$C^*$-Algebras generated by semigroups of partial isometries

Ilija Tolich

Authors:
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Definition
A bounded operator $T$ on a Hilbert space $H$ is a partial isometry if $\|Th\| = \|h\|$ for all $h \in (\ker T)\perp$. 

Lemma
The following are equivalent:
1. $T$ is a partial isometry.
2. $TT^*T = T$
3. $TT^*$ is a projection onto $\text{range } T$.
4. $T^*$ is a partial isometry.
5. $T^*TT^* = T^*$
6. $T^*T$ is a projection onto $(\ker T)\perp$. 

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Partial Isometries

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6. $T^* T$ is a projection onto $(\ker T)\perp$.
Doubly Quasi-lattice ordered groups

Let $P$ be a subsemigroup of a group $G$ such that $P \cap P^{-1} = \{e\}$.

The pair $(G, P)$ defines two partial orders on $G$:

A left partial order on $G$ defined by $x \leq l y$ if $x^{-1} y \in P$, $(\iff y \in xP)$

A right partial order on $G$ defined by $x \leq r y$ if $yx^{-1} \in P$, $(\iff y \in Px)$

If $G$ is abelian both orders are the same.

**Definition**

The partially ordered group $(G, P)$ is said to be doubly quasi-lattice ordered if, in both left and right orders, any pair $x, y \in G$ with a common upper bound in $P$ has a least common upper bound in $P$.

We denote the least upper bound of $x, y$ in the left order as $x \vee_l y$ and in the right order as $x \vee_r y$. 

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C*-Algebras generated by semigroups of partial isometries
Theorem (Crisp-Laca)

\((G, P)\) is quasi-lattice ordered in the left order if and only if every pair \(a, b \in P\) has a greatest right lower bound \(a \land_r b\).
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Corollary

The following are equivalent:

- \((G, P)\) is doubly quasi-lattice ordered.
- Any pair \(x, y \in G\) with a common left upper bound in \(P\) has a least common left upper bound in \(P\) and every pair \(a, b \in P\) has a greatest left lower bound \(a \land_l b\).
- Any pair \(x, y \in G\) with a common left upper bound in \(P\) has a least common right upper bound in \(P\) and every pair \(a, b \in P\) has a greatest right lower bound \(a \land_l b\).
Examples

\((\mathbb{Z}_2, \mathbb{N}_2)\) is a doubly quasi-lattice ordered group.

\((a, b) \leq (c, d)\) if \(a \leq c\) and \(b \leq d\).

\((a, b) \lor (c, d) = (\max\{a, c\}, \max\{b, d\})\).

Let \(F_2\) be the free group with generators \(\{a, b\}\), and let \(F_2^+\) be the free semigroup. Then \((F_2, F_2^+)\) is a doubly quasi-lattice ordered group.

Most elements have no common upper bound e.g. \(a, b\) have no common upper bound.

\((\mathbb{Q} \rtimes \mathbb{Q}^*, \mathbb{N} \rtimes \mathbb{N} \times \mathbb{N})\) is doubly quasi-lattice ordered. For \((m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N} \times \mathbb{N}\) we have \((m, a) \lor l(n, b) < \infty \iff (m + a N) \cap (n + b N) \neq \emptyset\).

However, \((m, a) \lor r(n, b) < \infty\) for all \((m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N}\).
Examples

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  $$a \leq_l ab \quad b \leq_r ab$$

  Most elements have no common upper bound e.g. $a, b$ have no common upper bound.
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  \((m, a) \lor_I (n, b) < \infty \iff (m + a\mathbb{N}) \cap (n + b\mathbb{N}) \neq \emptyset\).
Examples

- \((\mathbb{Z}^2, \mathbb{N}^2)\) is a doubly quasi-lattice ordered group. 
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- Let \(\mathbb{F}_2\) be the free group with generators \(\{a, b\}\), and let \(\mathbb{F}_2^+\) be the free semigroup. Then \((\mathbb{F}_2, \mathbb{F}_2^+)\) is a doubly quasi-lattice ordered group.
  
  \[ a \leq_l ab \quad b \leq_r ab \]

  Most elements have no common upper bound e.g. \(a, b\) have no common upper bound.

- \((\mathbb{Q} \rtimes \mathbb{Q}^*, \mathbb{N} \rtimes \mathbb{N}^\times)\) is doubly quasi-lattice ordered. For \((m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N}^\times\) we have

  \( (m, a) \lor_l (n, b) < \infty \iff (m + a\mathbb{N}) \cap (n + b\mathbb{N}) \neq \emptyset \).

  However, \( (m, a) \lor_r (n, b) < \infty \) for all \((m, a), (n, b) \in \mathbb{N} \rtimes \mathbb{N}^\times\).
Let $c, d \geq 0$ and

$$BS(c, d) := \langle a, b | ab^c = b^d a \rangle.$$ 

Let $BS(c, d)^+$ be the subsemigroup generated by $\{a, b, e\}$. Then $(BS(c, d), BS(c, d)^+)$ is a doubly quasi-lattice ordered group. (Spielberg)
Let $c, d \geq 0$ and

$$BS(c, d) := \langle a, b | ab^c = b^d a \rangle.$$ 

Let $BS(c, d)^+$ be the subsemigroup generated by $\{a, b, e\}$. Then $(BS(c, d), BS(c, d)^+)$ is a doubly quasi-lattice ordered group. \textbf{(Spielberg)}

If $d \geq 0$ then $BS(1, -d)$ is a quasi-lattice ordered group (in the left order) but not a doubly quasi-lattice ordered group.
Covariant Partial Isometric Representations

Definition

Let $(G, P)$ be a doubly quasi-lattice ordered group. A partial isometric representation of $P$ is a map $W: P \rightarrow A$ such that $Wp$ is a partial isometry for all $p \in P$, $W=1$ and $WxWy=W_{xy}$ for all $x, y \in P$.

A partial isometric representation is left-covariant if it satisfies

$$W_{x}W_{x}^{*}W_{y}W_{y}^{*}=\begin{cases} W_{x} \vee l_{y}W_{x}^{*} \vee l_{y} & \text{if } x \vee l_{y} < \infty \\ 0 & \text{otherwise.} \end{cases}$$

A partial isometric representation is right-covariant if it satisfies

$$W_{x}^{*}W_{x}W_{y}W_{y}^{*}=\begin{cases} W_{x}^{*} \vee r_{y}W_{x} \vee r_{y} & \text{if } x \vee r_{y} < \infty \\ 0 & \text{otherwise.} \end{cases}$$

If a partial isometric representation is both left- and right-covariant we say that it is covariant. By convention we say $W_{\infty}=0$ to avoid case arguments.
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Let \((G, P)\) be a doubly quasi-lattice ordered group. A partial isometric representation of \(P\) is a map \(W : P \rightarrow A\) such that \(W_p\) is a partial isometry for all \(p \in P\), \(W_e = 1\) and \(W_x W_y = W_{xy}\) for all \(x, y \in P\).

A partial isometric representation is left-covariant if it satisfies
\[
W_x W_y W_x^* W_y^* = \begin{cases} 
W_x \lor_l y W_x^* \lor_l y & \text{if } x \lor_l y < \infty \\
0 & \text{otherwise}
\end{cases}
\]

A partial isometric representation is right-covariant if it satisfies
\[
W_x^* W_y W_x W_y^* = \begin{cases} 
W_x^* \lor_r y W_x \lor_r y & \text{if } x \lor_r y < \infty \\
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A partial isometric representation is *left-covariant* if it satisfies

\[
W_x W_x^* W_y W_y^* = \begin{cases} 
W_{x \lor y} W_{x \lor y}^* & \text{if } x \lor y < \infty \\
0 & \text{otherwise.}
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Covariant Partial Isometric Representations

Definition
Let \((G, P)\) be a doubly quasi-lattice ordered group. A \textit{partial isometric representation} of \(P\) is a map \(W : P \to A\) such that \(W_p\) is a partial isometry for all \(p \in P\), \(W_e = 1\) and \(W_x W_y = W_{xy}\) for all \(x, y \in P\).

A partial isometric representation is \textit{left-covariant} if it satisfies

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W_x W_x^* W_y W_y^* = \begin{cases} 
W_{x \lor_I y} W_{x \lor_I y}^* & \text{if } x \lor_I y < \infty. \\
0 & \text{otherwise.}
\end{cases}
\]

A partial isometric representation is \textit{right-covariant} if it satisfies

\[
W_x^* W_x W_y^* W_y = \begin{cases} 
W_{x \lor_R y} W_{x \lor_R y}^* & \text{if } x \lor_R y < \infty \\
0 & \text{otherwise.}
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\]

If a partial isometric representation is both left- and right-covariant we say that it is \textit{covariant}.

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Covariant representations properties

We can rewrite the covariance identities as:

\[ W_x^* W_y = W_x^* W_{x \vee y} W_y^{-1}(x \vee y) \]

\[ W_x W_y^* = W_{(x \vee r y)x^{-1}} W_{x \vee r y} W_y \]
Covariant representations properties

We can rewrite the covariance identities as:

\[ W_x^* W_y = W_x^* W_{x \vee l} y W_y^{-1} (x \vee l y) \]
\[ W_x W_y^* = W_{(x \vee r y) x^{-1}} W_{x \vee r y} W_y^* \]

**Lemma**

Let \( W : P \to A \) be a covariant partial isometric representation. Any product of the form \( W_{n_1} W_{n_2}^* W_{n_3} W_{n_4}^* \ldots \) where \( n_i \in P \) is either 0 or may be expressed as \( W_p^* W_q W_r^* \) for some \( p, q, r \in P \) satisfying \( p \leq_l q \) and \( r \leq_r q \).
Analogue of Truncated shifts

Definition
Let $A \subset P$. Define $J^A : P \to B(\ell^2(a))$ by

$$J^A_p \epsilon_a = \begin{cases} \epsilon_{pa} & \text{if } pa \in A \\ 0 & \text{otherwise} \end{cases}.$$

Lemma
1. $J^A_p J^A_q = J^A_{pq}$ if and only if for all $a, b \in A$ we have
   $$\{x \in P : a \leq_r x \leq_r b\} \subseteq A.$$
2. $J^A$ is left-covariant if and only if, for all $a, b \in A$ with a common right upper bound in $A$, $a \wedge_r b \in A$.
3. $J^A$ is right-covariant if and only if, for all $a, b \in A$ with a common right lower bound in $A$, $a \vee_r b \in A$. 
Direct sums of Truncated shifts

Let $(G, P)$ be a doubly quasi-lattice ordered group. For $a \in P$ let $I_a := \{ x \in P : x \leq_r a \}$. Let $\{ \epsilon_x \}$ be an orthonormal basis for $\ell^2(I_a)$. Then $J^a : P \rightarrow B(\ell^2(I_a))$ defined

$$J^a_p \epsilon_x = \begin{cases} 
\epsilon_{px} & \text{if } px \leq_r a \\
0 & \text{otherwise}
\end{cases}$$

is a covariant partial isometric representation.
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\epsilon_{px} & \text{if } px \leq_r a \\
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\]

is a covariant partial isometric representation.
Let \(J : P \to B(\bigoplus_{a \in P} \ell^2(I_a))\) be defined as \(J_p = \bigoplus J^a_p\). Let \(C^*(J)\) be the \(C^*\)-algebra generated by \(\{J_p : p \in P\}\).
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Let $J : P \to B(\bigoplus_{a \in P} \ell^2(I_a))$ be defined as $J_p = \bigoplus J^a_p$. Let $C^*(J)$ be the $C^*$-algebra generated by $\{J_p : p \in P\}$.

**Lemma**

The set $S := \{ J^*_p J^*_q J^*_r : p, q, r \in P, p \preceq I q, r \preceq_r q \}$ is linearly independent and $\text{span } S$ is a dense unital $*$-subalgebra of $C^*(J)$. 
Proposition

There is a $C^*$-algebra $C^*(G, P)$ generated by partial isometries $\{v_p : p \in P\}$ which has the following property: for every covariant partial isometric representation $W : P \to A$ there is a unital homomorphism $\pi_W : C^*(G, P) \to A$ such that $\pi_W(v_p) = W_p$. 
Faithful Representations of $C^*(G, P)$

When is $\pi_J : C^*(G, P) \to C^*(J)$ faithful?

**Proposition**

*There is a norm-decreasing linear idempotent $E : C^*(G, P) \to \text{span}\{v_p^* v_p v_r v_r^* : p, r \in P\}$ such that*

$$E(\sum \lambda_{p,q,r} v_p^* v_q v_r) = \sum \lambda_{p,pr,r} v_p^* v_{pr} v_r.$$

**Definition**

A doubly quasi-lattice ordered group $(G, P)$ is amenable if $E$ is faithful for positive elements, in the sense that $E(a^* a) = 0$ implies $a = 0$.

**Theorem**

The homomorphism $\pi_J : C^*(G, P) \to C^*(J)$ is faithful if and only if $(G, P)$ is amenable.
Faithful Representations of $\mathbb{C}^*(G, P)$

When is $\pi_J : \mathbb{C}^*(G, P) \to \mathbb{C}^*(J)$ faithful?

**Proposition**

*There is a norm-decreasing linear idempotent $E : \mathbb{C}^*(G, P) \to \overline{\text{span}\{ v_p^* v_p v_r v_r^* : p, r \in P \}}$ such that*

$$E(\sum \lambda_{p,q,r} v_p^* v_q v_r v_r^*) = \sum \lambda_{p,pr,r} v_p^* v_{pr} v_r^*. $$

**Definition**

A doubly quasi-lattice ordered group $(G, P)$ is *amenable* if $E$ is faithful for positive elements, in the sense that $E(a^* a) = 0$ implies $a = 0$. 
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When is $\pi_J : C^*(G, P) \rightarrow C^*(J)$ faithful?

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A doubly quasi-lattice ordered group $(G, P)$ is **amenable** if $E$ is faithful for positive elements, in the sense that $E(a^*a) = 0$ implies $a = 0$.

**Theorem**

The homomorphism $\pi_J : C^*(G, P) \rightarrow C^*(J)$ is faithful if and only if $(G, P)$ is amenable.
Faithful representations

Definition

Let $W : P \to A$ be a covariant partial isometric representation. Let $L^W_{(x_1, x_2)} = W_{x_1} W_{x_1}^* W_{x_2}^* W_{x_2}$. A covariant partial isometric representations $W : P \to A$ sees all projections if, for every finite set $F \subset P_r \times P_l$ and $(x_1, x_2) \notin F$ such that $(x_1, x_2)$ is a lower bound for $F$, we have

$$\prod_{y \in F} (L^W_{(x_1, x_2)} - L^W_y) \neq 0.$$
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Let \( W : P \to A \) be a covariant partial isometric representation. Let \( L^W_{(x_1, x_2)} = W_{x_1} W^*_{x_1} W^*_{x_2} W_{x_2} \).

A covariant partial isometric representation \( W : P \to A \) sees all projections if, for every finite set \( F \subset P_r \times P_l \) and \( (x_1, x_2) \notin F \) such that \( (x_1, x_2) \) is a lower bound for \( F \), we have

\[
\prod_{y \in F} (L^W_{(x_1, x_2)} - L^W_y) \neq 0.
\]

Theorem
Let \( (G, P) \) be an amenable group and let \( W : P \to A \) be a covariant partial isometric representation. Further, let \( \pi_W \) be the corresponding homomorphism of \( C^*(G, P) \). If \( W \) sees all projections then \( \pi_W \) is faithful.
Amenable doubly quasi-lattice ordered groups.

Definition

Suppose that \((G, P)\) and \((K, Q)\) are doubly quasi-lattice ordered groups. A controlled map is an order preserving homomorphism \(\phi : (G, P) \rightarrow (K, Q)\) such that

1. the restriction \(\phi|_P : P \rightarrow Q\) is finite-to-1,
2. for all \(x, y \in P\) satisfying \(x \lor_l y \neq \infty\) we have \(\phi(x) \lor_l \phi(y) = \phi(x \lor_l y)\), and
3. for all \(x, y \in P\) satisfying \(x \lor_r y \neq \infty\) we have \(\phi(x) \lor_r \phi(y) = \phi(x \lor_r y)\).

Theorem

Let \((G, P)\) and \((K, Q)\) be doubly quasi-lattice ordered groups with a controlled map \(\phi : (G, P) \rightarrow (K, Q)\). If \(K\) is amenable then \((G, P)\) is amenable and \(C^*\)(\(G, P\)) is nuclear.
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Let \((G, P)\) and \((K, Q)\) be doubly quasi-lattice ordered groups with a controlled map \(\phi : (G, P) \to (K, Q)\). If \(K\) is amenable then \((G, P)\) is amenable and \(C^*\)-(\(G, P\)) is nuclear.
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3. for all \(x, y \in P\) satisfying \(x \vee_r y \neq \infty\) we have \(\phi(x) \vee_r \phi(y) = \phi(x \vee_r y)\).

Theorem
Let \((G, P)\) and \((K, Q)\) be doubly quasi-lattice ordered groups with a controlled map \(\phi : (G, P) \rightarrow (K, Q)\). If \(K\) is amenable then \((G, P)\) is amenable and \(\mathbb{C}^*(G, P)\) is nuclear.
If $G$ is an amenable group then $(G, P)$ is amenable. The abelianization map $\phi: (F_n, F_n^+ \cap n) \to (\mathbb{Z}_n, N_n)$ given by $\phi(a_i) = e_i$ is a controlled map.
If $G$ is an amenable group then $(G, P)$ is amenable.
Amenable doubly quasi-lattice ordered groups

- If $G$ is an amenable group then $(G, P)$ is amenable.
- $(\mathbb{F}_n, \mathbb{F}_n^+)$ is amenable. The abelianization map
  $\phi : (\mathbb{F}_n, \mathbb{F}_n^+) \to (\mathbb{Z}^n, \mathbb{N}^n)$ given by $\phi(a_i) = e_i$ is a controlled map.
If $G$ is an amenable group then $(G, P)$ is amenable.

$(\mathbb{F}_n, \mathbb{F}_n^+) is amenable. The abelianization map
\[ \phi : (\mathbb{F}_n, \mathbb{F}_n^+) \rightarrow (\mathbb{Z}^n, \mathbb{N}^n) \]
given by $\phi(a_i) = e_i$ is a controlled map.