Proving Quantum Indeterminism: Measurements of Value Indefinite Observables Are Unpredictable

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Workshop on Mathematics and Computation
CARMA, University of Newcastle, 19 June 2015
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  - pseudo-randomness
  - coin-tossing (chaoticity)
  - Omega number
  - Schrödinger equation
  - cellular automata, non-deterministic Turing machines
Randomness

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EPR: “If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.”
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Eigenvalue-eigenstate principle: A system in a state $|\psi\rangle$ has a definite property of an observable $A$ if and only if $|\psi\rangle$ is an eigenstate of $A$. 
The Kochen-Specker theorem

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A value assignment function $\nu : \mathcal{O} \to \{0, 1\}$ models the measurement of an observable.

**Kochen-Specker Theorem.** In $n \geq 3$ Hilbert space there is a finite set of (projection) observables $\mathcal{O}$ such that no value assignment function $\nu : \mathcal{O} \to \{0, 1\}$ can have the following three properties:

1. **Value definiteness (VD):** $\nu$ is total, i.e., $\nu(P)$ defined for all $P \in \mathcal{O}$.
2. **Noncontextuality (NC):** $\nu$ is a function of $P$ only.
3. **Quantum mechanics predictions (QM):** For every context $C \subset \mathcal{O}$: $\sum_{P \in C} \nu(P) = 1$. 
A possible choice

Either, we reject:
▶ QM (but then we depart from quantum theory), or
▶ NC (definite values depend on measurement context), or
▶ VD (some observables are value indefinite).

A (rather accepted) option is to assume QM and NC and adopt value indefiniteness as a model of quantum indeterminacy. In this case some observables are value indefinite, hence some quantum measurements are indeterminate.
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In this case *some observables are value indefinite*, hence *some quantum measurements are indeterminate.*
How much value indefiniteness is reasonable?

- Rather than assuming that value indefiniteness apply uniformly, can we prove it from “simpler” assumptions?
- To this aim we need to localise the VD hypothesis:
  - VD: Every observable is assigned a defined value.
  - VD': One observable is assigned a defined value.
  - VD'': An observable \( P \) is assigned 1, and a non-compatible observable \( P' \) is value definite.
  - It is reasonable to expect that a system in state \( |\psi\rangle \) has \( v(P_\psi) = 1 \).
- One direction of eigenvalue-eigenstate principle.
- Intuitively, expect everything outside this ‘star’ to be value indefinite.
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![Diagram](attachment:image.png)
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- We need explicit assumptions.
Consider value assignment partial functions $\nu : \mathcal{O} \rightarrow \{0, 1\}$: $\nu(P)$ undefined if $P$ value indefinite.

- If $\nu(P)$ is value definite, then its value is noncontextual.
- Value indefinite observables are considered contextual.
- Use “admissibility” to model the condition that for all $C \subseteq \mathcal{O}$:
  (a) if there exists a $P \in C$ with $\nu(P) = 1$, then $\nu(P') = 0$ for all $P' \in C \setminus \{P\}$;
  (b) if there exists a $P \in C$ with $\nu(P') = 0$ for all $P' \in C \setminus \{P\}$, then $\nu(P) = 1$. 

A formal framework
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QM: Use “admissibility” to model the condition that for all $C$, $\sum_{P \in C} v(P) = 1$ if some $v(P)$ may be undefined.
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**Admissibility of $v$**

A value assignment function $v$ is **admissible** whenever for every context $C \subset \mathcal{O}$:

(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

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Admissibility provides a way of deducing the value definiteness of observables.
Failure of existing Greechie diagrams

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*Does there exist a set of observables $\mathcal{O}$ such that there is no admissible value assignment function with two non-compatible observables $P, P' \in \mathcal{O}$ and $v(P) = 1$ and $P'$ value definite?*
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*Does there exist a set of observables $O$ such that there is no admissible value assignment function with two non-compatible observables $P, P' \in O$ and $v(P) = 1$ and $P'$ value definite?*

Classical Greechie orthogonality diagrams proving the Kochen-Specker theorem fail to prove this statement.
Localised value indefiniteness

**Theorem 1.** Let \( n \geq 3 \) and \( |\psi\rangle, |\phi\rangle \in \mathbb{C}^n \) be states such that \( 0 < |\langle \psi | \phi \rangle| < 1 \). Then we effectively construct a finite set of observables \( \mathcal{O} \) containing \( P_\psi \) and \( P_\phi \) for which there is no admissible value assignment function on \( \mathcal{O} \) such that \( v(P_\psi) = 1 \) and \( P_\phi \) is value definite.
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Proof

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1. We first prove the explicit case that $|\langle \psi | \phi \rangle| = \frac{1}{\sqrt{2}}$.
2. We prove a reduction for $0 < |\langle \psi | \phi \rangle| < \frac{1}{\sqrt{2}}$ to the first case.
3. We prove a reduction for the last case of $\frac{1}{\sqrt{2}} < |\langle \psi | \phi \rangle| < 1$ case.
Almost all observables are value indefinite

**Theorem 2.** The set of value indefinite observables has constructive measure 1.
These results are purely mathematical. How should we interpret them physically?

**Eigenstate value definiteness**

If a system is in a state $|\psi\rangle$, then $v(P_\psi) = 1$ for any *admissible* value assignment function $v$.

**Interpretation**

If a system is in a state $|\psi\rangle$, then the result of measuring an observable $A$ is indeterministic unless $|\psi\rangle$ is an eigenstate of $A$.

We assumed one direction of the eigenvalue-eigenstate principle, but derived the other direction.
The Kochen-Specker theorem shows (via the adopted interpretation) that quantum-mechanics is indeterministic.

**Theorem 1** shows the *extent* of this indeterminism and indicates precisely which observables are value indefinite.

Indeterminism does not imply randomness. However, unpredictability is a requirement of randomness. So,

*are quantum mechanical measurements unpredictable?*
Consider a physical experiment $E$ producing a single bit.
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A non-probabilistic model of prediction

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With a particular trial (instantiation) of $E$ we associate the real parameter $\lambda$ which fully describes it. While $\lambda$ is not in its entirety an obtainable quantity, it contains any information that may be pertinent to prediction and we may have practical access to finite aspects of this information.
An extractor is a physical device selecting a finite amount of information included in \( \lambda \) without altering the experiment \( E \). Mathematically, an extractor is a (deterministic) function \( \lambda \mapsto \xi(\lambda) \in \{0, 1\}^* \) where \( \xi(\lambda) \) is a finite string of bits.
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A predictor for $E$ is an algorithm (computable function) $P_E$ which halts on every input and outputs 0 or 1 or prediction withheld.

$P_E$ can utilise as input the information $\xi(\lambda)$, but, as required by EPR, must be passive, that is, it must not disturb or interact with $E$ in any way.
A non-probabilistic model of prediction (cont.)

A predictor $P_E$ provides a correct prediction using the extractor $\xi$ for an instantiation of $E$ with parameter $\lambda$ if, when taking as input $\xi(\lambda)$, it outputs 0 or 1 (i.e. it does not refrain from making a prediction) and this output is equal to $x$, the result of the experiment.
A non-probabilistic model of prediction (cont.)

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The predictor $P_E$ is $k$-correct for $\xi$ if there exists an $n \geq k$ such that when $E$ is repeated $n$ times with associated parameters $\lambda_1, \ldots, \lambda_n$ producing the outputs $x_1, x_2, \ldots, x_n$, $P_E$ outputs the sequence

$$P_E(\xi(\lambda_1)), P_E(\xi(\lambda_2)), \ldots, P_E(\xi(\lambda_n))$$

with the following two properties:

1. no prediction in the sequence is incorrect, and
A non-probabilistic model of prediction (cont.)

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with the following two properties:

1. no prediction in the sequence is incorrect, and
2. in the sequence there are $k$ correct predictions.
If $P_E$ is $k$-correct for $\xi$ for all $k$ then $P_E$ is correct for $\xi$. The infinity used in the above definition is *potential* not actual: its role is to guarantee arbitrarily many correct predictions.
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The outcome $x$ of a single trial of the experiment $E$ performed with parameter $\lambda$ is predictable (with certainty) if there exist an extractor $\xi$ and a predictor $P_E$ which is correct for $\xi$, and $P_E(\xi(\lambda)) = x$. 
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The outcome $x$ of a single trial of the experiment $E$ performed with parameter $\lambda$ is predictable (with certainty) if there exist an extractor $\xi$ and a predictor $P_E$ which is correct for $\xi$, and $P_E(\xi(\lambda)) = x$.

Accordingly, $P_E$ correctly predicts the outcome $x$, never makes an incorrect prediction, and can produce arbitrarily many correct predictions.
Unpredictability and strong incomputability

**Theorem 3.** If $E$ is an experiment measuring a quantum value indefinite observable, then for every predictor $P_E$ using any extractor $\xi$, $P_E$ is not correct for $\xi$. 
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Theorem 4. In an infinite repetition of the experiment $E$ measuring a quantum value indefinite observable which generates the infinite sequence $x_1 x_2 \ldots$, no single bit $x_i$ can be predicted with certainty.
An open problem

- Assume noncontextuality.
- **Theorem 1** doesn’t hold in two-dimensional Hilbert space.
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- Assume noncontextuality.
- **Theorem 1** doesn’t hold in two-dimensional Hilbert space.
- Does **Theorem 4** hold in two-dimensional Hilbert space?
References


