta d\mu_p.$$

In 2009, Murty and Sinha investigated the rate of convergence and proved the following theorem.

Theorem (Murty-Sinha)

For a fixed prime p and any interval $[\alpha, \beta] \subset [-2, 2]$,

$$\frac{1}{|S_k|} \# \{ f \in S_k : \lambda_f(p) \in [\alpha, \beta] \} = \int_\alpha^\beta \mu_p + O\left(\frac{\log p}{\log k} \right),$$

where the implied constant is absolute.

Inspired by Murty and Sinha's work, we investigate the rate of convergence in the Sato-Tate conjecture on average of primitive holomorphic cusp forms and obtain an error term for Nagoshi's result.

Theorem (Lau-W.)

Suppose that $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. For any interval $[\alpha, \beta] \subset [-2, 2]$, we have

$$\frac{1}{|S_k|\pi(x)} \# \{\lambda_f(p) \in [\alpha, \beta] : f \in H_k \text{ and } p \leq x\}$$
$$= \int_\alpha^\beta d\mu + O\left(\frac{\log x}{\log k} + \frac{(\log x)(\log\log x)}{x}\right)$$

where the implied constant is absolute.

Moreover, our result also holds for Maass forms. This implies that the Sato-Tate conjecture for Maass forms (which is still open) is true on average of Maass Hecke eigenforms. To begin with, we brief the setting of Maass forms.

Let $\mathcal{C}$ be the Hilbert space consisting of all Maass cusp forms with respect to the inner productor

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} y^{-2} f(z)\overline{g}(z)dxdy.$$

Complete Orthonormal Basis

Let $\{u_j : j \geq 0\}$ a complete orthonormal basis $\{u_j : j \geq 0\}$ in $\mathcal{C}$ consisting of eigenfunctions of the hyperbolic Laplace operator Δ and all the Hecke operators T_n, $n = 1, 2, \ldots$, where u_0 is a constant function. We call u_j a Maass Hecke eigenform.

Let $\lambda_j(n)$ be the eigenvalue of u_j under the n-th Hecke operator T_n and $1/4 + t_j^2$ be the eigenvalue of u_j under Δ with $0 < t_1 \leq t_2 \leq \cdots$.

Weyl's Law

$$r(T) := \#\{j : 0 < t_j \leq T\} = \frac{1}{12}T^2 + O(T \log T).$$

The Generalized Ramanujan's Conjecture

Similar to the primitive holomorphic cusp forms, we also have the generalized Ramanujan conjecture for Maass Hecke eigenforms which predicts that

$$|\lambda_j(p)| \leq 2 \text{ for all } j \text{ and all primes } p.$$

Unfortunately, this is far out of reach.

Bounds Towards the Generalized Ramanujan's Conjecture

The best result towards the generalized Ramanujan conjecture for Maass forms is due to Kim and Sarnak (2003). They proved for all primes p,

$$|\lambda_j(p)| \leq p^{\theta} + p^{-\theta}$$

where $\theta = 7/64$. The conjecture asserts $\theta = 0$.

The possible "exceptional" eigenvalues (whose absolute values are bigger than 2) cause a substantial difficulty that has not been managed in the work of Murty and Sinha. We have to control the contribution of the possible "exceptional" eigenvalues such as the upper bound of the total number of "exceptional" eigenvalues.

The Number of "Exceptional" Eigenvalues

In 1987, Sarnak figured out that for any fixed prime p,

$$\#\{1 \leq j \leq r(T) : |\lambda_j(p)| \geq \alpha \geq 2\} \ll T^{2-\frac{\log(\alpha/2)}{\log p}}.$$

Unfortunately, Sarnak's result is not enough for our purpose. If we take $\alpha = 2 + 1/\log T$, then the above bound is trivial by Weyl's law.

The Number of "Exceptional" Eigenvalues

In 1987, Sarnak figured out that for any fixed prime p,

$$\#\{1 \leq j \leq r(T) : |\lambda_j(p)| \geq \alpha \geq 2\} \ll T^{2-\frac{\log(\alpha/2)}{\log p}}.$$

Unfortunately, Sarnak's result is not enough for our purpose. If we take $\alpha = 2 + 1/\log T$, then the above bound is trivial by Weyl's law.

Theorem (Lau-W)

For any fixed prime p, we have

$$\#\{1 \leq j \leq r(T) : |\lambda_j(p)| > 2\} \ll T^2 \left(\frac{\log p}{\log T}\right)^2.$$

This implies that for any fixed prime p, the "exceptional" eigenvalues have density zero.

The Sato-Tate Conjecture

Similar to primitive holomorphic cusp forms, we also have the Sato-Tate conjecture for Maass Hecke eigenforms which predicts that for any u_j with $j > 0$ and any interval $[\alpha, \beta] \subset (-\infty, \infty)$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \{p \le x : \lambda_j(p) \in [\alpha, \beta]\} = \int_\alpha^\beta d\mu',$$

where $\pi(x)$ denotes the number of primes not bigger than x and

$$d\mu' = \begin{cases} \frac{1}{2\pi}\sqrt{4-t^2}dt & \text{if } |t| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

As we mentioned before, we have the following result.

Theorem (Lau-w)

Suppose that $T = T(x)$ satisfies $\frac{\log T}{\log x} \to \infty$ as $x \to \infty$. For any $[\alpha, \beta] \subset (-\infty, \infty)$, we have

$$\frac{1}{r(T)\pi(x)} \# \{\lambda_j \in [\alpha, \beta] : 1 \leq j \leq r(T) \text{ and } p \leq x\}$$
$$= \int_\alpha^\beta d\mu' + O\left(\frac{\log x}{\log T} + \frac{(\log x)(\log\log x)}{x}\right)$$

where $d\mu'$ is defined as above and the implied constant is absolute.

A Vertical version of Sato-Tate Conjecture

Similar to the primitive holomorphic cusp forms, we also have a "vertical" version of Sato-Tate Conjecture. In 1987, Sarnak proved the following theorem.

Theorem (Sarnak)

For any integer N and distinct primes $p_1, \ldots, p_N$,

$$\lim_{T\to\infty} \frac{1}{r(T)} \#\{1 \le j \le r(T) : \ (\lambda_j(p_1), \ldots, \lambda_j(p_N)) \in I\}$$

$$= \int_I \prod_{n=1}^{N} d\mu'_{p_n}.$$

Here $I = \prod_{n=1}^{N} [a_n, b_n] \subset (-\infty, \infty)^N$ *and* $d\mu'_p$ *is defined as above.*

A Quantitative Version of Sarnak's Theorem

Inspired by Murty and Sinha's work, Lau and I proved a quantitative version of Sarnak's theorem.

Theorem (Lau-W)

For any any integer N and distinct primes $p_1, \ldots, p_N$,

$$\frac{1}{r(T)} \# \{1 \leq j \leq r(T) : \ (\lambda_j(p_1), \ldots, \lambda_j(p_N)) \in I\}$$
$$= \int_I \prod_{n=1}^{N} d\mu'_{p_n} + O\left(\frac{N \log(p_1 p_2 \cdots p_N)}{\log T}\right)$$

holds uniformly for $I = \prod_{n=1}^{N} [a_n, b_n] \subset (-\infty, \infty)^N$.

We are going to prove

$$\#\{1 \le j \le r(T) : |\lambda_j(p)| > 2\} \ll T^2 \left(\frac{\log(p)}{\log T}\right)^2.$$

The idea of the proofs of our other results is similar to this but much more delicate.

One Crucial Observation

Let p be a fixed prime. If $|\lambda_j(p)| > 2$, then we have $|\lambda_j(p^M)| > M + 1$ for any positive integer M. This implies that

$$1 - \frac{\lambda_j(p^M)^2}{(M+1)^2} < 0$$

for these j with $|\lambda_j(p)| > 2$. Therefore, for all $1 \leq j \leq r(T)$

$$1 - \frac{\lambda_j(p^M)^2}{(M+1)^2} \leq \chi_I(\lambda_j(p)) \leq 1 + \frac{\lambda_j(p^M)^2}{(M+1)^2},$$

where $\chi_I(x)$ is the characteristic function of $I = [-2, 2]$.

Approximation of the Number of Eigenvalues in I

Define

$$N(T) = \#\left\{1 \leq j \leq r(T) : \ |\lambda_j(p)| \leq 2\right\}.$$

Then we obtain that

$$\sum_{1\leq j\leq r(T)} \left(1 - \frac{\lambda_j(p^M)^2}{(M+1)^2}\right) \leq N(T) \leq \sum_{1\leq j\leq r(T)} \left(1 + \frac{\lambda_j(p^M)^2}{(M+1)^2}\right).$$

Therefore,

$$r(T) - N(T) \leq \sum_{1\leq j\leq r(T)} \frac{\lambda_j(p^M)^2}{(M+1)^2}.$$

To estimate the last sum, we have to apply an unweighted Kuznetsov trace formula.

Kuznetsov Trace Formula

In 1981, Kuznetsov proved for any two positive integers m, n and a complex function h satisfying certain conditions,

$$\sum_{j=1}^{\infty} \alpha_j \lambda_j(n)\lambda_j(m)h(t_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2} h(r)dr$$

$$= \frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty} r\tanh(\pi r)h(r)dr + \sum_{\ell=1}^{\infty}\frac{1}{\ell}S(m,n;\ell)\hat{h}(\frac{4\pi\sqrt{mn}}{\ell}),$$

where $\alpha_j = |\rho_j(1)|^2/\cosh \pi t_j$, $\sigma_\nu(n) = \sum_{\ell|n}\ell^\nu$, $S(n,m;\ell)$ is the classical Kloosterman sum and

$$\hat{h}(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} \frac{rh(r)}{\cosh \pi r} J_{2ir}(x)dr,$$

with J_ν being the J-Bessel function of order ν.

By standard techniques, we proved an unweighted version for our purpose.

Lemma

Let $\kappa_0 = \frac{11}{155}$, $\eta_0 = \frac{43}{620}$ and m, n be any positive integers. For arbitrarily small $\epsilon > 0$, we have

$$\sum_{j=1}^{r(T)} \lambda_j(m)\lambda_j(n) = \frac{1}{12} T^2 \delta_{mn=\square} \frac{\sigma((m,n))}{\sqrt{mn}} + O_\epsilon\left(T^{2-\kappa_0+\epsilon}(mn)^{\eta_0+\epsilon}\right),$$

where $\sigma(\ell) = \sum_{d|\ell} d$ and $\delta_{\ell=\square} = 1$ if ℓ is a square and $\delta_{\ell=\square} = 0$ otherwise.

By the above lemma with $m = n = p^M$, we obtain

$$r(T) - N(T) \ll \sum_{j=1}^{r(T)} \frac{\lambda_j(p^M)^2}{(M+1)^2} \ll \frac{T^2 + T^{2-\kappa_0+\epsilon} p^{2M(\eta_0+\epsilon)}}{(M+1)^2}.$$

Taking $M = \left[\frac{\kappa_0 \log T}{10\eta_0 \log p}\right]$ with sufficiently large T, we obtain

$$r(T) - N(T) \ll T^2 \left(\frac{\log p}{\log T}\right)^2.$$

This completes the proof.

Thank you!