Simplicity of groupoid $C^*$-algebras

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Outline

1. Groupoids
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Groupoids

A groupoid is a small category $G$ with inverses: for each $\gamma \in G$ there exists $\gamma^{-1} \in G$ such that $\gamma \gamma^{-1} = r(\gamma)$ and $\gamma^{-1} \gamma = s(\gamma)$.

It’s a group with an identity crisis. The set of identity elements (“unit space”) is denoted $G^{(0)}$.

In a topological group, $G$ is given a locally compact Hausdorff topology.

- Composition is continuous from $G \ast G \subseteq G \times G$ to $G$.
- Inversion is continuous from $G$ to $G$.

$G$ is étale if $r, s : G \to G^{(0)}$ are local homeomorphisms. This forces $G^{(0)}$ open in $G$. 
Examples

1. Groups: these are groupoids with one object. Étale means discrete.

2. An $R \subseteq X \times X$ is a groupoid: define $r(x, y) = x$, $s(x, y) = y$, $(x, y)^{-1} = (y, x)$ and $(x, y)(y, z) = (x, z)$.

3. If a group $G$ acts on a space $X$, then $G \times X$ is a groupoid with $r(g, x) = g \cdot x$, $s(g, x) = x$, $(g, x)^{-1} = (g^{-1}, g \cdot x)$ and $(g, h \cdot x)(h, x) = (gh, x)$; it’s étale if $G$ is discrete.

By analogy with the last example, we think of groupoids as “acting” on their unit spaces.

Say $G$ is *topologically principal* if $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$. Like a topologically free action.
Bisections

A bisection of $G$ is a subset $U \subseteq G$ such that $r$ and $s$ restrict to homeomorphisms on $U$.

Every étale groupoid has a basis consisting of precompact open bisections.

An étale groupoid is effective if $\text{Int}\{g \in G : r(g) = s(g)\} = G^{(0)}$. Like an effective group action.

Theorem (Renault)

Let $G$ be an étale locally compact Hausdorff groupoid. If $G$ is topologically principal then it is effective. If $G$ is second countable then the converse holds.
$C^*$-algebras

A $C^*$-algebra is a complete (complex) normed $*$-algebra satisfying the $C^*$-identity $\|a^*a\| = \|a\|^2$.

Gelfand-Naimark: every $C^*$-algebra is isomorphic to a closed $C^*$-subalgebra of $B(\mathcal{H})$.

Key example: if $G$ is a locally compact Hausdorff group, then $C_c(G)$ has a universal $C^*$-completion $C^*(G)$. If $G$ is amenable, then this is the only completion.

$C^*(G)$ is universal for continuous unitary representations of $G$.

A $C^*$-algebra $A$ is simple if every nonzero homomorphism of $A$ is injective. $C^*(G)$ is only simple if $G = \{e\}$ (consider the 1-dimensional representation of $G$).
Groupoid $C^*$-algebras

To construct the groupoid $C^*$-algebra, consider $C_c(G)$. Operations:

$$f * g(\gamma) = \sum_{\alpha \beta = \gamma} f(\alpha)g(\beta) \quad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

There is a universal $C^*$-completion $C^*(G)$ which is essentially unique if $G$ is suitably amenable; Renault’s Disintegration Theorem says that representations of $C^*(G)$ correspond precisely to representations (in the appropriate sense) of $G$.

Question: when is $C^*(G)$ simple?
Simplicity

A groupoid $G$ is minimal if $r(Gu) = G(0)$ for every $u \in G(0)$. Think of a minimal action: every orbit is dense.

Renault proved (early ’80’s): if $G$ is amenable, topologically principal and minimal, then $C^*(G)$ is simple; further, minimality is necessary.

The full converse was unknown. Proved in various special cases by: Deaconu-Renault, Kumjian-Pask-Raeburn, Archbold-Spielberg, Exel-Vershik, Robertson-S.

Theorem (Brown-Clark-Farthing-S)

Suppose that $G$ is étale, second-countable and amenable. Then $C^*(G)$ is simple if and only if $G$ is topologically principal and minimal.
Other results

Also obtain nice $C^*$-algebraic characterisations of when $G$ is (individually) minimal and topologically principal.

There is also a class of abstract algebras, called Steinberg algebras associated to étale groupoids with totally disconnected unit space. We obtain a characterisation of simplicity for Steinberg algebras.

**Theorem**

*Suppose that $G$ is étale with totally disconnected unit space. Then $A(G)$ is simple if and only if $G$ is both effective and minimal.*

In this case, “effective” is a strictly weaker hypothesis than “topologically principal.”