Expectations over Fractal Sets

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Synapse spatial distributions

Classical box integrals

$B_n(s)$ is the order-$s$ moment of separation between a random point and a vertex of the $n$-cube:

$$B_n(s) := \langle |x|^s \rangle_{x \in [0,1]^n} = \int_{x \in [0,1]^n} |x|^s \, Dx$$
Classical box integrals

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$\Delta_n(s)$ is the order-$s$ moment of separation between two random points in the $n$-cube:

$$\Delta_n(s) := \langle |x - y|^s \rangle_{x,y \in [0,1]^n} = \int_{x,y \in [0,1]^n} |x - y|^s \, Dx\, Dy$$

R.S. Anderssen, R.P. Brent, D.J. Daley and P.A.P. Moran, Concerning $\int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{1/2} \, dx_1 \cdots dx_k$ and a Taylor Series Method. SIAM Journal on Applied Mathematics 30 (1976).

Outline:

1. Expectations over String-generated Cantor Sets (SCSs).

2. Expectations over Iterated Function System (IFS) attractors.

String-generated Cantor Set (SCS) expectations
String-generated Cantor Sets

Ternary expansion for coordinates of \( x = (x_1, \ldots, x_n) \in [0, 1]^n \) (with \( x_{jk} \in \{0, 1, 2\} \)):

\[
U(c) := \#\{\text{1's in ternary vector } c\}
\]

\[
x_1 = 0.x_{11} x_{12} x_{13} \ldots
\]

\[
x_2 = 0.x_{21} x_{22} x_{23} \ldots
\]

\[\vdots\]

\[
x_n = 0.x_{n1} x_{n2} x_{n3} \ldots
\]

\[
\uparrow \quad \uparrow \quad \uparrow
\]

\[
c_1 \quad c_2 \quad c_3 \quad \ldots
\]
String-generated Cantor Sets

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$$
\begin{align*}
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X_1 &= 0.x_{11}x_{12}x_{13} \ldots \\
X_2 &= 0.x_{21}x_{22}x_{23} \ldots \\
\vdots \\
X_n &= 0.x_{n1}x_{n2}x_{n3} \ldots
\end{align*}
$$

Definition (String-generated Cantor set)

*Given an embedding space $[0, 1]^n$ and an entirely-periodic string $P = P_1P_2\ldots P_p$ of non-negative integers with $P_i \leq n$ for all $i = 1, 2, \ldots, p,$*
String-generated Cantor Sets

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\vdots
\]
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x_n = 0. x_{n1} x_{n2} x_{n3} \ldots
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\( U(c) := \#\{1's \text{ in ternary vector } c\} \)

Definition (String-generated Cantor set)

Given an embedding space \([0, 1]^n\) and an entirely-periodic string \( P = P_1 P_2 \ldots P_p \) of non-negative integers with \( P_i \leq n \) for all \( i = 1, 2, \ldots, p \), the **String-Generated Cantor Set (SCS)**, denoted \( C_n(P) \), is the set of all admissible \( x \in [0, 1]^n \), where

\[
x \text{ admissible} \iff U(c_k) \leq P_k \quad \forall k \in \mathbb{N}
\]
String-generated Cantor Sets

Classical box integrals

$B_n(s)$ is the order-$s$ moment of separation between a random point and a vertex of the $n$-cube:

$$B_n(s) := \langle |x|^s \rangle_{x \in [0,1]^n} = \int_{x \in [0,1]^n} |x|^s \, Dx$$

$\Delta_n(s)$ is the order-$s$ moment of separation between two random points in the $n$-cube:

$$\Delta_n(s) := \langle |x - y|^s \rangle_{x,y \in [0,1]^n} = \int_{x,y \in [0,1]^n} |x - y|^s \, DxDy$$
Fractal Box Integrals

\[ B(2, C_2(1)) = \frac{11}{16} = 0.6875 \]

\[ \Delta(2, C_2(1)) = \frac{3}{8} = 0.375 \]

\[ B(2, C_2(2)) = \frac{2}{3} = 0.66 \ldots \]

\[ \Delta(2, C_2(2)) = \frac{1}{3} = 0.33 \ldots \]
Definition (Expectation over an SCS)

The expectation of \( F : \mathbb{R}^n \to \mathbb{C} \) on an SCS \( C_n(P) \) is defined by:

\[
\langle F(x) \rangle_{x \in C_n(P)} := \lim_{j \to \infty} \frac{1}{N_1 \cdots N_j} \sum_{U(c_i) \leq P_i} F\left(\frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_j}{3^j}\right)
\]

\[
\langle F(x - y) \rangle_{x, y \in C_n(P)} := \lim_{j \to \infty} \frac{1}{N_1^2 \cdots N_j^2} \sum_{U(c_i) \leq P_i} \sum_{U(d_i) \leq P_i} F\left(\frac{c_1 - d_1}{3} + \cdots + \frac{c_j - d_j}{3^j}\right)
\]

when the respective limits exist.
Functional equations for expectations

Proposition (Functional equations for expectations)

For \( x, y \) in \( \mathbb{R}^n \) and appropriate \( F \) the expectations pertaining to the box integrals \( B \) and \( \Delta \) satisfy the functional equations:

\[
\langle F(x) \rangle_{x \in C_n(P)} = \frac{1}{\prod_{j=1}^p N_j} \sum_{U(c_k) \leq P_k} \left\langle F \left( \frac{x}{3^p} + \sum_{j=1}^p \frac{c_j}{3^j} \right) \right\rangle
\]

\[
\langle F(x - y) \rangle_{x, y \in C_n(P)} = \frac{1}{\prod_{j=1}^p N_j^2} \sum_{U(b_k) \leq P_k} \left\langle F \left( \frac{x - y}{3^p} + \sum_{j=1}^p \frac{(b_j - a_j)}{3^j} \right) \right\rangle
\]
The functional expectation relations lead directly to:

**Theorem (Closed forms for \( B(2, C_n(P)) \) and \( \Delta(2, C_n(P)) \))**

For any embedding dimension \( n \) and SCS \( C_n(P) \) the box integral \( B(2, C_n(P)) \) is rational, given by the closed form:

\[
B(2, C_n(P)) = \frac{n}{4} + \frac{1}{1 - 9^{-p}} \sum_{k=1}^{p} \frac{1}{9^{k}} \frac{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}(n-j)}{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}}
\]
Special case - second moments

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$$B(2, C_n(P)) = \frac{n}{4} + \frac{1}{1 - 9^{-p}} \sum_{k=1}^{p} \frac{1}{9^k} \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j} (n-j) \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}$$

and the corresponding box integral $\Delta(2, C_n(P))$ is also rational, given by:

$$\Delta(2, C_n(P)) = 2B(2, C_n(P)) - \frac{n}{2}$$
Special case - second moments

The classical box integrals over the unit $n$-cube are:

$$B_n(2) = \frac{n}{3} \quad \text{and} \quad \Delta_n(2) = \frac{n}{6}$$

which matches the output of our closed forms when $P = n$. 

The classical box integrals over the unit $n$-cube are:
Iterated Function System (IFS) attractor expectations
IFS Attractors

Let \((X, d)\) be a metric space and let \((\mathcal{H}(X), h(d))\) be the associated space of non-empty compact subsets of \(X\) equipped with the Hausdorff metric \(h(d)\).
IFS Attractors

Let \((X, d)\) be a metric space and let \((\mathcal{H}(X), h(d))\) be the associated space of non-empty compact subsets of \(X\) equipped with the Hausdorff metric \(h(d)\).

**Definition**

For each \(i \in \{1, 2, \ldots, m\}\) (where \(m \geq 2\)), let \(f_i : X \to X\) be a contraction mapping with contractivity factor \(0 < c_i < 1\) (so \(d(f_i(x), f_i(y)) \leq c_i \cdot d(x, y)\)) and associated probability \(0 < p_i < 1\) (where \(\sum_{i=1}^{m} p_i = 1\)). A **hyperbolic iterated function system (IFS)** is the collection

\[
\{X; f_1, \ldots, f_m\}
\]
IFS Attractors

Theorem
Let \( \{X; f_1, \ldots, f_m\} \) be a hyperbolic IFS. Then the transformation \( F : \mathcal{H}(X) \to \mathcal{H}(X) \) defined by \( F(S) = \bigcup_{n=1}^{m} f_n(S) \) for all \( S \in \mathcal{H}(X) \) is a contraction mapping on \( \mathcal{H}(X) \) with contractivity factor \( C = \max\{c_1, \ldots, c_m\} \).
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Theorem (The Contraction Mapping Theorem)
The mapping \( F \) possesses a unique fixed point \( A \in \mathcal{H}(X) \), which satisfies:
\[
A = F(A) = \bigcup_{n=1}^{m} f_n(A)
\]
and which is referred to as the attractor of the IFS.

We will take as our 'deterministic fractals' those sets that can be so expressed as an IFS attractor.
IFS Attractors

**Theorem**
Let \( \{X; f_1, \ldots, f_m\} \) be a hyperbolic IFS. Then the transformation \( \mathcal{F}: \mathcal{H}(X) \to \mathcal{H}(X) \) defined by \( \mathcal{F}(S) = \bigcup_{n=1}^{m} f_n(S) \) for all \( S \in \mathcal{H}(X) \) is a contraction mapping on \( \mathcal{H}(X) \) with contractivity factor \( C = \max \{c_1, \ldots, c_m\} \).

**Theorem (The Contraction Mapping Theorem)**
The mapping \( \mathcal{F} \) possesses a unique fixed point \( A \in \mathcal{H}(X) \), which satisfies:
\[
A = \mathcal{F}(A) = \bigcup_{n=1}^{m} f_n(A)
\]
and which is referred to as the **attractor** of the IFS.

We will take as our ‘**deterministic fractals**’ those sets that can be so expressed as an IFS attractor.
IFS Attractors

\[ f_1(x, y) = \left( \frac{x}{2}, \frac{y}{2} \right) \]
\[ f_2(x, y) = \left( \frac{x + 1}{2}, \frac{y + \sqrt{3}}{2} \right) \]
\[ f_3(x, y) = \left( \frac{x + 2}{2}, \frac{y}{2} \right) \]
SCS in IFS framework

Any given SCS can be expressed as the attractor of an IFS in the following manner:

**Proposition**

The IFS corresponding to the SCS \( C_n(P) \) is:

\[
\{ [0, 1]^n \subset \mathbb{R}^n; f_1, f_2, \ldots, f_i, \ldots, f_m \}
\]

where \( f_i(x) = \left( \frac{1}{3} \right)^p x + \left( \frac{1}{3} \right)^2 c_{1i} + \ldots + \left( \frac{1}{3} \right)^p c_{pi} \) for \( i \in \{ 1, 2, \ldots, m \} \) ranging over all admissible columns \( c_k \), where \( m = \prod_{k=1}^{p} N_k \) and \( N_k = \sum_{j=0}^{p} \binom{n}{j} 2^{n-j} \).
Code Space

Definition

Given an IFS \( \{X; f_1, \ldots, f_m\} \), the associated **code space** \( \Sigma_m \) is defined as:

\[
\Sigma_m := \{ \sigma = \sigma_1\sigma_2 \ldots | \sigma_i \in \{1, 2, \ldots, m\} \quad \forall i \in \mathbb{N} \}
\]
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\]

The **address function** \( \phi : \Sigma_m \rightarrow X \) is defined by:

\[
\phi(\sigma) := \lim_{k \to \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \ldots \circ f_{\sigma_k}(x)
\]

for any \( x \in A \).
Code Space

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Given an IFS \( \{X; f_1, \ldots, f_m\} \), the associated code space \( \Sigma_m \) is defined as:

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\]

for any \( x \in A \). The attractor of the IFS can also be represented by:

\[
A = \{ \phi(\sigma) | \sigma \in \Sigma_m \}
\]
Expectations on IFS Attractors

Definition (Fundamental definition of expectation)

Let \( \{X; f_1, \ldots, f_m\} \) be an IFS with attractor \( A \in H(X) \). Let \( F : X \to \mathbb{C} \) be a complex-valued function over \( X \). The expectation of \( F \) over \( A \), \( \langle F(x) \rangle_{x \in A} \), is defined as:

\[
\langle F(x) \rangle_{x \in A} := \lim_{j \to \infty} \frac{1}{m^j} \sum_{\sigma_{|j|} \in \Sigma_m} F(\phi(\sigma_{|j|}))
\]

when the limit exists.
Corollary

(Fundamental definition of separation (using code-space)) Let \( \{X; f_1, \ldots, f_m\} \) be an IFS with attractor \( A \in H(X) \). Let \( F : X \to \mathbb{C} \) be a complex-valued function over \( X \). The separation expectation of \( F \) over \( A \), \( \langle F(x - y) \rangle_{x,y \in A} \), is defined as:

\[
\langle F(x - y) \rangle_{x,y \in A} := \lim_{j \to \infty} \frac{1}{m^{2j}} \sum_{\sigma_{|j|} \in \Sigma_m} \sum_{\tau_{|j|} \in \Sigma_m} F \left( \phi(\sigma_{|j|}) - \phi(\tau_{|j|}) \right)
\]

when the limit exists.
The invariant IFS measure

Definition
Let $B$ be a Borel subset of a metric space $(X, d)$. The residence measure is defined as:

$$
\mu(B) := \lim_{n \to \infty} \frac{1}{n} \left( \# \left\{ k : f^k(x) \in B, \ 1 \leq k \leq n \right\} \right)
$$

The residence measure is a normalised, invariant measure over the attractor of any IFS.
The invariant IFS measure

Theorem (Elton’s Theorem - special case)

Let \((X, d)\) be a compact metric space and let \(\{X; f_1, \ldots, f_m\}\) be a hyperbolic IFS.
The invariant IFS measure

Theorem (Elton’s Theorem - special case)

Let \((X, d)\) be a compact metric space and let \(\{X; f_1, \ldots, f_m\}\) be a hyperbolic IFS. Let \(\{x_n\}_{n=0}^{\infty}\) denote a chaos game orbit of the IFS starting at \(x_0 \in X\),
The invariant IFS measure

Theorem (Elton’s Theorem - special case)

Let \((X, d)\) be a compact metric space and let \(\{X; f_1, \ldots, f_m\}\) be a hyperbolic IFS. Let \(\{x_n\}_{n=0}^\infty\) denote a chaos game orbit of the IFS starting at \(x_0 \in X\), that is,

\[ x_n = f_{\sigma_n} \circ \ldots \circ f_{\sigma_1}(x_0) \]

where the maps are chosen independently according to the probabilities \(p_1, \ldots, p_m\) for \(n \in \mathbb{N}\).
The invariant IFS measure

Theorem (Elton’s Theorem - special case)

Let \((X, d)\) be a compact metric space and let \(\{X; f_1, \ldots, f_m\}\) be a hyperbolic IFS. Let \(\{x_n\}_{n=0}^{\infty}\) denote a chaos game orbit of the IFS starting at \(x_0 \in X\), that is,

\[ x_n = f_{\sigma_n} \circ \ldots \circ f_{\sigma_1}(x_0) \]

where the maps are chosen independently according to the probabilities \(p_1, \ldots, p_m\) for \(n \in \mathbb{N}\). Let \(\mu\) be the unique invariant measure for the IFS. Then, with probability 1,

\[ \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} F(x_k) = \int_X F(x) \, d\mu(x) \]
Expectations over IFS attractors

Corollary
Let \( \{X; f_1, f_2, \ldots, f_m\} \) be a contractive IFS with attractor \( A \in \mathcal{H}(X) \). Given a complex-valued function \( F : X \to \mathbb{C} \), the expectation of \( F \) over \( A \) is given by the integral:

\[
\langle F(x) \rangle_{x \in A} = \int_X F(x) \, d\mu(x)
\]
**Proposition (Functional equations for expectations)**

For points $x, y$ in the attractor $A$ of a non-overlapping IFS, the expectations for a complex-valued function $F$ satisfy the functional equations:

\[
\langle F(x) \rangle = \frac{1}{m} \sum_{j=1}^{m} \langle F(f_j(x)) \rangle
\]

\[
\langle F(x - y) \rangle = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} \langle F(f_j(x) - f_k(y)) \rangle
\]
Functional equations

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\langle F(x - y) \rangle = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} \langle F(f_j(x) - f_k(y)) \rangle
\]

and more generally

\[
\langle F(x_1, x_2, \ldots, x_n) \rangle = \frac{1}{m^n} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_n=1}^{m} \langle F(f_{j_1}(x_1), f_{j_2}(x_2), \ldots, f_{j_n}(x_n)) \rangle
\]
Ongoing mathematics and computation
Exact evaluation of even moments

Substitute a given IFS and function $F$ into the functional equation:

$$\langle F(x - y) \rangle = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} \langle F(f_j(x) - f_k(y)) \rangle$$

Simplify the resulting expression into a (linear) combination of $n$ simpler expectations.

Feed these expectations back into the functional equation to generate a system of $n$ (linear) equations in the $n$ unknown expectations.

Solve the system of equations and hence determine the expectation.
Orthogonal Sierpinski Triangle

\[ B(2, A) = \frac{10}{27} \]
Orthogonal Sierpinski Triangle

\[ B(2, A) = \frac{10}{27} \]

\[ \Delta(2, A) = \frac{8}{27} \]
Equilateral Sierpinski Triangle

\[ B(2, A) = \frac{4}{9} \]
Equilateral Sierpinski Triangle

\[ B(2, A) = \frac{4}{9} \]

\[ \Delta(2, A) = \frac{2}{9} \]
Koch Curve

\[ B(2, A) = \frac{1}{3} \]
Koch Curve

\[ B(2, A) = \frac{1}{3} \]

\[ \Delta(2, A) = \frac{4}{27} \]
Barnsley Fern
Barnsley Fern

\[
B(2, \alpha) = 2049440803137681904580160660775546421 
\approx 3.5
\]
Barnsley Fern

$B(2, A) = 2049440803137681904580160660775546421 \approx 3.5$
Barnsley Fern

\[ B(2, A) = \frac{2049440803137681904}{580160660775546421} \approx 3.5 \]
Barnsley Fern
Barnsley Fern
Barnsley Fern

\[ \Delta(2, A) = \frac{1561818604387599983932186}{541130352321871535527225} \approx 2.9 \]
Current research: Poles for box integrals

For classical box integrals over unit hypercubes, we have the following:

Theorem (Absolutely-convergent analytic series for $B_n(s)$)

For all $s \in \mathbb{C}$,

$$B_n(s) = \frac{n^{1+s/2}}{s + n} \sum_{k=0}^{\infty} \gamma_{n-1,k} \left( \frac{2}{n} \right)^k$$

where the $\gamma_{m,k}$ are fixed real coefficients defined by the two-variable recursion:

$$(1 + 2k/m) \gamma_{m,k} = (k - 1 - s/2) \gamma_{m,k-1} + \gamma_{m-1,k}$$

for $m, k \geq 1$, with initial conditions $\gamma_{0,k} := \delta_{0,k}$, $\gamma_{m,0} := 1$.

Note the single pole at $s = -n$, the negated dimension of the embedding space.
For classical box integrals over unit hypercubes, we have the following:

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*where the $\gamma_{m,k}$ are fixed real coefficients defined by the two-variable recursion:*

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*for $m, k \geq 1$, with initial conditions $\gamma_{0,k} := \delta_{0,k}$, $\gamma_{m,0} := 1$.*

Note the single pole at $s = -n$, the negated dimension of the embedding space.
Theorem (Fractal Dimension with the Open Set Condition)

Suppose that the open set condition holds for the IFS \( \mathcal{F} = \{ \mathbb{R}^n; f_1, f_2, \ldots, f_m \} \) (with associated contraction factors \( \{c_1, c_2, \ldots, c_m\} \)). That is, the attractor contains a non-empty set \( O \subset A \) which is open in the metric space \( A \) such that

1. \( f_i(O) \cap f_j(O) = \emptyset \) for all \( i, j \in \{1, 2, \ldots, m\} \) with \( i \neq j \)
2. \( \bigcup_{i=1}^{m} f_i(O) \subset O \)

Then the Hausdorff dimension and Minkowski box-counting dimension of the attractor of the IFS are equal and take the value \( \delta \), where:

\[
\sum_{i=1}^{m} (c_i)^\delta = 1.
\]
Current research: Poles for box integrals

Using the functional expectation relations, we can prove:

Proposition (SCS: Pole of $B(s, C_n(P))$)

For any SCS $C_n(P)$, the box integral $B(s, C_n(P))$ has a pole at

$$s = -\delta(C_n(P)).$$
Current research: Poles for box integrals

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*For any SCS $C_n(P)$, the box integral $B(s, C_n(P))$ has a pole at*

\[ s = -\delta(C_n(P)). \]

**Proposition (IFS: Pole of $B(s)$ over Uniform Affine IFSs)**

*Let $\mathcal{F} = \{ X; f_1, f_2, \ldots, f_m \}$ be a contractive affine IFS satisfying the open set condition with uniform contraction factors; that is, $c_1 = c_2 = \ldots = c_m$. Then the box integral $B(s, A)$ over the attractor $A \in \mathbb{H}(X)$ has a pole at*

\[ s = -\delta(A) \]
Current research: Poles for box integrals

Proposition (IFS: Bounds on pole of $\Delta(s)$ over Similarity IFSs)

Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive similarity IFS satisfying the open set condition; that is, $|f_i(x) - f_i(y)| = c_i|x - y|$ for all $i$. Then, if the box integral $\Delta_A(s)$ over the attractor $A \in \mathbb{H}(X)$ has a pole on the real axis, the pole is bounded by:

$$\frac{\log(m)}{\log(c_{\text{max}})} \leq s \leq \frac{\log(m)}{\log(c_{\text{min}})}$$

where $c_{\text{max}} = c = \max\{c_1, \ldots, c_m\}$ and $c_{\text{min}} = \min\{c_1, \ldots, c_m\}$. 
Current research: Odd-order moments

\[
B(s, C_2(2)) = \left\{ \begin{array}{ll}
-\frac{1}{4} - \frac{\pi}{8} & \text{for } s = -4 \\
-\sqrt{2} & \text{for } s = -3 \\
\infty & \text{for } s = -2 \\
2 \log(1 + \sqrt{2}) & \text{for } s = -1 \\
\frac{1}{3} \sqrt{2} + \frac{1}{3} \log(1 + \sqrt{2}) & \text{for } s = 1 \\
\frac{7}{20} \sqrt{2} + \frac{3}{20} \log(1 + \sqrt{2}) & \text{for } s = 3
\end{array} \right.
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(s)</th>
<th>(B(s, C_2(2)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-4</td>
<td>(-\frac{1}{4} - \frac{\pi}{8})</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
<td>(-\sqrt{2})</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>(\infty)</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>(2 \log(1 + \sqrt{2}))</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(\frac{1}{3} \sqrt{2} + \frac{1}{3} \log(1 + \sqrt{2}))</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>(\frac{7}{20} \sqrt{2} + \frac{3}{20} \log(1 + \sqrt{2}))</td>
</tr>
</tbody>
</table>

**Table:** Closed-form results for \(B\) box integrals of various order \(s\) over the unit square and unit cube. For the unit \(n\)-cube all integer values for \(1 \leq n \leq 5\) have closed forms. \(Ti_2\) is a generalized tangent (polylog) value and \(G\) is Catalan’s constant.
Current research: Odd-order moments

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s$</th>
<th>$B(s, C_3(3))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-5</td>
<td>$-\frac{1}{6}\sqrt{3} - \frac{1}{12}\pi$</td>
</tr>
<tr>
<td>3</td>
<td>-4</td>
<td>$-\frac{3}{2}\sqrt{2}\arctan\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>$\infty$</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>$-3G + \frac{3}{2}\pi \log(1 + \sqrt{2}) + 3 \text{ Ti}_2(3 - 2\sqrt{2})$</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>$-\frac{1}{4}\pi + \frac{3}{2}\log(2 + \sqrt{3})$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log(2 + \sqrt{3})$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$\frac{2}{5}\sqrt{3} - \frac{1}{60}\pi + \frac{7}{20}\log(2 + \sqrt{3})$</td>
</tr>
</tbody>
</table>
Current research: Odd-order moments

Table: Closed-form results for Δ box integrals of various order s over the unit square and unit cube. For the unit n-cube all integer values for 1 ≤ n ≤ 5 have closed forms. Ti2 is a generalized tangent (polylog) value and G is Catalan’s constant.
Current research: Odd-order moments

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s$</th>
<th>$\Delta(s, C_3(3))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>−2</td>
<td>$2\pi - 12 , G + 12 , Ti_2 \left(3 - 2\sqrt{2}\right) + 6\pi \log \left(1 + \sqrt{2}\right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 2 \log 2 - \frac{5}{2} \log 3 - 8\sqrt{2} \arctan \left(\frac{1}{\sqrt{2}}\right)$</td>
</tr>
<tr>
<td>3</td>
<td>−1</td>
<td>$\frac{2}{5} - \frac{2}{3} \pi + \frac{2}{5} \sqrt{2} - \frac{4}{5} \sqrt{3} + 2 \log \left(1 + \sqrt{2}\right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 12 \log \left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right) - 4 \log \left(2 + \sqrt{3}\right)$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$- \frac{118}{21} - \frac{2}{3} \pi + \frac{34}{21} \sqrt{2} - \frac{4}{7} \sqrt{3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 2 \log \left(1 + \sqrt{2}\right) + 8 \log \left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right)$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$- \frac{1}{105} - \frac{2}{105} \pi + \frac{73}{840} \sqrt{2} + \frac{1}{35} \sqrt{3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \frac{3}{56} \log \left(1 + \sqrt{2}\right) + \frac{13}{35} \log \left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right)$</td>
</tr>
</tbody>
</table>
Current research: What are the odd moments?

Insight into the evaluation of odd moments can be gleaned from computer-assisted mathematics. In particular, we are considering a modified Richardson Extrapolation Technique combined with the PSLQ Integer Relation Algorithm to hunt for closed forms (joint work with Nathan Clisby).
A vector \( \mathbf{x} = (x_1, x_2 \cdots x_n) \) of real numbers has an integer relation if there exists integers \( a_i \), not all zero, such that:

\[
\sum_{i=1}^{n} a_i x_i = 0.
\]
Current research: High-precision numerics

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- Given a vector \( \mathbf{x} = (x_1, x_2 \cdots x_n) \), PSLQ iteratively constructs a sequence of integer-matrices \( B_n \) that reduce the vector \( \mathbf{x}B_n \).
- The process continues until either:
  1. The smallest entry of the latest \( B_n \) abruptly decreases to within drops to within \( \epsilon \) of 0. This signals the detection of an integer relation, which PSLQ will produce as one of the columns of the last \( B_n \).
  2. The available precision is exhausted. In this case, PSLQ will establish a bound on the size of any possible integer relation.
Current research: High-precision numerics

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In order to find an integer relation among \( n \) terms, **PSLQ** requires at least \( nd \) accurate digits in all terms, where \( d \) is the number of digits of the largest integer \( a_i \).
Current research: High-precision numerics

The **Richardson Extrapolation Technique** combines multiple lower-accuracy evaluations to eliminate the highest-order error terms and thereby obtain a higher-accuracy evaluation.

- Start with an approximation formula $A_1(h)$ for quantity of interest $x$, accurate to $O(h^1)$. That is,

\[
x = A_1(h) + O(h^1) = A_1(h) + c_1 h + c_2 h^2 + \ldots.
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Thus, halving the step-size $h$ in the power series,

$$x = A_1 \left( \frac{h}{2} \right) + c_1 \frac{h}{2} + c_2 \frac{h^2}{4} + \ldots.$$

Current research: High-precision numerics
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- Combine these equations to eliminate $c_1 h$, yielding:

$$x = A_2(h) - \frac{1}{2} c_2 h^2 - \frac{3}{4} c_3 h^3 + \ldots = A_2(h) + O(h^2)$$  

where $A_2(h) = A_1 \left( \frac{h}{2} \right) + \frac{1}{2^{1-1}} \left( A_1 \left( \frac{h}{2} \right) - A_1(h) \right)$.  

Current research: High-precision numerics

Successive iterations over smaller step-sizes yield
\[ x = A_k(h) + O(h^k), \]
where
\[
A_k(h) = A_{k-1} \left( \frac{h}{2} \right) + \frac{1}{2^{k-1} - 1} \left( A_{k-1} \left( \frac{h}{2} \right) - A_{k-1}(h) \right).
\]

Adapting this process to IFS attractors, Nathan Clisby has computed 112 digits of \( B(1, \text{Sierpiński Triangle}) \):
Current research: High-precision numerics

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28949927817241259628464958573670699106107561807 \ldots
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We are aiming to improve this method by constructing an analogue of Bulirsch-Stöer extrapolation modified for IFS attractors, wherein the sequence of estimates is fitted to a rational function of \( h \) and evaluated at \( h = 0 \).
Thank you!